ON SEQUENCES OF SETS OF INTERVALS CONTAINING A GIVEN SET OF POINTS

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1. In the prosecution of investigations into sets of intervals on the straight line, some of which are to be found in the *Proceedings of the London Mathematical Society*, I was naturally led to consider sequences of sets of intervals * formed in the following manner:—Taking any set of points E, describe round each point an interval having that point as internal point; diminish the length of each interval, according to any law, in such a way that all tend simultaneously towards zero, the original set of points always remaining internal to their corresponding intervals. What do we know about those points which are always internal to the intervals of every set (inner limiting set)? The chief properties of such a set may, I find, be summed up in the following six theorems:—

(1) The inner limiting set consists of E, together with certain points of the first derived set E'; the latter points may sometimes be absent.

(2) The inner limiting set may contain every point of E'.

(3) If the content of the intervals † is ever less than that of E', there is a more than countable set of points E' not contained in the inner limiting set.

(4) The potency of the inner limiting set is the same as the potency of E', unless E contains no component dense in itself, while E' is more than countable.

(5) If E contains no component dense in itself, while E' is more than countable, the inner limiting set may be either countably infinite

* A special case of this was considered by Borel in his *Leçons sur la Théorie des Fonctions*; cf. also Baire's sets of the second category, and the general theory of content and kindred subjects as developed by Riemann, Cantor, H. J. S. Smith, and others.

† That is the content of a set of non-overlapping intervals having the same internal points as the given set (*Proc. London Math. Soc.*, Vol. xxxv., p. 386). It is well known that when E is countable the intervals can be so constructed that their content becomes less than *any* assignable quantity.

or have the potency c; and we can so arrange the intervals that the inner limiting set consists of E alone.

(6) In general, we can so arrange the intervals that those points of the inner limiting set which are not points of E are limiting points only of U, the greatest component of E which is dense in itself.*

Theorems 1, 2, and 3 are almost obvious; for convenience I give the formal proof. Theorem 4 and the first part of Theorem 5 are to be found in a paper presented to the Sächsischer Gesellschaft der Wissenschaft†; Theorem 6, which includes the latter part of Theorem 5 as a special case, is here stated and proved for the first time. The theorem employed requires some consideration of the analysis of sets of points into what Cantor calls *adherents and coherents*, and I have found it necessary to go shortly into this matter for the purposes of the proof.

2. *Proof of Theorem* 1.

Suppose P to be a point of the inner limiting set not contained in E; then we can assign an interval from each successive set containing P as internal point, and each of these intervals contains a point of E; since the length of these intervals decreases without limit, it follows that P is a limiting point of points of E. Q. E. D.

3. *Proof of Theorem* 2.

Take all the intervals of any set equal in length, and let that length diminish without limit for the successive sets; then every limiting point of E must be internal to intervals of every set, and therefore the inner limiting set consists of E together with all those points of E' which were not points of E. Q. E. D.

4. *Proof of Theorem* 3.

Let I be the content of E', and let $J < I$ be the content of one of the sets of intervals, and e any small positive quantity less than $J-I$.

If possible, let those points of E' which do not belong to the inner limiting set be arranged in countable order $P_1, P_2, \ldots$. Round each point P_i as centre describe an interval of length $e/2^{i+1}$, and add these intervals to those of the set in question; then the content J of the latter is increased

* As in Theorem 4, if the content of the intervals be ever less than that of U' (the first derived of U), there will be a more than countable set of limiting points of U not contained in the inner limiting set. A similar theorem evidently holds for any closed component of E'.

† "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipz. Ber.*, pp. 287–293.—*August 1st*, 1903.

at most by $\frac{1}{2}e$. Thus we have shut up all the points of the closed set E' in a set of intervals of content less than I, which is known to be impossible. Hence the points in question cannot be countable. Q. E. D.

5. *Adherences and Coherences.*

Given any set of points E, Cantor denotes as an adherent of E any isolated point of E, and as a coherent any limiting point contained in E. The set of all the adherents he calls the *adherence*, and of all the coherents the *coherence*, and denotes these symbolically by the addition of an a or a c respectively to the symbol for the whole set: thus, $E = E_a + E_c$. If E_c is neither an isolated set nor dense in itself, it will have both adherents and coherents, and we may proceed a stage further, $E_c = E_{ca} + E_{cc}$. We notice that the points of E_{ca} *must* be limiting points of E_a.

Similarly, if E_{cc} be neither an isolated set nor dense in itself, we can proceed a stage further and, generally, as long as we do not arrive at a coherence which is either an isolated set or dense in itself, we can proceed on another stage in our analysis. During the process *each new adherence consists of points which are limiting points of every preceding adherence.* If the process never comes to an end, we can examine whether, or no, the infinite series of coherences

$$E_c,\ E_{cc},\ E_{ccc},\ \dots \tag{1}$$

has any common point. Since these sets are not necessarily closed, they may have no common point, and in this case the original set consists only of the countably infinite series of adherences

$$E_a,\ E_{ca},\ E_{cca},\ \dots, \tag{2}$$

and is, therefore, countable, and has no component dense in itself.

If, however, the coherences (1) have at least one point common, we can "deduce" from them a set, viz., that consisting of all their common points. E will then consist of the series of adherences (2), together with this "deduced coherence." We can then continue the process with this deduced coherence, and so on. It can then be shown that the adherence of this deduced coherence consists, as before, of points which are limiting points of every preceding adherence.

If, now, E have any component dense in itself, it is easily seen that it has a definite largest component dense in itself, U, such that U contains every component of E which is dense in itself, and that those points of E which are not points of U are countable; if there be no such component U, however, E is itself countable.

From this it is easy to show that, after a countable set of stages, the process of analysis into adherences and a coherence comes to an end, leaving us either with U as residuary coherence or, if there be no U, without any residuary coherence at all.

Arranging the successive adherences then in *countable* order $A_1, A_2, \ldots$, *we have the following general analysis :—* $E = U+A_1+A_2+\ldots$ *(where U may, of course, in a special case be absent, and the series of A's be finite or countably infinite), U denoting the greatest component of E which is dense in itself, and $A_1, A_2, \ldots$ isolated, and therefore countable, sets (adherences) such that each A_i consists of points which are limiting points of every adherence A_h which, in the natural order, precedes it.*

6. *Proof of Theorem* 6.

Having arranged the coherences in countable order, let us arrange the points in each adherence also in countable order, and let P_{ij} denote the j-th point of A_i.

We will now show how to construct the intervals in such a way that the only extraneous points which can remain in the inner limiting set are limiting points of U.

Let us assume any finite positive quantity l. Then, since A_i contains none of its limiting points, we can assign a definite largest interval, of length less than l, say $2d_{ij}$, with P_{ij} as centre, such that inside this interval there is no point of A_i, except P_{ij}; one or both of the end-points of this interval may be points of A_i or limiting points of A_i.

The law of intervals is now that, round each point P_{ij} as centre, we describe an interval of length d_{ij} for the first set of intervals, and $e^n d_{ij}$ for the n-th set of intervals, where e is any assigned small positive quantity less than 1.

The law of intervals for points of U (if it exists) may be any we please.

Now, if there be any point of the inner limiting set not belonging to E, it is, by Theorem 1, a point of the first derived set E'. Let then Q be any point of E' which is not a point of E.

If Q be not a limiting point of E_a, it cannot be a limiting point of any adherence, whence, considering all the stages successively, it follows that Q must be a limiting point of U. Assuming then that Q is not a limiting point of U, it must be a limiting point of E_a, and may be a limiting point of some adherent subsequent to E_a in the natural order. Let A_i be any adherence of which Q is a limiting point; then Q is a limiting point of every adherence A_h preceding A_i in the natural order. Hence, by the construction, Q is external to all the intervals d_{ij} whose centres belong to A_i, or to any adherence preceding A_i in the natural order.

Thus, if Q be a limit for every A_i, Q cannot be a point of the inner limiting set without remaining always interior to intervals described round points of U, which it cannot do, since, by hypothesis, Q was not a limiting point of U. Such a Q could not then belong to the inner limiting set.

We must then be able to assign an integer h, such that Q is a limiting point of A_h, but not of any adherence subsequent to A_h in the natural order. Let A_i denote the adherence next after A_h in the natural order. Then we can assign a definite largest interval, of length say d, with Q as centre, such that inside this interval there is no point of A_i and therefore (since any point of an adherence subsequent to A_i is a limiting point of A_i) no point of any adherence subsequent to A_i in the natural order.

Let us now determine an integer m, so that $e^m l < d$; then, for all values of i and j, $e^m d_{ij} < d$; therefore Q is external to all the intervals $e^m d_{ij}$ of the m-th set of intervals whose centres P_{ij} belong to adherences subsequent to A_h in the natural order. Also, since Q was shown to be external to all the remaining intervals of the m-th set, it follows that, if Q be a point of the inner limiting set, it must, from and after the m-th set, be interior to intervals described round points of U, which again is contrary to the hypothesis that Q was not a limiting point of U. Thus it is shown that no point Q belonging to E', but not to the first derived U' of U, can be a point of the inner limiting set of the intervals as we have constructed them. Q. E. D.

This is the proof of Theorem 6, and, by supposing U to be non-existent, of Theorem 5 also.