

FROM THE COMPLEX TO THE SIMPLE.

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In mathematics we progress from the days when we first counted on our fingers till we advance to and through calculus into the more complicated machinery. Our steps take us from the simple ideas to complex ones. This is the natural order, and no subject adheres to it more closely than mathematics. But we have made some mistakes by confusing *lengthy* processes with *complex* ones. A California redwood requires more time to mature than a violet or arbutus but the process is not therefore more complex.

Consider, for example, the subject of factoring and the order in which we are led from the simple problems to the more complicated. We begin with factoring $3x+3y$, learning in the first step how to detect a monomial factor and advancing to $6x^4y+9x^3y^2+27x^2y^5$. Then we begin a new type such as $x^2+7x+12$ and proceed through a list of problems, in each of which the coefficient of the square is unity, until we reach a quantity like $x^2-84x+243$. Again we start a type $x^2+4x-12$ which, to judge from the books, is more complicated than any of the preceding because the constant term is -12 instead of $+12$. And at the end of each of these lists we have similar quantities from which a monomial factor must first be extracted according to the printed "hints." Next we begin with $2x^2+7x+5$, the numbers 2 and 5 being chosen because they are prime numbers; and then lead up to $12x^2+41x+24$, since 12 and 24 have various factors, the problems afterwards being complicated by the introduction of minus signs and monomial factors. Somewhere in this arrangement we consider perfect square trinomials, and the differences of two squares, proceeding in each case from x^2+2x+1 to $4x^2+12xy^2+9y^4$ or from x^2-y^2 to $9x^4-16y^2$ and always ending each list with some monomial factors.

Such an arrangement we call pedagogically sound because it leads from the simple to the complex. The following paragraphs present a different order which may appear to proceed from the complex to the simple but which is nevertheless sound pedagogically.

The class, let us assume, has reached the stage where it can perform mentally such multiplications as $(6m^2-7n)(2m^2+3n)$ but has had no work at all in any kind of factoring. Some day, apparently as if by accident, I very suddenly interrupt the pupil

reciting such a product and, pretending that a burst of inspiration has unexpectedly come upon me, I ask, "I wonder if anyone can tell me what quantities I multiplied together if I tell what the product is?" On the blackboard I write $12x^2 - 7xy - 10y^2 = (\quad) (\quad)$. The $12x^2$ I must have obtained by multiplying $6x$ by $2x$, or $4x$ by $3x$, or $12x$ by x , the $-10y^2$ by multiplying $-5y$ by $2y$ or $5y$ by $-2y$, etc. We experiment to see which of these will produce $-7xy$ for a middle term. No rules are given, not even for signs as the class must surely know how to multiply any two terms. Then the class is sent to the blackboard to work problems like the one above, but not a bit easier. No coefficient is unity; every quantity contains two letters. A quantity like $x^2 + 7x + 12$ will not arise till several days later, and $x^2 + 6x + 9$ arouses no comment even though every pupil writes it as $(x+3)(x+3)$; and even $9x^2 - 4y^2$ causes no stir as the pupil tries for such combinations as will make the middle term $0xy$. Zero, I have frequently said, is just as good and respectable a citizen of the number family as any other citizen; and since we have never had occasion to divide by it, we assume it obeys the same laws as other citizens. Thus the work is begun not with the simplest problem of its type but with a more general type.¹

Then some day when the class is at the blackboard I read off $32x^2 + 8x - 12$ and the results will be either $(8x-4)(4x+3)$ or $(2x-1)(16x+12)$, followed by an argument as to which is right. We verify the result by multiplication and also by the substitution of some number, say $x = 2$. In verifying it, some of the pupils have obtained $132 = 12 \cdot 11$, others $132 = 3 \cdot 44$ concluding that both are correct. I then ask, "How, in arithmetic, can we show that $12 \cdot 11 = 3 \cdot 44$ without multiplying the factors?" We are thus led to consider prime factors and learn that $8x-4$ and $16x+12$ can be factored further. After seeing that the monomial factor is visible to all keen pilots in the original polynomial as well as in one of its factors, our next discussion hinges on the question "Which is the better method: to dislodge the monomial factor at the beginning of the process or at the end?" We conclude that it is best done at the beginning, for then our work will be more rapid because $8x^2$ has fewer factors than $32x^2$.

¹To add a few words on the subject of factoring, I might say that $x^2 - y^2$ is not to be regarded as the difference of two squares but as the sum of x^2 and $-y^2$, and this attitude must have been adopted and used previous to factoring. Similarly, $x^3 - y^3$ is not the difference of two cubes but the sum of x^3 and $-y^3$. The memorized rules for factoring $x^3 + y^3$ and $x^3 - y^3$ will then read: one factor is a binomial and is the sum of the two cube roots; the other factor is a trinomial, two of whose terms are the squares of the cube roots and the other term is the negative product of the two cube roots. However, it is better to postpone the factoring of the sum of two perfect cubes until such a quantity is reached in the study of the simplification of fractions. At that point the pupil is more ready to appreciate the fact that if we know one of the factors of a polynomial the other factor may be found by division.

After this discussion the next problem assigned is $18x^4y + 15x^3y^2 - 12x^2y^3$, not $2x^2 + 14x + 24$.

After using this method of proceeding from the complex to the simple for several years, I decided to experiment with other subjects to see if it was always necessary to begin with the apparently simple and lead up to the apparently complex, also aiming to uncover some general theory that would explain where and why the method works.

Consider a subject in which the method will not work: quadratic equations. The following equation are typical of our steps in the subject: $7x^2 = 63$, $3x^2 = 25$, $x^2 + 6x + 9 = 25$, $x^2 + 4x = 5$, $x^2 + 10x = -21$, $x^2 + 6x + 7 = 0$, $2x^2 + 8x - 9 = 0$, $4x^2 - 20x + 19 = 0$, etc. Here, obviously, we can not reverse the order and begin with the last equation. The reason is equally obvious. To solve $ax^2 + bx + c = 0$ the pupil must know how to solve $x^2 + px + q = 0$ and to do this he must know $x^2 + 2kx + k^2 = m^2$. Each step in the solution of $ax^2 + bx + c = 0$ involves a reduction of the equation to a simpler standard form. On the other hand, factoring $8x^4 + 2x^2y^2 - 15y^6$ does not involve reducing this to a quantity the coefficient of whose leading term is unity.

We need proceed through the simple problems A, B, C, D, to the complex problem E only when the individual steps A, B, C, D arise in the solution of E; i. e., when the problem consists in reducing E to D, D to C, C to B, etc.

For example, when beginning fractional equations we have the steps: A, $2x/3 = 8$ in which we learn the multiplication axiom. Then B, $x/5 + x/6 = 11$ in which we learn to select a L. C. M. Next C, $x/7 + x/14 = 9$ wherein the L. C. M. is not the product of the denominators. Then step D, $3x/8 + 2x/5 = 1/10$ etc. These are followed a semester later by $(5x+12)/6 - 4(2x+7)/11 = -1/3$ which in turn leads to $6/x + 5/2x = 1/6$ and $(x+6)/(x-3) = 1/8 + (3x-4)/5$. Does the solution of any of these equations consist in reducing it to one of the type immediately preceding? Since it does not, we may begin anywhere in this list just as well as at the first. The word L. C. M. need not be used at all. Until and unless the pupil reaches a problem like

$$\frac{x+6}{4x-14} - \frac{2x+3}{9x+15} = \frac{6x+1}{6x^2-11x-35}$$

he need not worry about the *least* common multiple very much. The multiplier is merely a broom which properly applied to the fractions is guaranteed to sweep away any and all denominators.

Its only important characteristic is that it must be *divisible* by every denominator. To avoid using a carpet broom when a whisk broom will do the work, the pupil must first observe whether the denominators are different or involve repetitions. Then, it is just as easy to find a quantity divisible by $2x$ and 5, or by $(x-3)$ and 8 as to find one for 7 and 14. And at the time when a pupil is ready to work with fractional equations it is no more difficult for him to multiply binomial quantities than monomial terms.

The objection will be raised that there are other things to be taken into consideration; for example, even though a pupil may begin at the step E, will he be able to work the prose problems whose equations are of the type E without first working prose problems whose equations are of the kind in the steps A, B, C, D? To this question we can answer that if we include in our drill problems only such quantities or equations as may arise in prose problems we may as well throw away half of our algebra, as the exercises for drill are always much more complex than anything which arises in the prose problems. Ability to solve prose problems comes only gradually and their difficulty lies not in the algebra but in the lack of imagination or visualizing ability of the pupil. The pupil would have less trouble with them if he could see the automobile starting two hours later and then overtaking the train, could draw a picture of the Washington Monument, had made a few thousand dollar investments or had mixed some brands of tea. In fact, the more practical we make our problems the more difficult do they become. Hence this paper applies to the drill problems mostly. Mostly, but not entirely; perhaps some of our prose problems also need revision. As an illustration of this, let us consider "age" problems and some time, rate, and distance problems.

The ages of John and Henry we will say are x and $2x$ respectively; then an equation is formed comparing their ages five years hence. In other words, only two years, 1921 (now) and 1926 (5 years hence), are involved in the problem. Is the problem any more difficult if we compare Henry's age in 1931 with John's age in 1926, or Henry's in 1927 with John's in 1908? Is any advantage gained by thus complicating (?) the problem? There are several. One of the things which the pupil is supposed to learn from age problems is the translation into algebra of such words as "five years hence," "ten years ago," etc. If both of these terms occur in the same problem, the pupil's attention

is called to the difference between the plus and minus sign more sharply than if the idea "hence" occurs in one problem and the "ago" in another. Another advantage is seen when we write out the equation for the problem: A father is nine times as old as his son; nine years hence he will be three times as old. Here the equation is $9x+9 = 3(x+9)$. Note the prominence of the number 9. There are in this problem really two different nines, one arising from "nine times as old" and the other from "nine years hence." The pupil may easily confuse the two ideas because he is not called upon to distinguish between them². The trouble is that, without introducing fractions, we can make up few "age" problems which do not mix and repeat a number. But this trouble does not arise when using both "ago" and "hence" in the same problem. Neither would it arise if we did not feel compelled to make the problem simple by saying always that one age is a multiple of the other, but said that one age exceeded a certain multiple of the other by some amount.

When the subject of rate problems is to be considered and the pupil has become acquainted with the relations between d , r and t , the problems presented to him are variously graded. And it seems that a problem which can be printed in two lines must be simpler than one requiring three or four lines. But whether the men travel in the same or in opposite directions, toward or from each other, or whether the unknown is d , r or t , it is useless to try to grade the problems into types so that the solution of one involves a reduction to a previous type. Hence, as in factoring, we may as well begin with a general type instead of considering each type separately. And our grading will be based not on the number of lines required to print the problem but on the extent to which the problem calls on the pupil's imagination or visualizing ability. Thus, it is easy to draw a picture of the distances involved in the following problem³: A and B travel toward each other from points 200 miles apart, A starting 3 hours before B. A's rate per hour is 6 miles more than twice B's. When B has

²The problem may be compared to that of asking for the square of a^2 . From the pupil's answer we can not tell whether he knows the laws of exponents or not.

³Most teachers, I think, believe that the above problem must be approached gradually through a list in which A's rate must at first be the same as B's, then double B's, then six more than B's or six less, etc.; that A's time must first equal B's, then one hour more, then two hours less, etc.; that their distances traveled must first be the same, then A's 60 miles more than B's etc. That is one of the mistakes we have been making. Such approaches may be greatly minimized if the pupil, not only during the first month but occasionally thereafter, has drill in such exercises as "What quantity is 6 more than twice another?" or "Write a quantity which exceeds b by the square of a ." Such translations ought to be reviewed whenever the pupil for the first time meets an exponent, a fraction, parenthesis, radical, or any other mathematical symbol.

been on the road 4 hours he learns that A is 68 miles away. Find the rate of each. This problem would have been increased, not lessened, in difficulty if it had said that B meets A in 4 hours. The pupil has a harder time seeing that their distance apart is zero miles than 68 miles. The *zero* ought to arise later just as it does for the middle term when factoring a polynomial.

Having seen how some of the prose problems may need revision let me point out a few places where we need to change the drill work.

The axiom of division is usually illustrated by an equation like $3x = 12$. The equation $3x = 14$ would be better especially if the pupil is required to answer 4 $\frac{2}{3}$ instead of $\frac{14}{3}$ as only then can we be sure that he is actually dividing and not guessing. The equation $1.4x = 3.738$ and others like it ought not to come at the end of the list but at the beginning.

The axiom of multiplication is usually begun with an equation like $x/3 = 5$. The pupil may answer $x = 15$ without having done any multiplication as he is merely recalling having seen $15/3 = 5$. If we begin with $2x/3 = 12$ the pupil's impression is merely "multiply by one of the numbers and divide by the other to get the answer" and the next day he is just as likely to multiply by 2 and divide by 3. It is better to begin with $2x/3 - 5/7 = 13$, writing the next step as $21(2x/3 - 5/7) = 21.13$. This implies that the multiplication axiom will not be used until after the introduction of parentheses.

The drill in long division should not begin with dividing $x^2 + 7x + 12$ by $x + 4$. Nothing is gained and much is lost by having the remainder zero, or the problem so simple that the answer can be guessed. Algebra becomes formal and mechanical when the problems do not require thinking or attention. Begin with $(10x^2 - 7xy + 8y^2) \div (2x - 3y)$ or $(8a^3 - 2a^2 - 27a + 20) \div (2a - 3)$.

When drilling on the multiplication of binomials mentally, the type $(2a - 3b)(2a + 3b)$ should not be considered as distinct from any other binomials, and the type $(2a + 3b)^2$ should be considered *last* of all, not first.

In an earlier paper⁴ I pointed out why the simplification of $6(3x - 2)(5x + 4) - 2(x + 5)(3x - 4)$ should precede $(x + 2)(x + 3) - (x + 4)(x + 5)$

When quadratics are solved by factoring $18x^2 - 9x - 20 = 0$ ought to precede $x^2 - 3x + 10 = 0$.

The simplification of fractions ordinarily begins with $6/4$

⁴SCHOOL SCIENCE AND MATHEMATICS, May, 1920, page 436.

and $2a/6$ and leads up to a^2/ab , $42x^2/14xy$, $2a/(a^2+ab)$, $(x^2-6x+8)/(x^2+x-6)$, etc. Much hard work is always necessary to convince the pupil that he may not cancel the x s in $(x^2-1)/(xy+y)$ but may in x^2/xy . This difficulty can be lessened by beginning with $(15x^2-x-2)/(3x^2+13x+4)$ and using in the list of exercises some quantities which can not be factored at all (and hence can be reduced only by long division) and some quantities in which no factors are alike so that again no cancellation is possible. After the pupil has seen that cancellation is not always possible then $a^2/(ab+a^2)$ is considered, and a^2/ab last of all.

It may well be that I have been carried away by my own enthusiasm; that, having used a method successfully in factoring or in fractional equations, I have stretched the method too far, and forced it to work where it was clumsy. But to those who believe in trying a method for its possible merits, I suggest the above plans.

AN EXPERIMENT IN SOCIAL HYGIENE AT CARLETON COLLEGE

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Carleton College at Northfield, Minnesota, is one of the few so-called small colleges in the country doing aggressive work in social hygiene at the present time. The experiment here described was made possible during the year 1919-1920 because of the previous work done in physiology and sex hygiene by Dr. Neil S. Dungay, who has built up a splendid department of biology in the college. The department does work along four distinct lines: Zoology, botany, general biology and public health. It was in the latter work that I had the most interest. Coming as I did fresh from a year of war work, the need for social hygiene loomed strong. Two courses were established in the college for advanced pupils, one social hygiene and one for the teaching of sex hygiene. A third course, called "The Human Body"—a three hour course—was required of all Freshmen. The course, after a preliminary study of a few forms of plant and animal life as a background, took up the physiology and hygiene of the human body. It met a practical need of the students, many of whom were thrown, for the first time, on their own resources. Prior to my coming to Carleton, Dr. Dungay had devoted part of the time toward the close of the course to a specific treatment of sex hygiene. I took two out of three