



## LXV. On the beats of consonances of the form $h : 1$

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( $\gamma$ ) The refraction-equivalents of organic compounds so closely depending on the chemical constitution of those bodies shows the very close relation which exists between the particles which transmit light and the elements of the body.

( $\delta$ ) The study of optical rotation affords an insight into the close relationship between matter and light.

( $\epsilon$ ) The action of light in altering the electrical resistance of bodies and also in producing sound shows that there is no break in continuity between sound and light.

LXV. *On the Beats of Consonances of the Form  $h : 1$ .*

By R. H. M. BOSANQUET.

[Concluded from p. 435.]

*Combination-Tones arising from Terms of Orders higher than the first, in the Transforming-structure of the Ear.*

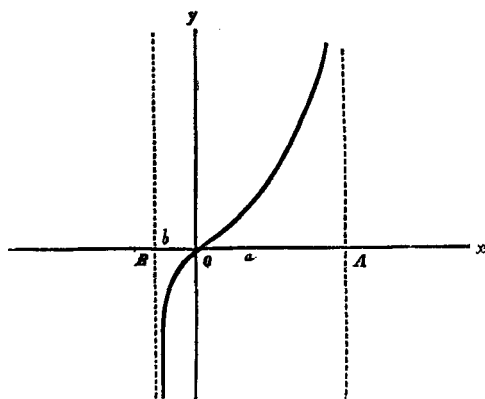
57. **H**ELMHOLTZ pointed out the way in which the hypothesis of asymmetry in the transmitting mechanism of the ear gives rise to the combination-tone of the first order, which he called the difference-tone. This asymmetry was represented in his investigation by a term of the second order in the force called into play by a given displacement. Helmholtz further indicated the tones to which the existence of terms of the third order gives rise, and pointed out that tones of the fourth order &c. would give rise to other combination-tones not further specified.

58. In specifying those combination-tones which arise from the terms of the third order, Helmholtz pointed out that one of them was a combination-tone of the second order "in the sense indicated by Hallström." This is also the sense in which the expression "combination-tone of the second order" is used by Helmholtz himself in the text of his work; and it means apparently a combination-tone which arises from a combination-tone or combination-tones of the first order when combined with any other notes present, or with each other, according to the law of combination-tones of the first order.

59. It is clear, however, that the principal combination-tones which arise from the terms of higher orders in the transmitting mechanism of the ear, are derived directly from the primary tones, and are not materially influenced by the secondary series of tones. This is obvious, on the one hand, since all the resultant tones, according to the principles of their origin, are of the nature of small quantities compared with the primary tones, and, on the other hand, because the tones derived from the terms of higher orders are in fact produced

with the greatest intensity when the tones derived from terms of lower orders are weak or evanescent. This fact has been used by König as one of the most powerful objections to the theory of combination-tones as hitherto expounded; and, indeed, the objectionable character of some of the hypothetical derivations by combination given by Helmholtz\* must have struck many readers independently.

60. I shall now examine Helmholtz's hypothesis of asymmetry in a little more detail; and I think it will appear that it leads, by tolerably simple mathematical treatment, to the development of the combination-tones of the higher orders, under the circumstances under which they actually exist, and independently of the combination-tones of the lower orders.



61. Let 0 represent the position of rest of a point free to move along the line  $0x$  between the points A and B, subject to certain forces in that line. Suppose that these forces are of the nature of springs tending to resist the departure from 0, and that on arrival at the points A, B, at distances  $a$ ,  $b$ , from 0 on either side, the springs ultimately go up against dead walls, so that further displacement is resisted with an infinite force. If we set off the forces as ordinates, they may be represented as in the figure; and analytically they may be expressed by such an assumption as

$$y = \frac{kx}{\left(1 - \frac{x}{a}\right) \left(1 + \frac{x}{b}\right)}$$

Expanding this function in a series proceeding by powers of

\* *Tonempfindungen*, 4th ed. p. 329; also p. 327, where the combinations are supposed to be formed with the partials of the primaries as well.

$x$ , we have an example of a law of force expressed by such a series, which for small displacements coincides with the pendulum law, while for displacements of but moderate extent the higher terms rapidly become prominent.

62. Assuming the existence of a law of this kind in the transmitting mechanism of the ear, we should have for the force corresponding to displacement  $u$  such an expression as

$$-(n^2u + \alpha u^2 + \beta u^3 + \gamma u^4 + \dots).$$

Suppose the system acted upon by two harmonic forces,

$$E \cos pt, \quad F \cos (qt - \epsilon),$$

and the mass = unity, or included in the coefficients ; the equation of motion is

$$\frac{d^2u}{dt^2} + n^2u = -\alpha u^2 - \beta u^3 - \gamma u^4 \dots + E \cos pt + F \cos (qt - \epsilon)$$

(following the notation of Lord Rayleigh on Sound, i. p. 65).

For the first approximation we neglect powers of  $u$  above the first ; then

$$u = e \cos pt + f \cos (qt - \epsilon),$$

where

$$e = \frac{E}{n^2 - p^2}, \quad f = \frac{F}{n^2 - q^2}.$$

63. We may here mention that, in the present case,  $n$  is negligible. This is easily seen, since, if  $n$  had any value corresponding to a frequency within the limits of the ordinary range of hearing, there would be a series of notes strengthened by the correspondence. But the only notes thus strengthened are those which are supposed to correspond to the ear-cavity. They are so high in the scale that the connexions of the internal ear would require to be nearly as rigid as brass or steel to produce them. A further reason for  $n$  not being large will be arrived at in speaking of combination-tones. And we shall assume that  $n$  is smaller than any values of  $p$  or  $q$  which occur in practice.

64. The first approximations to the subsequent terms may be now all made by substituting in them the value of  $u$  above obtained. The process to be followed for  $u^2$  coincides with that commonly adopted ; and the result is given in Lord Rayleigh's book, i. p. 66.

In the cases of  $u^3$  and higher powers the process is simpler than that which has been previously indicated.

$$\frac{d^2u}{dt^2} = -(\alpha u^2 + \beta u^3 + \dots) + E \cos pt + F \cos (qt - \epsilon).$$
$$\begin{aligned} u &= D^{-2} (E \cos pt + F \cos (qt - \epsilon)) \\ &= - \left( \frac{E}{p^2} \cos pt + \frac{F}{q^2} \cos (qt - \epsilon) \right) \\ &= - (e \cos pt + f \cos (qt - \epsilon)) \text{ say.} \end{aligned}$$
$$\begin{aligned} \frac{d^2 u}{dt^2} = & E \cos pt + F \cos (qt - \epsilon) \\ & - \alpha \left( e^2 \cos^2 pt + f^2 \cos^2 (qt - \epsilon) + 2ef \cos pt \cos (qt - \epsilon) \right) \\ & - \beta \left( e^3 \cos^3 pt + f^3 \cos^3 (qt - \epsilon) + 3e^2 f \cos^2 pt \cos (qt - \epsilon) \right. \\ & \quad \left. + 3ef^2 \cos pt \cos^2 (qt - \epsilon) \right) \\ & - \gamma \left( e^4 \cos^4 pt + f^4 \cos^4 (qt - \epsilon) + 4e^3 f \cos^3 pt \cos (qt - \epsilon) \right. \\ & \quad \left. + 6e^2 f^2 \cos^2 pt \cos (qt - \epsilon) + 4ef^3 \cos pt \cos (qt - \epsilon)^3 \right) \end{aligned}$$

66. There is, no doubt, a difficulty as to the absolute neglect of the term  $n^2u$ . The effect is to make the vibrating-point apparently rest in a position which is not one of equilibrium. Nevertheless the application of the facts to Helmholtz's hypothesis requires this proceeding; and it makes no difference whether it is done finally or at first. I think it very probable that damping terms, depending on the second and higher powers of the velocity, play an important part in the real explanation. The source of the terms, however, is of secondary importance in the present state of the question. The point is to show that those resultant sounds which depend on terms of higher orders can become great independently of those which depend on terms of lower orders.

67. Collecting the terms up to the fourth order, transform-

ing them into multiple arcs, and writing  $pt = \theta$ ,  $qt - \epsilon = \phi$ , the equation becomes

$$\begin{aligned} \frac{d^2u}{dt^2} = & E \cos \theta + F \cos \phi \\ & - \left[ \frac{\alpha}{2} (e^2 + f^2) + \frac{3}{8} \gamma (e^4 + f^4 + 4e^2 f^2) \right. \\ & + \frac{3}{4} \beta \{ (e^2 + 2f^2) e \cos \theta + (2e^2 + f^2) f \cos \phi \} \\ & + \frac{e^2}{2} \{ \alpha + \gamma (e^2 + 3f^2) \} \cos 2\theta + \frac{f^2}{2} \{ \alpha + \gamma (3e^2 + f^2) \} \cos 2\phi \\ & + \frac{\beta}{4} (e^3 \cos 3\theta + f^3 \cos 3\phi) \\ & + \frac{\gamma}{8} (e^4 \cos 4\theta + f^4 \cos 4\phi) \\ & + ef \{ \alpha + \frac{3}{2} \gamma (e^2 + f^2) \} \{ \cos (\theta + \phi) + \cos (\theta - \phi) \} \\ & + \frac{3}{4} \beta e^2 f \{ \cos (2\theta + \phi) + \cos (2\theta - \phi) \} \\ & + \frac{3}{4} \beta e f^2 \{ \cos (\theta + 2\phi) + \cos (\theta - 2\phi) \} \\ & + \frac{3}{4} \gamma e^2 f^2 \{ \cos 2(\theta + \phi) + \cos 2(\theta - \phi) \} \\ & + \gamma \frac{e^3 f}{2} \{ \cos (3\theta + \phi) + \cos (3\theta - \phi) \} \\ & \left. + \gamma \frac{e f^3}{2} \{ \cos (\theta + 3\phi) + \cos (\theta - 3\phi) \} \right]. \end{aligned}$$

68. On performing the double integration, we shall find the constant term in the above multiplied by  $t^2$ , an inadmissible result. It is only necessary to look back to the result of the complete process, when we find that the term in question is represented after integration by  $\frac{\text{constant}}{n^2}$ , where  $n^2$  is the small coefficient of the term we have neglected. This indicates that the position of equilibrium is indeed displaced, but through a finite amount; as this does not affect our results, we omit the term in question.

69. Remembering that  $\theta = pt$  and  $\phi = qt - \epsilon$ ,

$$e = \frac{E}{p^2}, \quad f = \frac{F}{q^2},$$

the remainder of the equation becomes

$$\begin{aligned}
 u = & -e \cos \theta - f \cos \phi \\
 & + \frac{3}{4} \beta \left\{ \frac{e^2 + 2f^2}{p^2} e \cos \theta + \frac{2e^2 + f^2}{q^2} f \cos \phi \right\} \\
 & + \frac{e^2}{2} \frac{\alpha + \gamma(e^2 + 3f^2)}{4p^2} \cos 2\theta + \frac{f^2}{2} \frac{\alpha + \gamma(3e^2 + f^2)}{4q^2} \cos 2\phi \\
 & + \frac{\beta}{4} \left( \frac{e^3}{9p^2} \cos 3\theta + \frac{f^3}{9q^2} \cos 3\phi \right) \\
 & + \frac{\gamma}{8} \left( \frac{e^4}{16p^2} \cos 4\theta + \frac{f^4}{16q^2} \cos 4\phi \right) \\
 & + ef \left\{ \alpha + \frac{3}{2} \gamma(e^2 + f^2) \right\} \left( \frac{\cos(\theta + \phi)}{(p+q)^2} + \frac{\cos(\theta - \phi)}{(p-q)^2} \right) \\
 & + \frac{3}{4} \beta e^2 f \left\{ \frac{\cos(2\theta + \phi)}{(2p+q)^2} + \frac{\cos(2\theta - \phi)}{(2p-q)^2} \right\} \\
 & + \frac{3}{4} \beta e f^2 \left\{ \frac{\cos(\theta + 2\phi)}{(p+2q)^2} + \frac{\cos(\theta - 2\phi)}{(p-2q)^2} \right\} \\
 & + \frac{3}{4} \gamma e^2 f^2 \left\{ \frac{\cos 2(\theta + \phi)}{4(p+q)^2} + \frac{\cos 2(\theta - \phi)}{4(p-q)^2} \right\} \\
 & + \gamma \frac{e^3 f}{2} \left\{ \frac{\cos(3\theta + \phi)}{(3p+q)^2} + \frac{\cos(3\theta - \phi)}{(3p-q)^2} \right\} \\
 & + \gamma \frac{e f^3}{2} \left\{ \frac{\cos(\theta + 3\phi)}{(p+3q)^2} + \frac{\cos(\theta - 3\phi)}{(p-3q)^2} \right\};
 \end{aligned}$$

so that there are six summation-tones and six difference-tones produced by direct transformation of the primaries, when the effect of terms up to the fourth order is considered.

70. The effect of the neglect of  $n^2$  in the denominators of all these terms, is to place the principal development of any term such as the difference-tone  $p-q$  at the point where  $p-q=0$ , whereas if the complete solution were retained the condition for the principal development would be

$$n^2 - (p-q)^2 = 0.$$

No known phenomenon enables us to distinguish between these two cases. Every thing happens, so far as we know, precisely as if the simpler condition were that which is really important.

71. If we proceed to terms of higher orders in the same way, we shall always have, in the result of terms of the  $n+1$ th order, the two following terms representing  $n$ th difference-tones, which alone are important for our present purpose

( $\alpha_{n+1}$  is the  $\overline{n+1}$ th coefficient),

$$\frac{(n+1)\alpha_{n+1}}{2^n} \left\{ \frac{e^{nf} \cos(n\theta - \phi)}{(np - q)^2} + \frac{e^{fn} \cos(\theta - n\phi)}{(p - nq)^2} \right\},$$

besides other terms analogous to those shown above.

72. In the neighbourhood of any consonance of the form  $h:1$ , the terms having the denominators  $(hp - q)^2$  become large; this is Helmholtz's explanation of the origin of difference-tones, generalized.

73. As the argument from the analytical expressions fails to give perfect satisfaction unless the nature of the causes involved be more directly demonstrated, I shall try to show more simply how it is that this comes about.

In periodic functions such as  $\cos pt$ ,  $\cos(qt - \epsilon)$ , the quantities  $p$ ,  $q$  are such that, if  $\tau$ ,  $\tau'$  be the periodic times,

$$p\tau = q\tau' = 2\pi, \text{ or } p = \frac{2\pi}{\tau}, \quad q = \frac{2\pi}{\tau'}.$$

If, then,

$$M\tau = 1,$$

$$N\tau' = 1,$$

$M$ ,  $N$  are the frequencies of the primaries, and

$$p = 2\pi M, \quad q = 2\pi N.$$

In the case of a mistuned consonance of the form  $h:1$ , the denominator of the  $h$  difference-tone term in the above expression will be  $4\pi^2(hM - N)^2$ . And  $hM - N$  is the frequency of the beat which gives rise to the transformation according to all theories (putting  $k=1$  in the more general formula  $hM - kN$ ).

$\therefore \frac{1}{hM - N}$  is the time of duration of the beat of the resultant form, whether we call it the Smith's beat, or the bow of the pendulum curves. As the denominator diminishes, the time or duration of the beat increases.

74. What happens, then, is that a force is developed, by the influence of the higher terms in question, which acts for a time corresponding to the duration of the beat. And it is matter of ordinary mechanical knowledge that, under these circumstances, the space traversed is proportional to the square of the time during which the action lasts; so that when the beat is lengthened the effect of the transformation is strengthened.

75. It is possible to found an independent treatment of the subject on these considerations, the course of which would be somewhat as follows.

In mistuned consonances of the form  $h:1$  there are alter-



nate increases and diminutions of the maximum resultant displacement, the duration of which can be arrived at by the considerations employed by Smith in determining the duration of beats. The duration of one such increase and diminution can be shown by the known formula to be  $\frac{1}{hM - N}$ .

Assume that the transmitting mechanism of the ear possesses such powers of transformation that any regular sequence of increases and diminutions of maximum resultant displacement is capable of giving rise, by transformation, to a subjective note having the same period as that of one increase and diminution. This assumption only differs from that made above in definiteness of form; for the algebraic series which is above proved to give rise to transformations of this description, is itself an assumption.

It immediately follows, by considerations differing little from those made use of in the ordinary investigation of the motion of a particle under the action of a uniform force, that the coefficient of the term in question will contain the square of the periodic time—that is to say, the coefficient  $\frac{1}{(hM - N)^2}$ ; and this is the essential point proved by the more complete analytical investigation above given.

76. Though perhaps defective as a complete demonstration of the *rationale* of the origin of difference-tones, these considerations render the general meaning of the coefficients of the difference-tone terms in the above equations tolerably clear. And we have thus sketched a method, in which the doctrine of transformation arising out of the Smith's beats, as the resultant forms pass through the transmitting mechanism of the ear, forms the basis for the further explanation of the phenomena of beats as we find them.

*The Resultant Wave-forms of Mistuned Consonances.*

77. I am principally acquainted with these forms as drawn by means of Donkin's harmonograph. The curves (Plates IV.–VII.) that accompany this paper exhibit all the points on which it will be necessary to touch.

78. It is hardly possible to be acquainted with these curves without seeing that the figures formed by the vertices which occur in the curves are in some way related to the phenomena of the mistuned consonances. And as I had myself considerable difficulty in coming to definite conclusions as to the real nature of this relation, and do not know of any published discussion of the subject, I add this article dealing with the rela-

tion in question so far as it is connected with the subject of the paper.

79. The curves are referred to an axis of  $x$ , along which the wave-lengths are measured, and an axis of  $y$  parallel to which the displacements are measured.  $\lambda$  and  $\lambda'$  are the wave-lengths on the paper of the two primary curves. If it is required to consider a question of frequency, the paper must be supposed to be drawn past the observer with velocity  $v$ , when the frequencies will be  $\frac{v}{\lambda}$ ,  $\frac{v}{\lambda'}$  respectively.

80. The tangent of the inclination of a curve to the  $x$ -axis will be spoken of shortly as the "slope."

It is assumed that  $q\lambda' = p\lambda + \delta$ , where  $p$ ,  $q$  are integers,  $q > p$ , and  $\delta$  is small.

81. The equation of the resultant of two primary curves may then be written

$$y = E \cos 2\pi \frac{x}{\lambda} + F \cos \frac{2\pi}{\lambda'} (x - \alpha).$$

The slopes of the two single curves are

$$-\frac{2\pi}{\lambda} E \sin 2\pi \frac{x}{\lambda}, \quad -\frac{2\pi}{\lambda'} F \sin \frac{2\pi}{\lambda'} (x - \alpha).$$

The ratio of the coefficients is

$$\frac{\lambda' E}{\lambda F} = \frac{p E}{q F} \text{ nearly.}$$

When this ratio is much greater than unity, the resultant slope is nearly that of the first term. When it is much less than unity, the resultant slope is nearly that of the last term.

82. The general expression for the resultant slope is given by

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda} E \sin 2\pi \frac{x}{\lambda} - \frac{2\pi}{\lambda'} F \sin \frac{2\pi}{\lambda'} (x - \alpha).$$

The vertices of the resultant curve are obtained by equating  $\frac{dy}{dx}$  to zero, whence

$$\frac{E}{\lambda} \sin \frac{2\pi x}{\lambda} + \frac{F}{\lambda'} \sin \frac{2\pi}{\lambda'} (x - \alpha) = 0.$$

83. Case I., where  $F$  is great, and the first term negligible compared with the second ( $\frac{\lambda' E}{\lambda F}$  small).

Here the vertices are those of the second component of the curve. Consequently, in every cycle of  $p$  and  $q$  vibrations of  $\lambda$  and  $\lambda'$  respectively, the  $q$  vertices of  $\lambda'$  appear, those of  $\lambda$

being smoothed out. The sole effect of the term involving  $\lambda$  is in this case to modify slightly the positions of the vertices.

84. If, then,  $p\lambda$  were exactly  $=q\lambda'$ , then after a certain distance, which may be called a short cycle, the vertices would recur for precisely the same values of  $y$ . And the corresponding vertices in successive short cycles would lie on  $q$  straight lines, or on  $2q$  straight lines if the lower vertices be included.

This short cycle is obviously  $p\lambda = q\lambda'$  in duration.

85. Since, however, in our general case  $p\lambda + \delta = q\lambda'$ , the coincidence after the short cycle is not exact; but the vertex determined by equating the second term of the inclination to zero has, in the first term, a different correction to the value of  $y$  from that which existed before the short cycle.

86. At the vertex before the short cycle let  $x = \alpha$ , so that the second term of the inclination vanishes; then, before the short cycle,

$$y_0 = E \cos \frac{2\pi\alpha}{\lambda} + F;$$

after one short cycle,  $x = q\lambda' + \alpha = p\lambda + \delta + \alpha$ ,

$$y_1 = E \cos \frac{2\pi}{\lambda} (\alpha + \delta) + F;$$

after two short cycles,  $w = 2q\lambda' + \alpha = 2(p\lambda + \delta) + \alpha$ ,

$$y_2 = E \cos \frac{2\pi}{\lambda} (\alpha + 2\delta) + F;$$

and so on, till after  $n$  short cycles, where  $n\delta = \lambda$ , nearly or exactly,

$$y_n = E \cos \frac{2\pi\alpha}{\lambda} + F;$$

and the ordinate of the vertex in question has gone through a complete period of a pendulum-curve in the space

$$\begin{aligned} nq\lambda' &= n(p\lambda + \delta) \\ &= n\lambda \left( p + \frac{1}{n} \right), \text{ since } n\delta = \lambda \\ &= \lambda(np + 1). \end{aligned}$$

87. Now we have seen that there are  $q$  of these vertices, each of which gives rise to one of these curves. Consequently this space,  $(np + 1)\lambda = nq\lambda'$ , presents, both above and below,  $q$  projecting bows, and each bow is of length

$$\frac{np + 1}{q}\lambda \text{ or } n\lambda'.$$

This is the length of Smith's beat, or of the beat as given by a well-known formula. This is easily verified as follows:—

88. Let  $v$  be the velocity of sound corresponding to wavelengths  $\lambda$  and  $\lambda'$ , and  $M$ ,  $N$  the corresponding frequencies; then

$$\frac{v}{\lambda} = M, \quad \frac{v}{\lambda'} = N,$$

and  $q\lambda' = p\lambda + \delta$  becomes ( $n\delta = \lambda$ ),

$$\frac{q}{N} = \frac{p}{M} + \frac{1}{nM};$$

$$\therefore pN - qM = \frac{N}{n} = \frac{v}{n\lambda'};$$

which connects the expression above obtained with the ordinary formula for the frequency of the beat. Hence the Smith's beat in this case corresponds in period to the projecting bow formed by the  $\frac{1}{q}$ th part of the whole periodic curve of slow disturbance of one of the vertices.

89. Case II., where  $\frac{E}{\lambda} = \frac{F}{\lambda'}$ , so that the condition for a vertex reduces to

$$\sin \frac{2\pi x}{\lambda} + \sin \frac{2\pi}{\lambda'} (x - a) = 0.$$

This condition gives the following series of values:—

$$\begin{aligned} \frac{x}{\lambda} &= \frac{-x + a}{\lambda'}, & x &= \frac{a\lambda}{\lambda + \lambda'}, \\ &= \frac{-x + \lambda' + a}{\lambda'}, & &= \frac{(a + \lambda')\lambda}{\lambda + \lambda'}, \\ &\dots\dots\dots & &\dots\dots\dots \\ &= \frac{-x + \nu\lambda' + a}{\lambda'}, & &= \frac{(a + \nu\lambda')\lambda}{\lambda + \lambda'}, \end{aligned}$$

until

$$\nu\lambda' = \lambda + \lambda';$$

and if

$$q\lambda' = p\lambda,$$

$$\nu = \frac{p + q}{p}.$$

Where this is not a whole number, the condition will be

$$\nu\lambda' = k(\lambda + \lambda'),$$

$$\nu = k \cdot \frac{p+q}{p};$$

and if  $p : q$  is in its lowest terms,

$$k = p, \text{ and } \nu = p + q,$$

$p + q$  is therefore the number of independent vertices arising from these terms.

90. Another series of values satisfies the condition ; these are as follows (since  $\sin x = \sin (\pi - x)$  &c.) :—

$$\begin{aligned} \frac{x}{\lambda} &= \frac{1}{2} + \frac{x-\alpha}{\lambda'}, & x &= \frac{\lambda' - 2\alpha}{2(\lambda' - \lambda)} \cdot \lambda, \\ &= \frac{3}{2} + \frac{x-\alpha}{\lambda'}, & &= \frac{3\lambda' - 2\alpha}{2(\lambda' - \lambda)} \cdot \lambda, \\ &\dots\dots\dots & &\dots\dots\dots \\ &= \frac{2\nu - 1}{2} + \frac{x-\alpha}{\lambda'}, & &= \frac{(2\nu - 1)\lambda' - 2\alpha}{2(\lambda' - \lambda)} \cdot \lambda, \end{aligned}$$

until

$$(2\nu)\lambda' = 2(\lambda - \lambda'),$$

$$\nu = \frac{\lambda' - \lambda}{\lambda'},$$

$$= \frac{p - q}{p}.$$

And if this be not a whole number,

$$(2\nu)\lambda' = 2k(\lambda - \lambda'),$$

$$\nu = k \left( \frac{q}{p} - 1 \right),$$

$$\nu = \frac{k}{p} (p - q);$$

and if  $p : q$  is in its lowest terms,

$$k = p, \quad \nu = p - q, \text{ or } q - p, \text{ since } q > p;$$

$q - p$  is therefore the number of independent vertices arising from these terms.

91. The relation of these different sets of vertices may be otherwise exhibited by putting the expression for the inclination into the form

$$\sin \frac{2\pi x}{\lambda} + \sin \frac{2\pi}{\lambda'} (x - \alpha)$$

$$= 2 \sin \pi \left\{ x \left( \frac{1}{\lambda} + \frac{1}{\lambda'} \right) - \frac{\alpha}{\lambda'} \right\} \cos \pi \left\{ x \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right) + \frac{\alpha}{\lambda'} \right\} = 0.$$

The zero values of the sine give the  $p+q$  vertices of the first set; and the values which make the cosine vanish give the  $q-p$  vertices of the other set.

92. Each of these vertices occupies, as in the former case, its special position in the short cycle  $q\lambda' = p\lambda$ , and lies always on a straight line  $y = \text{constant}$  when such an exact relation holds.

93. Also, as before, when the above relation is changed into  $q\lambda' = p\lambda + \delta$ , it may be shown, by examining the successive arguments of the vertices, that they shift their position in successive short cycles, so that they lie on pendulum-curves of long period; also that the period of these curves is, for the  $p+q$  system,  $p+q$  times that of the Smith's beat, and for the  $p-q$  system,  $p-q$  times that of the Smith's beat.

94. The curves of both these systems, with the Smith's beats which form part of them, are readily recognized in all those of the pendulum-curve illustrations which approximately satisfy the condition

$$pE = qF \text{ or } \lambda'E = \lambda F.$$

In the case of the major third, where there are many vertices in each short cycle, and the figure of the short cycle is itself complicated, these curves are not easily recognized. It is necessary to mark a set of corresponding vertices in order to recognize the curve in this case. By the time we arrive at the fifth the curves are quite plain.

The curves of the  $p+q$  system are large and bold, extending completely from top to bottom of the illustrations; each curve comprises the bow of a Smith's beat both above and below. These may be spoken of as the external system.

The curves of the  $q-p$  system are smaller, and lie nearer the axial line of the illustrations. These may be spoken of as the internal system. In the particular case where  $q-p=1$ , such as  $q=2$ ,  $p=1$ , the internal system exhibits complete periodic curves having the period of the Smith's beat.

95. Case III., where  $F$  is so small that  $qF$  is small compared with  $pE$ . This would fall under the argument of the first case, with the signification of the letters reversed. But as we made the convention  $q > p$ , there arise some special points of difference.

96. Where  $q$  is much greater than  $p$ , as is the case of high harmonics combined with a fundamental,  $F$  has to be very small indeed in order that  $qF$  may be small compared with  $pE$ . In this case, unless  $F$  is almost evanescent, it is not generally true that the only vertices are those of  $E$  (the fundamental); for in these cases the vertex of the fundamental curve becomes

almost a straight line in the short space occupied by a wave of the higher curve; and under these circumstances the vertices of the higher curve continue to be visible wherever they come upon the vertex of the lower, especially where the two vertices are turned opposite ways.

The conditions of case I. not being strictly fulfilled, the consequences there deduced do not strictly follow. The considerations as to the number of different vertices which develop curves are not materially affected. And it remains true that there are always  $p$  curves (in case I.  $q$  curves) actually developed; but it is not true that there are no traces of any of the other  $q$  curves of the entire external set of  $p+q$  of case II. On the contrary, it is seen in several of the illustrations, where for the most part  $p=1$ , that, instead of the outline being one pendulum-curve embracing the outlines of all the Smith's beats, the internal vertices of the long curves present traces of the crossing of two pendulum-curves of longer period—an effect which is seen to survive from the more general cases, on comparing the illustrations to case II. As the amplitude of the higher note diminishes, this curve assumes a trochoidal form, the external vertices being less sharp than the internal, where there is the survival from the crossing. Ultimately, no doubt, the outline would become theoretically a pendulum-curve.

But, in the case of indefinite diminution of the coefficient  $\frac{qF}{pE}$ , where  $\frac{q}{p}$  is great,  $\frac{F}{E}$  is of the order of the product of two small quantities; consequently the effects on the displacements, or the curves we are examining, would themselves tend to become evanescent before their peculiarities; consequently the curve enveloping the Smith's beats would never in this way be reduced to a pendulum-curve having the period of those beats.

In the application of these considerations we have, further, to remember that the resultant tones which present pendulum-curves having the periods of Smith's beats are only heard when both notes are pretty loud; and under these circumstances the indefinite diminution of the ratio above supposed is not admissible. The only case, therefore, in which a locus of vertices is a pendulum-curve of the same complete period as the period of Smith's beat, is that of an internal system under case II., where  $q-p=1$ . As the existence of this system depends on the accurate adjustment of the coefficients to the law  $pE=qF$ , it cannot be referred to even as an illustration of a phenomenon of general occurrence.

97. We conclude, in conformity with the explanation at the end of the former part, (1) that the forms exhibited by the

resultant of two pendulum-curves do not, as a rule, exhibit any appearances corresponding to pendulum-curves having the period of the Smith's beat, except in a very small number of cases, the conditions for which can only be fulfilled by accident ; (2) that the increases and diminutions of the maximum displacement which form what we have called the bows of the harmonic curves, correspond in duration with the Smith's beat, but not in the period of the harmonic curves of which they form part.

98. We infer from the previous parts of this paper:— (3) that the variations of maximum displacement which are represented in these figures by the bows of the harmonic curves, give rise by transformation to pendulum-vibrations having the same frequency as those variations—these being the notes which König calls beat-notes, and Helmholtz difference-tones of various orders; and (4) that the actual beats of mistuned consonances of the form  $n : 1$ , as heard by the ear, are given rise to by the interference of these beat-notes or difference-tones with the lower note of the combination.

99. The upper numbers prefixed to the Plates of curves are the ratios of the wave-lengths; the lower ones the ratios of the amplitudes.

Typical curve of case I. .  $\frac{27}{80}$   
9 : 8

Typical curve of case II. .  $\frac{27}{80}$   
3 : 1

Typical curve of case III.  $\frac{27}{80}$ , or rather  $\frac{1}{3} \times \frac{80}{99}$   
 $9 : 2$   $10 : 1$

POSTSCRIPT.—The curves shown in the Plates are all illustrations of the subject of this paper, with the exception of three sets; namely, the combinations of vibrations whose wave-lengths are nearly as 4:5, as 2:3, and as 2:5. These have been given for the sake of completeness in the collection of curves, and that readers may have the opportunity of seeing the nature of the difference between such curves as these, which may be said to belong to mistuned consonances of the form  $h:k$ , and our normal forms which belong to mistuned consonances of the form  $h:1$ .