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geometry of curves and surfaces, and in kinematics and dynamics. The reader cannot fail to be impressed with the directness of the method in all these applications, especially if he is familiar with the ordinary modes of attack. In virtue of this directness of attack and the extraordinary conciseness of notation more detail can be packed into one quaternion page than into three or four pages of ordinary analysis. By what other method, for example, could systematic discussions of line, surface and volume integrals, of spherical harmonics, heterogeneous strain, elastic vibrations, and electro-magnetic theory be given in less than fifty pages? In the variety of the mathematical and physical subjects taken up there are only two other books which can compare with Professor Joly's *Manual*, and these are Hamilton's *Elements* and Tait's *Treatise*.

The greater part of the book is necessarily a development of much that is to be found in the pages of Hamilton, Tait, and M'Aulay; but Professor Joly has a characteristic style of his own, more nearly akin to Hamilton's than to Tait's. In the last two chapters especially are the author's additions more in evidence. These are on Projective Geometry and Hyperspace. The former is based upon a new interpretation of the quaternion; and in the latter Professor Joly gives a sketch of the properties of associative algebras applicable to n-dimensional space.

Professor Joly has certainly succeeded in his aim of providing the student with a *working* book. He takes excursions into many fields of mathematics pure and applied, and the treatment is not superficial. Important applications are worked out in detail; and numerous examples are given by which the student may test his progress. Let the reader accept on trust the initial assumptions and developments, and work earnestly through the succeeding chapters. He will come out in the end a practised quaternionist. C. G. KNOTT.

Leçons sur les fonctions de variables réelles, par E. BOREL; **Leçons sur les fonctions discontinues**, par RENÉ BAIRE; **Le calcul des residus et ses applications à la théorie des fonctions**, par E. LINDELÖF. (Paris, Gauthier-Villars, 1905, 3 f. 50 c. each.)

M. Borel's book is the sixth of his series of monographs on the theory of functions, of which the first appeared as recently as 1898. M. Borel is only human, and by now he has decided to leave to his pupils the work of preparing his lectures for publication. It cannot be said that this method has proved in every case an unqualified success. The first few volumes, prepared by M. Borel himself, and particularly the admirable *Leçons sur les fonctions entières*, were remarkable alike for their originality, for the judgment shewn in the selection of material, and for the lucidity and proportion of the exposition. This high standard has not been maintained in all of the later volumes, some of which have been rather scrappy, and have given the impression of hasty, and at times perfunctory composition. In these respects, however, the present volume is an improvement upon its immediate predecessors, M. Maurice Fréchet having performed his task unusually well. But I cannot help thinking that M. Borel would be fortunate if he could find the time to write his books himself.

The principal problem with which M. Borel deals in this volume is that of the representation of functions by means of series of polynomials, a form of representation the importance of which was first shown by Weierstrass's well-known theorem that every continuous function of a real variable can be expanded in such a series. M. Borel confines himself to functions of real variables, reserving the complex theory for the next volume of his series; and for the most part he is concerned with continuous functions only, the short chapter on the representation of discontinuous functions containing little more than a discussion of the comparatively simple case in which the aggregate of points of discontinuity is enumerable, and a reference to the results obtained by M. Baire. The consequence is that the book is rather disconnected; for the first two chapters, which deal with the theory of aggregates and continuity and discontinuity in general, contain a good deal which, though very interesting in itself, is really not required for M. Borel's purpose. The central chapters (3 and 4) are excellent,[1] and the long note added by M. Painlevé is perhaps the most interesting feature of the book, though its natural place would not be in this volume, but in the next one.

There is one criticism which will occur to every regular reader of this series, in which M. Borel has now enlisted the collaboration of a number of other eminent mathematicians. There is an amount of repetition which a judicious general editor should be able to diminish; and that the plan of the series is open to this criticism M. Borel, to judge from his remarks in the preface, appears to recognise: 'Il a paru préférable d'admettre parfois quelques brèves redites plutôt que de renoncer à l'indépendance des Volumes de la Collection, chacun d'eux devant pouvoir être lu isolément par un lecteur ayant des connaissances générales d'Analyse. Sans ce principe d'indépendance, on aurait eu tous les inconvénients d'un grand Traité, sans en avoir les avantages.'

In principle, no doubt M. Borel is right. Each author should be asked to deal with some definite question, and he should have full liberty to preface his discussion of it with a general account of those modern developments of analysis which are necessary for his purpose, and with which a reader who has not read the other volumes of the series cannot be expected to be familiar. But he should be very careful to make this general account as short as is consistent with clearness, and to limit it strictly to results which will afterwards be required. This is the course adopted by M. Baire, who, if he is at times a little diffuse, is careful not to encumber his book with unnecessary matter. M. Borel has not set his colleagues so good an example. Why, for example, should he think it necessary to introduce a few pages concerning M. Lebesgue's generalisation of the notion of the definite integral? Most interesting and most important this generalisation certainly is; but it has already been expounded by M. Lebesgue himself in an earlier volume of the series, and no allusion whatever is made to it throughout the remainder of the book.

It will not be necessary to say much about the volume contributed by M. Baire. It is in substance a popular edition of his remarkable

[1] The argument of p. 66 has become inverted in some curious way. The function is '*more* continuous' when $\phi(\epsilon)$ decreases *less* quickly.

memoir, *Sur les fonctions de variables réelles*, published as a thesis in 1899, and afterwards in the *Annali di Matematica.* A good deal of introductory matter has been added, and the argument has been simplified and condensed. The problem of finding the *necessary and sufficient* conditions that a function whose points of discontinuity are given should be capable of representation as the sum of a series of continuous functions is one which most mathematicians would have regarded as hopeless if M. Baire had not completely solved it; and that M. Baire's researches should be made more accessible to the ordinary reader was much to be desired. But it is difficult to resist the impression that these two volumes might well have been condensed into one. M. Baire's results are particularly interesting when applied to differential coefficients of continuous functions. If $f(x)$ is continuous, so is

$$\frac{f(x+h)-f(x)}{h}$$

considered as a function of x. Now, if the differential coefficient exists for every value of x under consideration, and we denote by h_1, h_2, h_3 ... a series of positive quantities whose limit is zero,

$$f'(x)=\lim_{n=\infty}.\frac{f(x+h_n)-f(x)}{h_n},$$

and is therefore representable as the limit of a sequence of continuous functions, or (what is the same thing) the sum of a series of such functions. M. Baire's results, therefore, give us much important information as regards the possible discontinuities of the differential coefficients of continuous functions.

For M. Lindelöf's *Calcul des Residus* I have nothing but praise. The applications of Cauchy's "calculus" to the theory of functions, and in particular to the summation of series and the theory of analytic continuation, are of the most far-reaching character, and, so far as I know, no one before M. Lindelöf has attempted to give a systematic account of them. Laurent, it is true, published in 1865 a *Théorie des Residus*, which hardly deserved to have been so soon forgotten, but the applications of the theory have multiplied ten times since then. Some new account was urgently necessary, and M. Lindelöf has given us exactly what was wanted. One admirable feature of his book is the thoroughness and exactitude of his historical references, especially to the writings of Cauchy. M. Lindelöf is one of the few mathematicians who have found life long enough to make 'une étude détaillée' of Cauchy's works.

M. Lindelöf very wisely does not trouble himself with all the difficulties as to the *minimum* of assumption required to establish Cauchy's theory, which centre round Goursat's proof of Cauchy's theorem. These difficulties, of course, have absolutely no bearing on the applications with which M. Lindelöf is chiefly concerned. Assuming the continuity of the differential coefficient, he proceeds to show that any analytic function $f(x)$ is itself the differential coefficient of an analytic function $F(x)$ determinate save for an additive constant, a conclusion from which, combined with the definition of the definite integral along a curvilinear path, Cauchy's theorem immediately follows. There is certainly a great deal to be said for presenting the proof of the theorem in this way.

A short but clear account follows of some familiar applications of the formula

$$f(x)=\frac{1}{2\pi i}\int\frac{f(t)}{t-x}dt$$

and of some others which are not so familiar, such as occur in the proof of Jensen's theorem, the theory of the Bernoullian and Eulerian functions, the factorisation of such functions as $\sin x - ax\cos x$, and the transformation of slowly convergent series. In Chapter III. he comes to the ground which he has made particularly his own. He proves a whole series of general formulae, of which the formula of Plana and Abel,

$$\Sigma f(x) = -\tfrac{1}{2}f(x)+\int^{x}f(x)dx+\frac{1}{i}\int_0^{\infty}\frac{f(x+it)-f(x-it)}{e^{2\pi t}-1}dt,$$

was historically the first. The applications of these formulae to different regions of analysis—Gauss's sums, the Zeta and Gamma functions, the continuation of power series, and the asymptotic behaviour of integral functions—are so numerous that it is impossible to enumerate them here. I will only cite the beautiful formula which defines the behaviour of the function

$$F(x,\,s)=\frac{x}{1^s}+\frac{x^2}{2^s}+\frac{x^3}{3^s}+\dots$$

near $x=1$, viz.,

$$F(x,\,s)=\Gamma(1-s)\left(\log\frac{1}{x}\right)^{s-1}+\sum_0^{\infty}\zeta(s-\nu)\,\frac{(\log x)^{\nu}}{\nu\,!},$$

which solves completely a problem which has exercised the ingenuity of many mathematicians.

M. Borel's series of monographs is almost indispensable for anyone who is engaged in research in the theory of functions, and with the possible exception of M. Borel's own first two volumes, M. Lindelöf's contribution certainly seems to me the best. G. H. HARDY.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. JAMES, Ph.D., Instructor in Mathematics, Washington University, St. Louis. New York: Wiley & Sons. London: Chapman & Hall. 1905. Pp. 176. $1.

Presupposing some familiarity with elementary experimental mechanics, and a knowledge of the calculus, the author has written a sketch of the "dynamics of a particle," paying special attention to the effect of the earth's own motion on the motion of a particle on the earth's surface.

The entire absence of examples will appear to many as a defect, but otherwise the book may prove helpful in enabling a student to generalise some of the limited concepts gained by previous experimental work, and to grasp the difficulties of abstract dynamics. C. S. J.