ON THE CONVERGENCE OF SERIES OF ORTHOGONAL FUNCTIONS

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A NEW line of investigation in the theory of Fourier series and of other series of orthogonal functions was opened by the discovery made by Liapounoff and Hurwitz that the coefficients of the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

which corresponds to a function $f(x)$ such that $\{f(x)\}^2$ is integrable are such that $\tfrac{1}{2}a_0^2 + \sum_{n=1} (a_n^2 + b_n^2)$ converges to the value $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \{f(x)\}^2 dx$, independently of any knowledge as to the convergence of the Fourier's series. This theorem was established in its most general form by Fatou,* for the case of a function $f(x)$, which, whether limited or not, is such that $\{f(x)\}^2$ is summable, in the sense that it has a Lebesgue integral in the interval $(-\pi, \pi)$, the coefficients being also Lebesgue integrals. The theorem was extended to the case of series of other orthogonal functions, in connection with the theory of integral equations.

If the coefficients $c_1, c_2, \ldots, c_n, \ldots$ of a series $\sum_{n=1} c_n \phi_n(x)$, where the $\{\phi_n(x)\}$ form a sequence of normal orthogonal functions, are given, the question arises whether a function $f(x)$ exists such that the constants c_n are the coefficients $\displaystyle\int_0^1 f(x)\, \phi_n(x)\, dx$, formed in Fourier's manner, corresponding to the function $f(x)$.

It was established† by F. Riesz and Fischer that the necessary and sufficient condition that a function $f(x)$, whose square is summable, should

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* *Acta Math.*, Vol. 30 (1906). Other proofs have been given by Lebesgue and by W. H. Young (see his paper on "Successions of Integrals and Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11).

† F. Riesz, *Göttinger Nachrichten*, 1907, also *Comptes Rendus* of the French Academy, May 17, 1909; Fischer, *Comptes Rendus* of the French Academy, 1907, Vol. 144, p. 1022.

exist, such that the given constants $c_1, c_2, \ldots, c_n, \ldots$ are the Fourier's constants $\int_0^1 f(x)\,\phi_1(x)\,dx,\ \int_0^1 f(x)\,\phi_2(x)\,dx,\ \ldots,\ \int_0^1 f(x)\,\phi_n(x)\,dx, \ldots$ is that the series $c_1^2+c_2^2+\ldots+c_n^2+\ldots$ is convergent.* The further question arises whether, and how far, the series $\sum\limits_{n=1} c_n\phi_n(x)$ is convergent. The earliest result as regards this question was obtained, for the case of the ordinary Fourier's series, by Fatou (*loc. cit.*), who shewed that the series

$$\tfrac{1}{2}a_0+\sum_{n=1}(a_n\cos nx+b_n\sin nx)$$

is convergent at every point of the interval $(-\pi, \pi)$ with the exception at most of a set of points of zero measure, provided

$$\lim_{n=\infty} na_n = 0, \quad \text{and} \quad \lim_{n=\infty} nb_n = 0.$$

It was then proved† by Weyl, by means of an ingenious method originally due to Jerosch, and further developed by Weyl himself, that the series $\sum\limits_{n=1} c_n\phi_n(x)$ converges everywhere in the interval (0, 1), with the possible exception of a set of points of zero measure, in case the series $\sum\limits_{n=1} n^{\frac{1}{3}}c_n^2$ is convergent. Weyl also proved, by a similar method, that, in case the functions $\phi_n(x)$ are such that $|\phi_n(x)|$ is less than some fixed number, for all the values of n and x, the convergence of the series $\Sigma n^{\frac{1}{3}}c_n^2$ is sufficient to ensure the convergence of the series $\sum\limits_{n=1} c_n\phi_n(x)$ in the same sense as before; this condition is clearly satisfied if

$$\lim_{n=\infty} n^{\frac{2}{3}+\lambda}c_n = 0,$$

for some value of $\lambda\,(>0)$; or, in the case of the ordinary Fourier's series, if

$$\lim_{n=\infty} n^{\frac{2}{3}+\lambda}a_n = 0, \quad \text{and} \quad \lim_{n=\infty} n^{\frac{2}{3}+\lambda}b_n = 0,$$

for some value of $\lambda(>0)$.

In the present communication I have established the wider result that it is sufficient for the convergence of the series $\sum\limits_{n=1} c_n\phi_n(x)$ at all points of (0, 1) with the exception at most of those belonging to an exceptional set of measure zero, that a number $k\,(>0)$ should exist such that the series $\sum\limits_{n=1} n^k c_n^2$ is convergent. This condition is clearly satisfied, if a number

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* Various proofs of this theorem are discussed in a paper by W. H. and G. C. Young, *Quarterly Journal*, Vol. XLIV.

† See *Math. Annalen*, Vol. 67, p. 225.

$\lambda\,(>0)$ exists, such that

$$\lim_{n=\infty} n^{\frac{1}{2}+\lambda} c_n = 0,$$

or, in the case of the ordinary trigonometrical series, that

$$\lim_{n=\infty} n^{\frac{1}{2}+\lambda} a_n = 0, \quad \text{and} \quad \lim_{n=\infty} n^{\frac{1}{2}+\lambda} b_n = 0.$$

The proof is developed by a method different from the one employed by Weyl to establish his theorems corresponding to $k=\frac{1}{2}$, and $k=\frac{1}{3}$, in that no use is made of the *Majorantenreihen*. It appears that the restriction that $|\phi_n(x)|$ should be limited for all values of n and x, which Weyl employed for the case $k=\frac{1}{3}$, is entirely unnecessary.

1. Let $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... denote a set of normal orthogonal functions for the interval (0, 1); the functions accordingly satisfy the conditions

$$\int_0^1 \{\phi_n(x)\}^2 dx = 1, \quad \int_0^1 \phi_n(x)\,\phi_{n'}(x)\,dx = 0,$$

for $n \neq n'$.

Let it be assumed that the coefficients $c_1, c_2, \ldots, c_n$, of a series

$$c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots,$$

are such that the series $1^k c_1^2+2^k c_2^2+\ldots+n^k c_n^2+\ldots$

is convergent, for some value of k greater than zero. It will then be shewn that the series $\sum_{n=1}^{\infty} c_n\phi_n(x)$ is convergent for all values of x in the interval (0, 1) with the exception at most of those values that belong to a set of points of measure zero.

Let λ be a positive integer so chosen that $1/\lambda \leqslant k$, then it is clear that the series

$$1^{1/\lambda}c_1^2+2^{1/\lambda}c_2^2+\ldots+n^{1/\lambda}c_n^2+\ldots$$

is convergent.

The partial sum $c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)$ will be denoted by $s_n(x)$. But, for convenience of printing, in the investigation $s(x, n)$ will be written for $s_n(x)$; also $c(n)$ will be written for c_n.

We choose a positive number δ which may be arbitrarily small, and we then choose a positive integer r_1 so large that the series

$$r_1 c^2(r_1^\lambda)+(r_1^\lambda+1)^{1/\lambda}c^2(r_1^\lambda+1)+(r_1^\lambda+2)^{1/\lambda}c^2(r_1^\lambda+2)+\ldots$$

converges to a sum that is less than δ^3. In view of a later requirement,

r_1 will also be taken to be greater than all the binomial coefficients $\lambda,\ \frac{\lambda(\lambda-1)}{2!},\ \frac{\lambda(\lambda-1)(\lambda-2)}{3!},\ \ldots$ in the expanded form of $(1+x)^\lambda$.

Consider the set of indices

$$r_1^\lambda,\ (r_1+1)^\lambda,\ (r_1+2)^\lambda,\ \ldots,\ m^\lambda;$$

we have

$$\int_0^1 \{s(x,\ m^\lambda)-s(x,\ t^\lambda)\}^2\, dx = c^2(t^\lambda+1)+c^2(t^\lambda+2)+\ldots+c^2(m^\lambda),$$

where $t < m$. It follows, by letting $t = r_1,\ r_1+1,\ \ldots,\ m-1$, successively, and adding the equations, that

$$\int_0^1 \sum_{t=r_1}^{t=m-1} \{s(x,\ m^\lambda)-s(x,\ t^\lambda)\}^2\, dx$$

$$= \sum_{t=r_1^\lambda+1}^{(r_1+1)^\lambda} c^2(t)+2\sum_{t=(r_1+1)^\lambda+1}^{(r_1+2)^\lambda} c^2(t)+\ldots+(m-r_1)\sum_{t=(m-1)^\lambda+1}^{m^\lambda} c^2(t).$$

Since $m-r_1 < \{(m-1)^\lambda+1\}^{1/\lambda}$, we see that the expression on the right-hand side is less than $\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t)$.

It follows that the set of points at which

$$\sum_{t=r_1}^{t=m-1} \{s(x,\ m^\lambda)-s(x,\ t^\lambda)\}^2 \geqslant \delta^2$$

has a measure less than $\frac{1}{\delta^2}\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t)$. From this we see that in a set of points G_m of measure

$$> 1-\frac{1}{\delta^2}\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t) > 1-\delta,$$

all the numbers

$$|s(x,\ m^\lambda)-s(x,\ r_1^\lambda)|,\ |s(x,\ m^\lambda)-s\{x,\ (r_1+1)^\lambda\}|,\ \ldots,$$
$$|s(x,\ m^\lambda)-s\{x,\ (m-1)^\lambda\}|,$$

are less than δ. In this set G_m of points

$$s(x,\ t^\lambda)-s(x,\ t'^\lambda)| < 2\delta,$$

for all pairs of values of t and t', such that

$$r_1 \leqslant t \leqslant m, \quad \text{and} \quad r_1 \leqslant t' \leqslant m.$$

Let m have the values $m_1, m_2, m_3, \ldots$ successively, in an increasing sequence of integers. The corresponding sets of points being denoted by $G_{m_1}, G_{m_2}, G_{m_3}, \ldots$, it is clear that each of these sets contains the next, for any point of G_{m_2} belongs to G_{m_1}.

Since the measure of each of the sets is $> 1-\delta$, and since each one contains the succeeding ones, it follows that there exists a set G, of measure $\geqslant 1-\delta$, which consists of those points that belong to all the sets $G_{m_1}, G_{m_2}, G_{m_3}, \ldots$. In this set G, of measure $\geqslant 1-\delta$, we have

$$|s(x, t^\lambda)-s(x, t'^\lambda)| < 2\delta,$$

for all pairs of values of t and t', such that $t \geqslant r_1$, $t' \geqslant r_1$.

Next, let us consider the following groups of indices which are $\geqslant r^\lambda$, and $\leqslant (r+1)^\lambda$, where r is a number $\geqslant r_1$:—

$$\begin{array}{lcll}
r^\lambda,\ r^\lambda+1,\ r^\lambda+2, & \ldots, & r^\lambda+r; & H_1^{(1)}, \\
r^\lambda+r,\ r^\lambda+r+1,\ r^\lambda+r+2, & \ldots, & r^\lambda+2r; & H_2^{(1)}, \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \ldots & \ldots \quad \ldots & \\
r^\lambda+(P_1-1)r,\ \ r^\lambda+(P_1-1)r+1, & \ldots, & r^\lambda+P_1 r; & H_{P_1}^{(1)}, \\
r^\lambda+P_1 r,\ r^\lambda+P_1 r+1; & \ldots, & & H_{P_1+1}^{(1)},
\end{array}$$

where P_1 denotes $$\lambda r^{\lambda-2}+\frac{\lambda(\lambda-1)}{2!}r^{\lambda-3}+\ldots+\lambda,$$

so that $$r^\lambda+P_1 r+1 = (r+1)^\lambda.$$

The last index in each group is also the first index in the next, and each group except the last contains $r+1$ indices. Consider the group $H_m^{(1)}$, where $1 \leqslant m \leqslant P_1$. We have

$$\begin{aligned}
\int_0^1 \Big[& \{s[x, r^\lambda+(m-1)r]-s[x, r^\lambda+(m-1)r+1]\}^2 \\
& + \{s[x, r^\lambda+(m-1)r]-s[x, r^\lambda+(m-1)r+2]\}^2+\ldots \\
& + \{s[x, r^\lambda+(m-1)r]-s(x, r^\lambda+mr)\}^2\Big]\,dx \\
= {} & rc^2\{r^\lambda+(m-1)r+1\}+(r-1)c^2\{r^\lambda+(m-1)r+2\}+\ldots+c^2(r^\lambda+mr) \\
< {} & \sum_{t=r^\lambda+(m-1)r+1}^{t=r^\lambda+mr} t^{1/\lambda} c_t^2.
\end{aligned}$$

If $m = P_1+1$, we have

$$\int_0^1 \{s(x, r^\lambda+P_1 r)-s(x, r^\lambda+P_1 r+1)\}^2 dx = c^2(r^\lambda+P_1 r+1) < t^{1/\lambda} c_t^2,$$

where $$t = r^\lambda+P_1 r+1.$$

It follows that in a set of points of measure

$$> 1-\frac{1}{\delta^2} \sum_{t=r^\lambda+(m-1)r+1}^{t=r^\lambda+mr} t^{1/\lambda} c_t^2,$$

we have $$|s[x, r^\lambda+(m-1)r]-s[x, r^\lambda+(m-1)r+k]| < \delta,$$

for all the values 1, 2, 3, ..., r of k. Hence, in this set, we have

$$|s[x, r^\lambda+(m-1)r+t]-s[x, r^\lambda+(m-1)r+t']| < 2\delta,$$

for all pairs of values of t and t', such that $1 \leqslant t \leqslant r$, $1 \leqslant t' \leqslant r$. This holds for each of the groups $H_m^{(1)}$, $m = 1, 2, \ldots, P_1$; and also for the group $H_{P_1+1}^{(1)}$, in a set of measure

$$> 1-\frac{1}{\delta^2} t^{1/\lambda} c_t^2,$$

where $$t = (r+1)^\lambda,$$

we have $$|s(x, r^\lambda+P_1 r)-s[x, (r+1)^\lambda]| < \delta.$$

Taking all the values of m successively, we see that in a certain set E_1 of measure

$$> 1-\frac{1}{\delta^2} \sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$

the condition $$|s(x, a_1)-s(x, a)| < \delta$$

is satisfied for all integers a, such that

$$r^\lambda \leqslant a < (r+1)^\lambda,$$

when a_1 denotes the first index in that group $H_m^{(1)}$ to which a belongs.

Next, consider the indices r^λ, $r^\lambda+r$, $r^\lambda+2r$, ..., $r^\lambda+r^2$, $r^\lambda+r^2+r$, ..., $r^\lambda+P_1 r$. As before, we define a number of groups of indices, each of which, except the last, contains $r+1$ indices. Thus, if $P_1 = rP_2+\lambda$, where P_2 denotes $\lambda r^{\lambda-3}+\frac{\lambda(\lambda-1)}{2!}r^{\lambda-4}+\ldots+\frac{\lambda(\lambda-1)}{2}$, the groups are taken to be

$$\begin{array}{llr}
r^\lambda,\ r^\lambda+r,\ r^\lambda+2r, & \ldots,\ r^\lambda+r^2; & H_1^{(2)}, \\
r^\lambda+r^2,\ r^\lambda+r^2+r,\ r^\lambda+r^2+2r, & \ldots,\ r^\lambda+2r^2; & H_2^{(2)}, \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots & \\
r^\lambda+(m-1)r^2,\ r^\lambda+(m-1)r^2+r, & \ldots,\ r^\lambda+mr^2; & H_m^{(2)}, \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots & \\
r^\lambda+(P_2-1)r^2,\ r^\lambda+(P_2-1)r^2+r, & \ldots,\ r^\lambda+P_2r^2; & H_{P_2}^{(2)}, \\
r^\lambda+P_2r^2, & \ldots,\ r^\lambda+P_2r^2+\lambda r; & H_{P_2+2}^{(2)}.
\end{array}$$

We see that, taking the group $H_m^{(2)}$,

$$\begin{aligned}
\int_0^1 \Big[&\{s[x, r^\lambda+(m-1)r^2]-s[x, r^\lambda+(m-1)r^2+r]\}^2 \\
&+\{s[x, r^\lambda+(m-1)r^2]-s[x, r^\lambda+(m-1)^2+2r]\}^2+\ldots \\
&+\{s[x, r^\lambda+(m-1)r^2]-s(x, r^\lambda+mr^2)\}^2\Big]\,dx \\
= r\,\{&c^2[r^\lambda+(m-1)r^2+1]+\ldots+c^2[r^\lambda+(m-1)r^2+r]\} \\
&+(r-1)\{c^2[r^\lambda+(m-1)r^2+r+1]+\ldots+c^2[r^\lambda+(m-1)r^2+2r]\} \\
&+\ldots \\
&+c^2(r^\lambda+mr^2).
\end{aligned}$$

The expression on the right-hand side is

$$< \sum_{t=r^\lambda+(m-1)r^2+1}^{t=r^\lambda+mr^2} t^{1/\lambda}c_t^2.$$

For the group $H_{P_2+1}^{(2)}$, the expression is

$$\lambda\{c^2(r^\lambda+P_2r^2+1)+\ldots+c^2(r^\lambda+P_2r^2+r)\} \\ +(\lambda-1)\{c^2(r^\lambda+P_2r^2+r+1)+\ldots\}+\ldots,$$

and since r_1 has been so chosen that $r_1^\lambda>\lambda$, we see that this is less than

$$\sum_{t=r^\lambda+P_2r^2}^{t=r^\lambda+P_2r^2+\lambda r} t^{1/\lambda}c_t^2.$$

As before, we conclude that there exists a set E_2, of measure greater than

$$1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda}c_t^2,$$

in which
$$|s(x, a)-s(x, a_2)| < \delta,$$
for all values of a which belong to the set $r^\lambda, r^\lambda+r, \ldots, (r+1)^\lambda$; where a_2 denotes the first index in that set $H_m^{(2)}$ to which a belongs.

We proceed, as before, with the indices
$$r^\lambda, r^\lambda+r^2, r^\lambda+2r^2, \ldots, r^\lambda+P_2 r^2;$$
and write
$$P_2 = rP_3+\frac{\lambda(\lambda-1)}{2}.$$

We define as before the sets $H_1^{(3)}, H_2^{(3)}, \ldots, H_{P_2+1}^{(3)}$, each of which contains $r+1$ indices, except that the last contains only $\frac{1}{2}\lambda(\lambda-1)$ indices, and this is by hypothesis $< r_1$. As before, we conclude that in a set E_3 of measure
$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$
the condition $|s(x, a)-s(x, a_3)|$ is satisfied for all values of a, such that a belongs to the set $r^\lambda, r^\lambda+r^2, \ldots, r^\lambda+P_2 r^2$, and where a_3 is the first index of that set $H_m^{(3)}$ to which a belongs.

We continue this process until we have to divide the indices
$$r^\lambda, r^\lambda+r^{\lambda-2}, \ldots, r^\lambda+r^{\lambda-1}, r^\lambda+r^{\lambda-1}+r^{\lambda-2}, \ldots, r^\lambda+\lambda r^{\lambda-1}+\tfrac{1}{2}\lambda(\lambda-1)r^{\lambda-2},$$
into sets
$$H_1^{(\lambda-1)}, H_2^{(\lambda-1)}, \ldots, H_{P_{\lambda-1}+1}^{(\lambda-1)}.$$
As before, there exists a set $E_{\lambda-1}$ of measure greater than
$$1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$
in which
$$|f(x, a)-f(x, a_{\lambda-1})| < \delta,$$
where a is any of the above indices, and $a_{\lambda-1}$ is the first index of the group $G_m^{(\lambda-1)}$ to which a belongs.

Lastly, we have to consider the set of indices $r^\lambda, r^\lambda+r^{\lambda-1}, \ldots, r^\lambda+\lambda r^{\lambda-1}$, which we denote by $H^{(\lambda)}$. As before, it is seen that there exists a set of points E_λ of measure
$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$
at every point of which $\quad |s(x, a)-s(x, a_\lambda)| < \delta,$

where, in this case, a_λ has the unique value r^λ.

Now the sets $E_1, E_2, E_3, \ldots, E_\lambda$ have each a measure
$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2;$$

therefore there exists a set of points F_r of measure

$$> 1 - \frac{\lambda}{\delta^2} \sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$

each point of which belongs to all the sets $E_1, E_2, \ldots, E_\lambda$.

Now let a denote any integer such that $r^\lambda \leqslant a < (r+1)^\lambda$, then a is in one of the sets $H_1^{(1)}, H_2^{(1)}, \ldots, H_{P_1+1}^{(1)}$; therefore there is a number $a^{(1)}$, belonging to the set $r^\lambda, r^\lambda + r, \ldots, r^\lambda + P_1 r$, such that in the set F_r, we have

$$| s(x, a) - s(x, a^{(1)}) | < \delta.$$

Again, $a^{(1)}$ belongs to one of the sets $H_1^{(2)}, H_2^{(2)}, \ldots, H_{P_2+1}^{(2)}$; therefore there exists a number $a^{(2)}$ among those numbers which are the first indices of these sets, such that in F_r, we have

$$| s(x, a^{(1)}) - s(x, a^{(2)}) | < \delta,$$

and so on. It follows since

$$| s(x, r^\lambda) - s(x, a) | \leqslant | s(x, a) - s(x, a^{(1)}) | + | s(x, a^{(1)}) - s(x, a^{(2)}) | + \ldots + | s(x, a^{(\lambda)}) - s(x, r^\lambda) |,$$

that
$$| s(x, r^\lambda) - s(x, a) | < \lambda\delta$$
for all points of F_r.

Now give to r the values $r_1, r_1+1, r_1+2, \ldots$, successively; we thus obtain sets of points $F_{r_1}, F_{r_1+1}, F_{r_1+2}, \ldots$; the measure of F_{r_1+s} being

$$> 1 - \frac{\lambda}{\delta^2} \sum_{t=(r_1+s)^\lambda}^{t=(r_1+s+1)^\lambda} t^{1/\lambda} c_t^2.$$

There exists a set of points K, of measure

$$> 1 - \frac{2\lambda}{\delta^2} \sum_{t=r_1^\lambda}^{\infty} t^{1/\lambda} c_t^2,$$

each point of which belongs to all the sets $F_{r_1}, F_{r_1+1}, \ldots$.

At a point x of this set K, we have

$$| s(x, a) - s(x, r^\lambda) | < \lambda\delta,$$

for all indices a which are $\geqslant r_1^\lambda$, where r^λ is such that

$$r^\lambda \leqslant a < (r+1)^\lambda.$$

In accordance with the choice which we have made of the integer r_1, the measure of K is $> 1 - 2\lambda\delta$. In the set G which is of measure $\geqslant 1 - \delta$, we have

$$| s(x, t^\lambda) - s(x, t'^\lambda) | < 2\delta,$$

for all values of t and t' which are not less than r_1. The sets K and G have in common a set L_δ of measure greater than $1-(2\lambda+1)\delta$. This set L_δ has the property that for any two integers a, a', neither of which is $< r_1^\lambda$, the condition

$$|s(x, a)-s(x, a')| < 2(\lambda+1)\delta;$$

for if t^λ, t'^λ are such that

$$t^\lambda \leqslant a < (t+1)^\lambda, \qquad t'^\lambda \leqslant a' < (t'+1)^\lambda,$$

we have $$|s(x, a)-s(x, a')| \leqslant |s(x, a)-s(x, t^\lambda)| + |s(x, a')-s(x, t'^\lambda)| + |s(x, t^\lambda)-s(x, t'^\lambda)|$$
$$< 2(\lambda+1)\delta.$$

If ϵ be an arbitrarily fixed positive number, we may take

$$\delta = \epsilon/(2\lambda+2);$$

therefore a set D_ϵ of points exists, of measure $> 1-\epsilon$, such that

$$|s_n(x)-s_{n'}(x)| < \epsilon$$

for all points of the set, and for all values of n and n' which are both greater than some fixed integer i_ϵ. Let ζ be an arbitrarily chosen positive number, then a sequence of values of ϵ can be so chosen that $\epsilon_1+\epsilon_2+\ldots$ converges to a value less than ζ. The sets D_{ϵ_1}, D_{ϵ_2}, ... have in common a set D of points which belong to all of them, and the measure of D is

$$> 1-(\epsilon_1+\epsilon_2+\ldots) > 1-\zeta.$$

In this set D the sequence $\{s_n(x)\}$ converges uniformly; for, at all points x of this sequence, $|s_n(x)-s_{n'}(x)|$ is less than the arbitrarily small number ϵ_m, for all values of n and n' which are not less than some fixed integer dependent on ϵ_m.

It has now been shewn that the series

$$c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots$$

converges uniformly in some set of points of measure $> 1-\zeta$, where ζ is an arbitrarily chosen positive number (< 1). Since ζ is arbitrarily small, it follows that the series is convergent at all points of a set of which the measure is equal to that of the interval (0, 1) of representation. This mode of convergence is said to be quasi-uniform convergence in the interval.* The following theorem has now been established:—

* This mode of convergence I have discussed in a previous paper, "On the Representation of a Summable Function by a Series of Finite Polynomials," see *ante*, p. 162.

If $\phi_1(x)$, $\phi_2(x)$, ... are a sequence of normal orthogonal functions, and if the series $1^k c_1^2 + 2^k c_2^2 + \ldots + n^k c_n^2 + \ldots$ converges for some value of k that is greater than zero, then the series $c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + \ldots$ converges at all points of the interval for which the orthogonal functions are defined, with at most the exception of a set of points of which the measure is zero. The convergence is quasi-uniform, in the sense that a set of points of measure less than, but as nearly equal as we please to, that of the interval can be determined so that the convergence of the series is uniform in the set of points.

It will be observed that no assumption is made in this theorem that the set of normal orthogonal functions is a complete set. Thus, if $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... is a complete set, the condition stated in the theorem is sufficient to ensure the convergence, in the sense stated in the enunciation, of all the series of the form

$$c_1\phi_{n_1}(x) + c_2\phi_{n_2}(x) + \ldots + c_r\phi_{n_r}(x) + \ldots,$$

where n_1, n_2, ..., n_r, ... is a sequence of increasing integers defined according to any prescribed law.

The particular case of the above theorem which arises when k has the value $\frac{1}{2}$ was established* by Weyl by a somewhat intricate method depending upon the use of *Majorantenreihen*. Weyl also established the theorem for the case $k = \frac{1}{3}$, on the assumption that the functions $\phi_n(x)$ are less in absolute value than some fixed positive number, for all the values of n and x; this last restriction has been shewn above to be unnecessary.

In the case of the ordinary trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

the convergence of the two series $\sum\limits_{n=1}^{\infty} n^k a_n^2$, $\sum\limits_{n=1}^{\infty} n^k b_n^2$ for some value of k greater than zero is sufficient to ensure the convergence of the series at every point of the interval $(-\pi, \pi)$ with the exception at most of a set of points of which the measure is zero.

2. In case a number p greater than $\frac{1}{2}$ exists, such that

$$\lim_{n=\infty} (n^{\frac{1}{2}+p} c_n) = 0,$$

we see that
$$c_n < A/n^{\frac{1}{2}+p},$$

* See *Mathematische Annalen*, Vol. 67 (1909), p. 225.

where A is some fixed positive number. Choosing the positive number k less than $2p$, we then see that the series $\sum_{n=1}^{\infty} n^k c_n^2$, being less than the convergent series $\sum_{n=1}^{\infty} \frac{A^2}{n^{1+2p-k}}$, is convergent, and thus the condition contained in the general theorem given above is satisfied. We thus have the following result :—

If, for some value of p greater than zero, $n^{\frac{1}{2}+p} c_n$ converges to zero as n is indefinitely increased, the series $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is convergent at all points of the interval for which the normal orthogonal functions $\phi_n(x)$ are defined, with the exception at most of the points of some set of which the measure is zero.

The conditions

$$\lim (n^{\frac{1}{2}+p} a_n) = 0, \qquad \lim (n^{\frac{1}{2}+p} b_n) = 0,$$

for some value of p greater than zero, are sufficient to ensure that the trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is convergent, with the same exception as above.