154. Continued Inversion by Coaxal Circles
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be omitted (compare the last paragraph of the *Report* quoted above). But, so long as such proofs must be taught, I would strongly recommend the arrangement given in J. Tannery's admirable *Introduction à la Théorié des Fonctions d'une Variable*, a text-book which deserves to be better known in this country than it appears to be.* T. J. I'a BROMWICH.

MATHEMATICAL NOTES.

153. [P. 3. b.] *On a Fundamental Theorem in Inversion.*

If two figures F, F' be inverses of each other, their inverses with respect to any pair of corresponding points P, P' are similar.

Demonstration. Let O be the centre, and ρ the radius of inversion of F, F'. Let Q, R be any two points of the figure F; and Q', R' the corresponding points of the figure F'. Also let q, r be the inverses of Q, R with respect to centre P, radius of inversion λ; and q', r' the inverses of Q', R' with respect to centre P', radius of inversion λ'. The theorem will be proved if we show that the ratio of qr to $q'r'$ is constant.

Use the following lemma: If A', B' be the inverses of A, B with respect to a centre O, radius of inversion ρ, then $A'B'/AB=\rho^2/OA\,.\,OB$.

We have by the lemma

$$\frac{qr}{QR}=\frac{\lambda^2}{PQ\,.\,PR}; \quad \text{and} \quad \frac{q'r'}{Q'R'}=\frac{\lambda'^2}{P'Q'\,.\,P'R'};$$

$$\begin{aligned}\therefore \frac{qr}{q'r'}&=\frac{\lambda^2}{\lambda'^2}\cdot\frac{QR}{Q'R'}\cdot\frac{P'Q'}{PQ}\cdot\frac{P'R'}{PR}\\&=\frac{\lambda^2}{\lambda'^2}\cdot\frac{\rho^2}{OQ'\,.\,OR'}\cdot\frac{OP'\,.\,OQ'}{\rho^2}\cdot\frac{OP'\,.\,OR'}{\rho^2}\\&=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'^2}{\rho^2}=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'}{OP}\ \text{since}\ \rho^2=OP\,.\,OP'\\&=\frac{\lambda^2}{OP}\div\frac{\lambda'^2}{OP'}.\end{aligned}$$

Thus the ratio of qr to $q'r'$ *is* constant and therefore the theorem is proved.

A second demonstration will be given later on. The metrical relation obtained above, namely, that, if λ, λ' be the radii of inversion at P, P' the corresponding lengths in the inverses of F, F' with respect to P, P' are to one another as $\dfrac{\lambda^2}{OP}=\dfrac{\lambda'^2}{OP'}$ should be noted, as it is of importance in the applications of the theorem to recent Geometry.

From the theorem it follows that if a figure F undergoes n successive inversions and is thereby converted into F_n, and to the point P in F corresponds the point P_n in F_n, then the inverse of F_n with respect to P_n is similar to the inverse of F with respect to P.

154. [P. 3. b.]. *Continued Inversion by Coaxal Circles.*

An interesting way of obtaining some of Mr. C. E. Youngman's results (p. 7) is by the stereographic projection. Let HK be the points common to the coaxal system. Bisect HK at O, and draw OV equal to OH and perpendicular to the plane of the system. Draw a sphere through V with O as

*It may not be out of place to refer to the fact that the real difficulty in finding the infinite product is not to prove that $x(1-x^2/\pi^2)$, etc. are factors; but to prove that *no factor of the form e^{ax} is present.* This remark is due to Stolz, but is not made in any of the English text-books.

centre. To find the position of P after successive inversions with respect to the circles (A), (B), (C), ... we join VP cutting the sphere in Q; bring Q to the position of Q' by successive reflections in the planes through HK perpendicular to VA, VB, VC, ...; and then join VQ' cutting the plane AHK in the required position P'. Since successive reflexions in two planes through HK inclined at an angle α are equivalent to a rotation through 2α about HK, and since the circles (A) and (B) cut at an angle AVB = the angle between the planes through HK perpendicular to VA and VB, we have at once $[AB]=[CD]$, provided the circles (A) and (B) cut at the same angle as (C) and (D). If (C) or (D) is the line HK we have Mr. R. F. Davis's result. Again, if (A) and (B) are orthogonal $[AB]=[BA]$, for rotations through π and through $-\pi$ about HK are equivalent, etc.

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155. [K. 20. a.]. *Definitions of Trigonometrical Ratios, and General Proof of Addition-Theorems for Sine and Cosine.*

The object of this note is to indicate a method of treating the fundamental theory of Trigonometry which, so far as simplicity and completeness are concerned, seems to have considerable merit. No absolute novelty can be claimed for or alleged against it. (See, *e.g.*, Casey's *Trigonometry*.)

We shall use the symbol $\equiv$ as denoting "has the same length, direction and sense as"; and the symbol $=$ will as usual denote both equality in magnitude and likeness in sign.

Prop. I. The projections on a *directed* line Ox of any two directed line-segments of equal length and the same direction, are equal to one another.

Proof. Let AB, CD be two such segments, and ab, cd their projections on Ox.

Let AE and CF drawn $\parallel$ to Ox meet Bb and Dd in E and F. Then the triangles AEB, CFD are equal in all respects, having their corresponding sides in the same directions.

Thus $$AE\equiv CF; \quad \therefore\ ab\equiv cd.$$

Prop. II. If a directed line meet the directed line Ox in A, and any two segments of the former, starting from A, be taken, then the ratio of projection on Ox to segment is the same for the two segments.

Proof. Let AB, AG be two such segments (which may have like or unlike senses) and Ab, Ag their projections on Ox.

Then $$\frac{gA}{Ab}=\frac{GA}{AB}; \quad \therefore\ \frac{Ag}{AG}=\frac{Ab}{AB}.$$

Prop III. The ratio of projection on Ox to segment is the same in magnitude and sign for any two segments of any two directed lines that make the same angle α with Ox.

Proof. If LM and PQ be such segments, and the line LM meet Ox in A; and if along this line we lay off segments $AN\equiv LM$ and $AR\equiv PQ$, then the ratio of projection to segment is the same for LM and for AN (Prop. I.) and for AR (Prop. II.) and for PQ (Prop. I.).

Def. I. The ratio of projection on Ox to segment, for any directed segment of a directed line making an angle α with the directed line Ox, is the *cosine of* α.

Note that a directed segment of a directed line is positive or negative according as its sense is the same as that of the line or opposite to it. We have shown in Props. I., II., and III. that this definition of $\cos\alpha$ is complete.

Def. II. The corresponding ratio when the projection is on a line making an angle $+90°$ with Ox is called $\sin\alpha$.