

ON SEMI-INTEGRALS AND OSCILLATING SUCCESSIONS OF FUNCTIONS

By W. H. YOUNG, Sc.D., F.R.S.

[Read June 9th, 1910.—Received July 16th, 1910.]

§§ 1-4. *Introductory.*

1. The introduction of the concept of upper and lower semi-continuity by Baire was a happy one. It has enabled us, not only to deal with continuity itself with much greater ease, but it has rendered possible the discovery and statement in concise form of theorems which might otherwise not have been suspected. One branch of the Theory of Functions of a Real Variable in which it is particularly useful is in the theory of oscillating series. To mention only one of the more recent results obtained by myself in this domain, we have the theorem that what may be called* semi-integration below of an oscillating succession is allowable if (i) the points in the neighbourhood of which the non-uniformity of the lower oscillation of the succession is infinite form a countable set, and (ii) the lower function of the integrals of the succession is upper semi-continuous on the left and lower semi-continuous on the right.

We owe to Jordan the recognition of the importance of a certain class of continuous functions, namely, the class consisting of those continuous functions which possess bounded variation (*fonction à variation bornée*). All continuous functions obtained by Lebesgue integration belong to a sub-class of this class. In fact, a function which is an integral, interpreting the word "integral" in the most general Lebesgue sense, is not only a function with bounded variation, but the increment over any set of intervals has zero as unique limit, when the content of the set of intervals diminishes indefinitely.

The recognition of this fact has enabled Vitali to obtain the following result.† If the points at which a sequence of functions converges non-

* If $l(x)$ is the lower function of the succession, and $L(x)$ of the succession of integrals, semi-integration below implies that $\int l(x) dx \leq L(x)$.

† Stated a little differently by Vitali.

uniformly with infinite measure form a set of zero content, then term-by-term integration is allowable, provided the integrals of the functions converge towards a function which is an integral. If it were not an integral term-by-term integration would lead to a result absurd on the face of it; thus we might say that under the circumstances stated with regard to the non-uniformity of the convergence, term-by-term integration is allowable, if it has any sense. Striking as this result is, we may say *en passant* that the determination of whether or not a function is an integral, especially when, as in the present case, it is defined as a limit of succession of functions, is so much more difficult than the determination of whether or not a function is continuous, that Osgood's theorem as to the term-by-term integration of sequences, of which the generalisation for oscillating successions has been stated above, remains unimpaired in importance, as well as in interest. Osgood's theorem may, indeed, be also thought of as giving a sufficient condition under which a sequence of functions each of which is an integral, has another integral as limiting function, viz., this is the case if (i) the limiting function is continuous, and (ii) the points of infinite non-uniform convergence are countable. The question then arises of itself, what is the generalisation for successions of functions of Osgood's theorem, stated in this last manner, more especially when we consider semi-integration of the succession above or below only? We naturally expect to be able to treat the concept of being an integral in the same way as that of continuity, viz., to be able to split it up into two constituent parts. When we have done this we may hope to be able to obtain the generalisation for oscillating successions of Vitali's theorem.

With this object in view I introduce in the present paper the concepts of upper semi-integral and lower semi-integral. This use of terms seems appropriate in view of the fact that it follows at once from the definition that an upper integral of a function (in Darboux's sense) is an upper semi-integral of the integrand, though it is not in general associated with it, using the word "associated" in a sense to be explained later on (§ 12). There is thus no confusion. A function which is both an upper semi-integral and a lower semi-integral is certainly an integral, and it will be found that the various properties of an integral are distributed between the classes of upper and lower semi-integrals.

The simplest definition which can be given of these new concepts is that an upper semi-integral is the sum of an integral and a positive monotone increasing function,* while a lower semi-integral is the

* "Positive" here means ≥ 0 , and "monotone decreasing" means monotone and not increasing. Similarly "negative" means ≤ 0 , and "monotone increasing" means monotone and not decreasing.

sum of an integral and a negative monotone decreasing function. Thus, as we naturally exclude infinite values and consider closed intervals, semi-integrals are, like integrals, functions with bounded variation.

On the other hand, it is important to remark that they are not necessarily continuous. We find it convenient, in fact, to simultaneously split up the concepts of integration and continuity; the latter concept is, however, split up in the special manner which the theorem concerning oscillating sequences, more than once referred to, would lead us to expect, viz., it follows from the definition that an upper semi-integral is upper semi-continuous on the left and lower semi-continuous on the right, while a lower semi-integral is lower semi-continuous on the left and upper semi-continuous on the right.

The disadvantage of the definition of semi-integrals just given is that it requires the previous definition of the term integral. Jordan's separation of the positive and negative variations, however, enables us immediately to give independent definitions, as soon as we adapt to these separate variations the statement of the property of an integral relative to the total variation. Thus *we define an upper (lower) semi-integral to be a function with bounded variation, whose negative (positive) increment over any set of intervals has zero as unique limit, when the content of the intervals decreases indefinitely.*

2. The introduction of these concepts enables us to split up into two constituent parts Lebesgue's "Fundamental Theorem of Integration." Moreover, in this way it becomes possible to give an independent account of the chief properties of semi-integrals. Among these we may mention the property that an upper semi-integral exceeds, and a lower semi-integral is less than, the common integral of any of its derivatives by a monotone increasing function. If the sum of two semi-integrals of the same kind is an integral, each of the semi-integrals is an integral.

The limit of an ultra-monotone* sequence of upper (lower) semi-integrals is an upper (lower) semi-integral, whether the sequence be ascending or descending, provided only the sequence of integrands does not diverge, except possibly at a set of points of content zero. If a function of bounded variation has one of its derivatives finite above (below)†

* An ultra-monotone sequence is defined to be a monotone sequence such that the difference between any two consecutive terms is a monotone function, the sense of the monotony being preserved.

† "Finite above" implies that the function is never $+\infty$, "finite below" that it is never $-\infty$.

it is a lower (upper) semi-integral, and the integral of that finite derivate over the set of points where it is positive (negative) is the positive (negative) variation of the function.*

3. These properties themselves, though easily proved by anyone who has mastered Lebesgue's work, seem to be sufficiently striking to appear to justify the introduction of the new terms; but their *raison d'être* appears more especially, as already explained, when we come to the theory of integration of oscillating successions. We have, for example, the theorem that, if the limiting functions of the integrals of a finitely oscillating succession of functions are greater (less) than or equal to the integral of the lower (upper) functions of the succession, or, with our phraseology, if semi-integration of the succession is legitimate below (above), then all these limits are necessarily upper (lower) semi-integrals, from which it of course follows that, if semi-integration is allowable, both above and below, the limits are integrals. Moreover, we have the theorem that the necessary and sufficient condition that a succession of functions whose points of infinite non-uniform oscillation below (above) form a set of zero content, should be semi-integrable below (above) is that the lower (upper) function of the succession of integrals should be an upper (lower) semi-integral. That under the same conditions semi-integration should be possible both above and below, it is necessary and sufficient that the upper and lower functions of the succession of integrals should both be integrals.

We have, moreover, the answer to the question as to the generalisation of the second portion of Osgood's result, viz., if the succession of integrands oscillates finitely, and the points at which it has non-uniform oscillation with infinite measure below (above) are countable, then the necessary and sufficient condition that the upper and lower limits of the integrals should be both upper (lower) semi-integrals, is that they should be upper (lower) semi-continuous on the right and lower (upper) semi-continuous on the left. Under the same conditions the condition that they should be integrals is that they should be continuous.

It is here shewn that the necessary and sufficient condition that a succession of functions, oscillating finitely, should be semi-integrable

* For a proof of this theorem in the modified form in question the reader may refer to "Functions of Bounded Variation," a forthcoming paper in the *Quarterly Journal*, by the author, in which, moreover, beside certain new results, new proofs (the first in English) will be found of theorems due to Lebesgue, and required in the present paper. These proofs differ from Lebesgue's, among other respects, in that, with one exception, they do not involve the use of Cantor's numbers.

above (below) is that given any positive ϵ , we can find E , such that the positive (negative) increment of each of the integrals of the succession of integrals taken over any set of intervals whose content is less than E should be less than ϵ . Moreover we have a generalisation of the other result of Vitali's in this connexion, viz., that, if the succession is bounded below (above), the condition in question is also necessary and sufficient for the semi-integrability (not absolute) of the succession term-by-term.

It will be seen that, though several of the generalisations with which this paper is concerned are almost intuitive to anyone accustomed to deal with oscillating successions, as soon as the preliminary step involved in the splitting up of the notion of integral has been taken, this is by no means the case with all of them. They are much more intuitive, apparently, than the theorems of my paper on "Term-by-Term Integration of Oscillating Series."* Yet, when the step in question has been taken, it is found that some of the theorems true for sequences do not admit of generalisation. This fact and the great importance of the subject will, I hope, be thought to have justified the detailed treatment here given to the matter.

4. Among the results of this paper to which attention may be directed we have the following remarkably convenient necessary and sufficient test for the term-by-term integrability of sequences:—a sequence is integrable term-by-term if two other integrable sequences can be found, one of which is greater than the given sequence, that is, such that the constituent function is greater than or equal to that of the given sequence, and the other sequence is less than the given sequence. This theorem will, I feel sure, be found in practice of great utility. It is noteworthy that it is not sufficient to find two sequences, one greater than the given sequence and integrable above, and the other less than the given sequence and integrable below, unless these two sequences are bounded respectively above and below. Moreover the corresponding tests for successions do not apply as they stand. The nature of the difficulty will be apparent, after reading the paper; it depends on the fact, here proved by an example, that if a succession of essentially positive functions is semi-integrable above, it is not necessarily absolutely semi-integrable above, whereas an integrable sequence of essentially positive function is necessarily absolutely integrable.

It is noteworthy that the test in question holds good, whether the range of integration be a finite or an infinite interval, and, though in

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 99-117.

practice this is less important, whether the integrals are absolutely convergent, or not. Thus it enables us not only to determine with ease in the great variety of cases whether or not an infinite series which is capable of term-by-term integration over a finite interval can also be integrated when one or both of the limits of integration are infinite; it enables us also to discuss questions of primary importance in the theory and practice of definite integrals between infinite limits and involving parameters.

Want of space alone has prevented me enlarging in the present paper on this point. I have, however, illustrated in more ways than one the use of the test, *inter alios* by obtaining new results with regard to the continuity of a parametric integral, and the possibility of differentiating under the sign of integration.

On Integrals and Semi-Integrals.

5. We use the term "integral" in the most general sense consistent with the definition of Lebesgue,* and assume that all functions which occur, bounded or unbounded, possess integrals.† One of the advantages of this generalisation, whether we adopt Lebesgue's definition, or one given by myself,‡ is that we are naturally led to consider the behaviour of the function at the points of certain sets, not necessarily filling up an interval, or a set of intervals. In this connection it is convenient if we define with him the integral of a function over a set of points contained in an interval as the integral over that interval of the function got from the original function by ascribing the value zero to it at the points of the set complementary to the set in question, and leaving it unaltered elsewhere. With this understanding we have the following theorem, which is needed in the sequel:—

THEOREM.—*If $f(x)$ is a summable function, its integral over any measurable set of points E exists, and, if D_1, D_2, \dots are sets of intervals, each lying inside the preceding, and having the points of E as common points, and no other common points, except perhaps a set of content zero, then the integral of $f(x)$ over the set E is the unique limit of its integral over the sets D_1, D_2, \dots .*

* H. Lebesgue, "Intégrale, Longueur, Aire," 1902, *Ann. di Mat.*

† Such functions are called by Lebesgue "summable functions," *Leçons sur l'intégration*, 1904.

‡ W. H. Young, "The General Theory of Integration," 1904, *Phil. Trans.*, Series A, Vol. 204, pp. 221–252. "A New Method in the Theory of Integration," 1910, *Proc. London Math. Soc.*, Series 2, Vol. 9, pp. 15–50.

6. Turning to the consideration of the integral itself, we remark that a function which is an integral has various well-known and striking properties which distinguish it from other continuous functions. Denoting the function which is equal to the integrand wherever positive (negative) and is elsewhere zero, as the positive (negative) part of the integrand, the integral is from the very definition the difference between the integrals of two essentially positive functions, viz., the positive part and the absolute value of the negative part of the integrand, and is therefore the difference between two monotone increasing positive continuous functions, or, which is the same thing, the sum of a monotone increasing and a monotone decreasing continuous function. Thus it also follows that an integral has bounded variation. Though, however, the integral of a summable function is necessarily continuous and possesses these properties, a continuous function possessing these properties is not, as has been pointed out by Lebesgue and Vitali, necessarily an integral. Lebesgue gives the following necessary and sufficient conditions that a function should be an integral:—

(a) *The function must have bounded variation.*

(b) *Its total increment over any set of non-overlapping intervals of content I , must have zero as its limit as I approaches zero.**

It will be seen from what follows that the following is an equivalent necessary and sufficient test, which, though logically inferior, may be in practice much more convenient:—

It is necessary and sufficient that (1) a function should exist which is an integral and exceeds the given function by a monotone increasing function, and (2) a function should exist which is an integral and which is less than the given function by a monotone increasing function.

It may be remarked that in this form the test is complete; if it is satisfied, the function is necessarily continuous and of bounded variation.

We note at once, moreover, how the very form of this test suggests the possibility of splitting up the property of being an integral into two parts. We shall say, in fact, that a function is a lower semi-integral if it satisfies the first part of the test, and an upper semi-integral if it satisfies the second.

We are thus naturally led to split up Lebesgue's test. Before doing

* H. Lebesgue, *Leçons sur l'intégration*, p. 129, footnote (statement only); "Sur la recherche des fonctions primitives par l'intégration," *Rend. dei Lincei*, Vol. xvi, 1907, pp. 286, 287; "Sur les Intégrales singulières," 1909, *Ann. de Toulouse*, 3rd Series, Vol. I, p. 43. G. Vitali, "Sulle funzioni integrali," *Atti d. Torino*, Vol. xl, 1905, pp. 1021-1034.

so, however, it will be convenient to give a proof of the *necessity* of Vitali's condition, which includes Lebesgue's condition (b) and entails condition (a) as a consequence,* viz.,

THEOREM.—*A function $F(x)$ which is an integral is absolutely continuous, in other words it is such that, if we take any finite or countably infinite set of non-overlapping intervals of content less than a suitably chosen quantity E , the positive and negative increments taken over this set of intervals are each numerically less than a pre-assigned arbitrary positive quantity e .*

When the integrand is bounded the theorem is obvious. To prove it in the general case we remark that, if a function exceeds another by a monotone increasing function, the positive increment of the former over any set of non-overlapping intervals is necessarily greater than or equal to that of the latter.

Denoting then, by f_1 and f_2 , the positive and negative parts of f , that is, the two positive functions agreeing numerically with f wherever $f \geq 0$ and $f \leq 0$ respectively, and zero elsewhere, these are both summable functions, and

$$f(x) = f_1(x) - f_2(x),$$

and, denoting the integrals by large letters,

$$F(x) = F_1(x) - F_2(x);$$

hence, by what was pointed out above, the positive increment of $F(x)$ over any set of intervals is less than or equal to the corresponding positive increment of $F_1(x)$. Thus the theorem for the positive increment of $F(x)$ is true, if it is true for $F_1(x)$. It is therefore only necessary to prove the theorem in the case when the integrand is positive.

Let $a_n(x) = f(x)$ wherever the positive function $f(x)$ is less than n , and elsewhere let $a_n(x) = 0$. Then

$$a_1(x) \leq a_2(x) \leq \dots \leq a_n(x) \leq \dots$$

is a monotone increasing sequence, having $f(x)$ as limit, and the integral of f is (by definition if this integral is an improper integral) the limit of the integral of $a_n(x)$ over any interval.

Hence pre-assigning a positive quantity e , we can find an integer n , such that

$$\int a_n(x) dx \leq \int f(x) dx \leq \frac{1}{2}e + \int a_n(x) dx;$$

* See "Functions of Bounded Variation," *loc. cit.*

and therefore
$$0 \leq \int \{f(x) - a_n(x)\} dx \leq \frac{1}{2}e.$$

Since this is true over the whole interval of integration it is true over every set of intervals D in the main interval, since the integrand is positive. Thus

$$0 \leq \int_D f(x) dx \leq \frac{1}{2}e + \int_D a_n(x) dx.$$

Now $a_n(x)$ lies between 0 and n , therefore this gives

$$0 \leq \int_D f(x) dx \leq \frac{1}{2}e + nD.$$

Thus, if the set of intervals was such that $D < \frac{1}{2}e/n$, this gives

$$0 \leq \int_D f(x) dx < e,$$

which proves the required result for the positive increment; similarly it may be proved for the negative increment.

7. The property just proved is not a property of every continuous function which is the sum of a monotone increasing and a monotone decreasing function. We notice, however, that every monotone increasing function, continuous or not, possesses one half of the property, since the negative increment is zero, while every monotone decreasing function possesses the other half. Moreover, it is evident that a function which is the sum of two functions both of which possesses the same half-property, will itself possess it. Bearing in mind certain facts to be proved shortly, we are thus led to formulate the following definition, which for the rest has already suggested itself.

DEFINITION.—A function which can be expressed as the sum of an integral and a monotone increasing function (continuous or not) is called an upper semi-integral, and one which can be expressed as the sum of an integral and a monotone decreasing function (continuous or not) is called a lower semi-integral.

8. With this definition, a theorem which I have proved elsewhere* shews that, in particular, a continuous function of bounded variation, having a derivate which is finite above (below) is an upper (lower) semi-integral, and the integral of the positive (negative) part of that derivate is the positive (negative) variation of the semi-integral over the segment considered.

Again, a semi-integral possesses a finite differential coefficient, except at a set of content zero, since, by a theorem of Lebesgue's* this is a pro-

* See "Functions of Bounded Variation," *loc. cit.*

perty of any function of bounded variation, and *the derivates of a semi-integral are all summable functions, whose common integral has, therefore, by another theorem of Lebesgue's* the same differential coefficient as the semi-integral except at a set of content zero.*

Hence, if $F(x)$ is an upper semi-integral, and $g(x)$ any one of its derivates, and $I(x)$ and $M(x)$ the integral and monotone increasing function whose sum is $F(x)$, we have, except at a set of content zero, differentiating

$$g(x) = I'(x) + M'(x),$$

whence, integrating,

$$\begin{aligned} \int_a^x g(x) dx &= \int_a^x I'(x) dx + \int_a^x M'(x) dx \\ &= I(x) + M(x) - \text{a monotone increasing function},* \\ &= F(x) - \text{a monotone increasing function}; \end{aligned}$$

from which we see, by what has already been pointed out, that this last monotone increasing function has a differential coefficient which is zero, except at a set of content zero.

Now a finite function which is zero except at a set of content zero cannot be a derivate without being identically zero, by Lebesgue's theorem, or by the analogous theorem stated at the beginning of the present article. Hence, if $F(x)$ is not itself an integral, the differential coefficient just mentioned has infinite values.

Moreover, by a familiar extension* of the theorems used, these infinities cannot form a reducible† set.

A similar discussion may be made for a lower semi-integral. Thus we have the following theorem:—

THEOREM.—*An upper (lower) semi-integral which is not an integral is equal to the common integral of its derivates plus a monotone increasing (decreasing) function having zero as its differential coefficient, except at a set of content zero, and an infinite derivate at an irreducible set of points.*

9. THEOREM.—*An upper (lower) semi-integral has the property that, if we take any finite or infinite set of non-overlapping intervals of content less than a suitably chosen positive quantity E , the negative (positive*

* See "Functions of Bounded Variation," *loc. cit.*

† [Note added February 18th, 1911.—They form indeed a non-countable set. Moreover there is at least one perfect set consisting entirely of such infinities. This follows from a theorem stated and proved in my "Note on the Fundamental Theorem of Integration," presented on November 4th, 1910, to the Cambridge Philosophical Society.]

increment of the function taken over those intervals is numerically less than any pre-assigned arbitrary positive quantity ϵ .

This statement corresponds to Vitali's form of the condition that a function should be an integral. Corresponding to Lebesgue's form we have the following alternative statement:—

THEOREM.—*An upper (lower) semi-integral is a function of bounded variation, and has the property that its negative (positive) increment over any set of non-overlapping intervals of content I , must have zero as its limit as I approaches zero.*

Either of these theorems, however, expresses only a portion of the information within our reach as to the nature of semi-integrals. The converse is also true, viz., the property, or pair of properties, in question is characteristic of semi-integrals.

Since a function which satisfies the conditions of the former theorem (Vitali's form of the condition) satisfies the requirements of the alternative theorem (Lebesgue's form of the condition), it will be sufficient to prove the following theorem, in order to justify the statement just made.

THEOREM.—*If $F(x)$ is a function of bounded variation, such that its negative (positive) increment over any set of non-overlapping intervals of content I has zero as its limit as I approaches zero, then $F(x)$ is an upper (lower) semi-integral.*

Since $F(x)$ has bounded variation it has a finite differential coefficient, except at a set of content zero. Let $F'(x)$ be the function which is equal to this differential coefficient except at the exceptional set, and there is zero. Then

$$\int F'(x)dx$$

has F' for differential coefficient except at a set of content zero. Therefore, if

$$G(x) = F(x) - \int F'(x)dx,$$

$G(x)$ has zero for differential coefficient, except at a set of zero content. But also the negative increment of $G(x)$ over any set of intervals of content less than ϵ has zero as limit, when ϵ diminishes indefinitely; therefore, describing round the exceptional points at which the derivatives of $G(x)$ are different from zero a set of non-overlapping intervals of sufficiently small content, the increment of $G(x)$ over these intervals is greater than $-\epsilon$, where ϵ is as small a positive quantity as we please. Hence, by the ordinary reasoning in which a Lebesgue chain of intervals is employed, it follows that $G(x) - G(a)$ is essentially positive, whatever interval (a, x)

be considered. Thus $G(x)$ is a monotone increasing function of x , and therefore $F(x)$ is the sum of an integral and a monotone increasing function of x .

*Hence also it follows that a function with bounded variation, whose positive and negative increments over a set of non-overlapping intervals of content I both have zero as limit when I is indefinitely decreased, is an integral.**

We also have the alternative statement:—

A function such that, pre-assigning any positive quantity e , we can find a positive quantity E , such that the positive and negative increments of the function over any set of non-overlapping intervals of content less than E are both less than e , is an integral.†

10. We may now enunciate the second definition of upper and lower semi-integrals, and it will be evident from what precedes that it will be concomitant with the first.

DEFINITION.—*A function of bounded variation, such that its negative (positive) increment over any set of non-overlapping intervals of content I has zero as limit, when I approaches zero, is called an upper (lower) semi-integral. Or, as an alternative, a function such that, pre-assigning any positive quantity e , we can find a positive quantity E , such that the negative (positive) increment of the function over any set of non-overlapping intervals of content less than E is numerically less than e , is called an upper (lower) semi-integral.*

11. The mode of proof adopted in § 9 leads again to the theorem given at the end of § 8.

We have, moreover, the following theorems:—

THEOREM.—*A function which is both an upper and a lower semi-integral is an integral.*

In other words, if a function is such that it is less than an integral by a monotone increasing function, and greater than an integral by a monotone increasing function, it is itself an integral. This gives a test, which may in certain cases be extremely convenient, for ascertaining whether a function is an integral.

THEOREM.—*If the sum of two upper (lower) semi-integrals is a lower*

* Lebesgue, *loc. cit.*

† Vitali, *loc. cit.*

(upper) semi-integral, each of the semi-integrals is an integral. In particular, if the sum of two monotone increasing (decreasing) functions is an integral, each of those functions is an integral, and of course continuous.

12. We have seen that semi-integrals are characterised by the fact that they are obtained from integrals by a process of addition or subtraction of a monotone increasing function. The same semi-integrals will in this way be obtained from various integrals; there are, however, semi-integrals which may be said to belong to, or to be associated with, a particular integral. We have seen, in fact, that a semi-integral shares with only one integral, to a constant *près*, the property of having a differential coefficient which coincides with that of the integral except at a set of content zero. Thus every integral has an associated set of upper and lower semi-integrals.

Thus we may conveniently give the following definition:—

DEFINITION.—*The associated semi-integrals of an integral are called the semi-integrals of the integrand.*

Thus every summable function has integrals and semi-integrals. The upper (lower) semi-integrals are all of them greater (less) than the definite integral, but their differential coefficients all exist and are finite and summable and agree with the integrand, except at a set of content zero.

13. It will now be necessary to consider certain portions of the theory of oscillating successions of functions. We say that such a succession *oscillates finitely above (below)*, if there is no point at which the upper (lower) function is $+\infty$ ($-\infty$). It is mostly with such successions that we shall have to deal, and if nothing is said to the contrary, it may be assumed that this is the case.

If a succession of functions is such that the limiting functions of the integrals of the succession are all less (greater) than, or equal to, the integral of the upper (lower) function of the original succession, this latter succession is said to be *semi-integrable above (below)*. The integrals in this case are supposed taken over any interval (a, x) whatever. If, and only if, the succession is semi-integrable both above and below, and further the upper and lower functions agree except at a set of content zero, all the limits of the integrals will coincide with the common integral of the upper and lower functions, in this case the succession is said to be *integrable term-by-term*, or, shortly, to be *integrable*.

14. THEOREM.—If $f_1 \leq f_2 \leq \dots$ is a monotone ascending sequence* of essentially positive functions, which diverges at a set of points of content greater than I , then we can assign an integer m , such that, for all values of $n \geq m$, the set of points at which $f_n(x) > H$ is of content greater than I .

For, if x is a point at which $f(x) > H$, it is a point at which, for some value of n , $f_n(x) > H$. Hence the set of points at which $f(x) > H$ is a subset of the generalised outer limiting set of the sets of points at which for fixed n , $f_n(x) > H$, as n assumes all positive integral values. *A fortiori* the set of points at which the sequence f_1, f_2, \dots diverges to $+\infty$, is a sub-set of that outer limiting set, so that the content of that outer limiting set is greater than I .

But the outer content of a generalised outer limiting set is the unique limit of the outer contents of the defining sets.† Therefore the limit of the content of the set of points at which $f_n(x) > H$ is greater than I , whence the required result at once follows.

COR.—If f_1, f_2, \dots is a monotone ascending sequence of essentially positive summable functions, and the sequence of integrals has for limit a finite function $F(x)$, then the original sequence converges, except possibly at a set of content zero, and is integrable term-by-term.‡

For, since the functions are never negative,

$$\int_a^b f_n(x) dx > HJ_n,$$

where J_n denotes the content of the set of points at which $f_n(x) > H$. Also, since the sequence is monotone increasing,

$$F(b) - F(a) \geq \int_a^b f_n(x) dx.$$

Hence J_n is less than $[F(b) - F(a)]/H$, and therefore, by the theorem, the content of the points of divergence $\leq [F(b) - F(a)]/H$, which, if H is sufficiently large, is as small as we please.

This proves that the f -sequence converges except at a set of content zero.

* In the present paper a sequence of functions is to be understood to mean a succession whose upper and lower functions agree everywhere, whether the common values are all finite or not.

† Young's *Theory of Sets of Points*, p. 104.

‡ G. Vitali, "Sull' Integrazione per Serie," 1907, *Rend. di Palermo*, Vol. xxiii, p. 151.

Hence, if at every point of this exceptional set we change the value of each $f_n(x)$ to zero, the new sequence will converge everywhere, and the limiting function $f(x)$ will differ from the upper and lower functions of the original sequence only at the exceptional set of content zero.

If now we put $g_n(x) = f_n(x)$ wherever less than A , and zero elsewhere, the g -sequence will converge to the function $g(x)$ got by the same rule from $f(x)$. Also since the g -succession is bounded, we know, by the Lebesgue theory, that $g(x)$ is summable, and that its integral is the unique limit of $\int g_n(x) dx$, and is therefore not greater than $F(x)$. Hence, by the definition of an integral, $f(x)$ is summable, and

$$\lim_{A=\infty} \int g(x) dx = \int f(x) dx \leq F(x).$$

But, since the f -succession is monotone ascending,

$$\int f(x) dx \geq \int f_n(x) dx,$$

whence, the sign of inequality in the preceding relation is inadmissible.* Therefore

$$\int f(x) dx = F(x),$$

which, since $f(x)$ only differs from the upper and lower functions of the original sequence at a set of content zero, proves the second statement of the corollary.

15. The following theorem has been given in a previous paper† in a less precise form; it is fundamental in the theory of oscillating successions.

THEOREM.—*If the integrals of a succession of functions which is bounded below (above) form a succession which nowhere diverges properly to $+\infty$ ($-\infty$), then (1) the original succession is semi-integrable below (above), and (2) it oscillates finitely below (above) except at a set of content zero.*

Let $l(x)$ be the lower function of the original succession f_1, f_2, \dots . Denote by w_n the function whose value at any point is the lower bound of

* This case, when the f -sequence converges, was proved by Beppo Levi.

† The first result of this theorem was proved in the case of finitely oscillating successions in my paper "On Term-by-Term Integration of Oscillating Series," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 111. A slight correction is needed in lines 14 and 15; they should read "integrals are finite; for, by (1), w_n is not less than s_n ."

the values of f_n, f_{n+1}, \dots at that point. Then

$$w_1 \leq w_2 \leq \dots \leq w_n \leq \dots$$

is a monotone increasing sequence, having as limit the lower function $l(x)$. Also, since

$$w_n(x) \leq f_n(x),$$

and the functions f_n are summable, w_n is summable, and its integral is not greater than that of f_n . Hence

$$\text{Lt}_{n=\infty} \int_a^x w_n(x) dx \leq \text{lower Lt}_{n=\infty} \int_a^x f_n(x) dx.$$

Since the lower limit on the right is by hypothesis finite, it follows from the preceding corollary that term-by-term integration of the w -sequence is allowable, so that $l(x)$ is summable. Hence $l(x)$ is finite except at a set of content zero, whence (2) follows, and

$$\text{Lt}_{n=\infty} \int_a^x w_n(x) dx = \int_a^x l(x) dx,$$

whence (1) follows, that is

$$\int_a^x l(x) dx \leq \text{lower Lt}_{n=\infty} \int_a^x f_n(x) dx,$$

16. THEOREM.—*If a succession of functions is semi-integrable below (above), the upper and lower functions of the succession of integrals are upper (lower) semi-integrals, being, in fact, greater (less) than the integral of the lower (upper) function of the original succession by a monotone increasing function.*

We proceed to prove the first of these two alternative statements. Since the lower limit of the difference of two functions is not greater than the difference of the upper limits, or of the lower limits, we have, denoting the upper and lower functions of the succession of integrals by $U(x)$ and $L(x)$,

$$\text{lower Lt}_{n=\infty} \int_x^{x+h} f_n(x) dx \leq U(x+h) - U(x),$$

and also

$$\leq L(x+h) - L(x).$$

From the former inequality, since the succession is semi-integrable below,

$$\int_x^{x+h} l(x) dx \leq U(x+h) - U(x),$$

whence

$$\int_a^{x+h} l(x) dx - U(x+h) \leq \int_a^x l(x) dx - U(x).$$

Similarly from the second of the above inequalities

$$\int_a^{x+h} l(x) dx - L(x+h) \leq \int_a^x l(x) dx - L(x),$$

which shews that the right-hand sides of both these last inequalities are monotone increasing functions, and proves the theorem in this case. Similarly the alternative theorem may be proved.

17. THEOREM.—*If the points* in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x) , is unbounded above (below) form at most a set of zero content, then the necessary and sufficient condition that the f -succession should be semi-integrable above (below) is that the upper (lower) function of the succession of integrals should be a lower (upper) semi-integral.*

That this condition is necessary has just been proved (§ 16), that it is sufficient follows easily. For, since the exceptional points form a closed set and it is, by hypothesis, of zero content, it can be enclosed in a finite number of intervals D , the sum of whose lengths is as small as we please.

Then, by the characteristic properties of an integral and a lower semi-integral, the integral of the upper function of the f -succession over these intervals is as small as we please numerically, and the positive increment of the upper function of the integral succession is also as small as we please. That is,

$$-e \leq \int_D u(x) dx \leq e,$$

$$\text{upper } \text{Lt}_{n=\infty} \int_D f_n(x) dx \leq e.$$

But in each of the complementary intervals d the f -succession is bounded and therefore semi-integrable above, that is

$$\text{upper } \text{Lt}_{n=\infty} \int_d f_n(x) dx \leq \int_d u(x) dx.$$

Hence, since the upper limit of a sum is \leq the sum of the upper limits,

$$\text{upper } \text{Lt}_{n=\infty} \left\{ \int_{D+d} f_n(x) dx \right\} \leq \int_{D+d} u(x) dx + 2e.$$

Since e is at our disposal, this proves the theorem.

COR.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a*

* These points are the points at which the peak (chasm) function of the succession has the value $+\infty$ ($-\infty$).

function of the ensemble (n, x) is unbounded form a set of content zero, the necessary and sufficient condition that the succession should be semi-integrable both above and below is that the lower and upper functions of the succession of integrals should be integrals.

This may be put another way:—we may say that, if the points at which the peak and chasm functions are infinite form a set of zero content, the succession will be semi-integrable both above and below if, and only if, all the upper functions and all the lower functions of all possible sub-successions are integrals. Moreover every unique limiting function of a sub-sequence, if such functions exist, is accordingly an integral.

The following example shews that the restriction imposed on the content of the points at which the succession is unbounded is essential. We have here a succession which converges everywhere to zero, while the succession of integrals converges to the unique limiting function x , which is an integral.

18. *Standard Example of a Succession which is not Semi-Integrable above.**

Divide the interval $(0, 1)$ into 2^n equal parts, and in each of these define $h_n(x)$ by the same rule, viz., divide that segment into 2^n parts, and let $h_n(x)$ be $= 2^n$ at every point of the most left-hand part, and be zero at all the remaining points. (Fig. 1.)

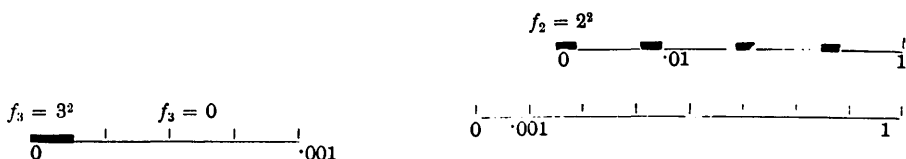


FIG. 1.

Then the sum of the lengths of the intervals in which $h_n(x) = 2^n$ is 2^{-n} ; therefore the sum of the lengths of all the intervals in which any of the functions from and after $h_n(x)$ are positive is 2^{1-n} ; and therefore the content of those intervals is less than 2^{1-n} , and has zero as limit, when n increases indefinitely.

Hence† the points belonging to an infinite number of these sets of

* Cp. Vitali, "Sull' Integrazione per Serie," 1906, *Rend. di Palermo*, Vol. XXIII, p. 155.

† Young's *Theory of Sets of Points*, Theorem 19, p. 96.

intervals form a set E of content zero, which includes, of course, all the binary points of division.

At the points of E the upper function of the h -succession is therefore infinite, and elsewhere it is zero. The lower function is always zero.

If now at all the points of E we put $f_n(x) = 0$, and elsewhere $f_n(x) = h_n(x)$, it is evident that *the f -succession will converge everywhere to zero*. But

$$\lim_{n=\infty} \int_0^x f_n(x) dx = \lim_{n=\infty} \int_0^x h_n(x) dx = x.$$

Thus the integral of the upper function

$$< \lim_{n=\infty} \int_0^x f_n(x) dx.$$

*Standard Example of an Oscillating Succession which is Semi-Integrable above without being Integrable above.**

19. Let all the functions of the g -succession be zero except in the intervals now to be specified.

$g_1 = 1$ in the interval $(\cdot 1, 1)$, using the binary notation, that is $(\frac{1}{2}, 1)$:

$g_2 = 1$ „ $(0, \cdot 1)$;

$g_3 = 1$ „ $(\cdot 11, 1)$;

$g_4 = 1$ „ $(\cdot 1, \cdot 11)$;

$g_5 = 1$ „ $(\cdot 01, \cdot 1)$;

$g_6 = 1$ „ $(0, \cdot 01)$,

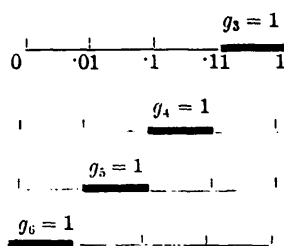


FIG. 2.

(Fig. 2), and so on; generally, if $n = 2^r - 1$, the $n+1$ functions

$$g_n, g_{n+1}, \dots, g_{2n},$$

are each $= 1$ in one of the $(n+1)$ equal parts into which we divide the segment $(0, 1)$, these segments being assigned to the functions in order from right to left. In this way *the upper function of the succession is clearly 1 and the lower function 0 everywhere*. The integral of the constituent function g_n , however, decreases without limit. Thus, as of course

* That is to say, without the upper limit of the integrals being the integral of the upper function.

must be the case, since the succession is bounded below and the integrals form a sequence, the integral of the lower function

$$= \lim_{n \rightarrow \infty} \int_0^x g_n(x) dx,$$

but the integral of the upper function

$$= x > \lim_{n \rightarrow \infty} \int_0^x g_n(x) dx.$$

20. THEOREM.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x) , is unbounded above (below) form at most a set of zero content, the f -succession is semi-integrable above (below), provided a g -succession which is semi-integrable above (below) can be found, such that, for all values of n , $f_n(x)$ is less (greater than or equal to $g_n(x)$).*

For, if the g -succession is semi-integrable above, the upper limit of the integrals is, by § 16, a lower semi-integral. Also, since

$$f_n(x) \leq g_n(x),$$

$$\int_a^x f_n(x) dx = \int_a^x g_n(x) dx$$

—a positive monotone increasing function ;

$$\text{and therefore } \lim_{n \rightarrow \infty} \int_a^x f_n(x) dx = \lim_{n \rightarrow \infty} \int_a^x g_n(x) dx$$

—a positive monotone increasing function.

Thus the upper limit of the integrals of the f -succession is also a lower semi-integral, whence, by § 17, the f -succession is semi-integrable above.

Similarly the alternative theorem may be proved.

COR.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x) , is unbounded, form at most a set of zero content, the f -succession is semi-integrable both above and below, provided two successions can be found, whose constituents are respectively \geq and \leq the corresponding constituents of the f -succession, the former of these successions being semi-integrable above and the latter semi-integrable below.*

It may be remarked that the tests here given are necessary as well as sufficient, on the assumption that the exceptional points do not form a set of positive content.

If the exceptional points of the auxiliary successions have zero content, those of the f -succession have, of course, zero content.

21. It will be seen that *the restriction imposed, by which the points at which the succession is unbounded above (below) form a set of content zero, is essential*. In fact, if we take the standard example, given in § 18, of an f -succession which is not semi-integrable above, and add to each constituent function $f_n(x)$ the corresponding constituent $g_n(x)$ of the standard g -succession semi-integrable above given in § 19, the succession $f_1 + g_1, f_2 + g_2, \dots$ is semi-integrable above. For, since the upper function of the f -succession is identically zero, the upper function of the $(f+g)$ -succession is the same as that of the g -succession, so that its integral from 0 to x has the value x . Since the integral of $f_n(x)$ has the limit x , and the value of the integral of $g_n(x)$ decreases with zero as limit as n increases, the limit of the integral of $[f_n(x) + g_n(x)]$ is x , and is equal to the integral of the upper function of the $(f+g)$ -succession, so that this latter succession is, as was stated, semi-integrable above. Moreover, since $g_n(x)$ is positive, the constituents of the $(f+g)$ -succession are greater than or equal to the corresponding constituents of the f -succession.

This proves that, if we omit the restriction as to the content of the exceptional points being zero, the preceding theorem is longer true.

It may be remarked that the $(f+g)$ -succession, though semi-integrable both above and below, is not integrable;* in fact, as is proved below (§ 26), if the $(f+g)$ -succession were integrable, the f -succession would necessarily be semi-integrable above, whether or not the exceptional points were of zero content.

22. The property of a succession of being semi-integrable above (below) relates to the integrals taken over any the same segment; if we integrate over any set of points, and the characteristic inequality is still true, irrespective of the particular set chosen, the succession is said to be *absolutely semi-integrable above (below)*. This distinction is only of importance when the succession is unbounded, as the following theorem shews:—

THEOREM.—*If the f -succession is bounded above (below), it is absolutely semi-integrable above (below).*

For, let S be any set of points, and g_n the function equal to f_n at every

* See § 13.

point of S and zero elsewhere, while v is the function got by the same rule from the upper function $u(x)$. Then the upper function of the g -succession is $v(x)$, and the g -succession is bounded above, if the f -succession is so.

Hence, by the theory of oscillating successions (§ 15),

$$\text{Llt}_{n=\infty} \int_a^b g_n(x) dx \leq \int_a^b v(x) dx,$$

that is,
$$\text{Llt}_{n=\infty} \int_S f_n(x) dx \leq \int_S u(x) dx.$$

Since this is true for every set S , this proves the first of the alternative theorems. Similarly the second may be proved.

23. THEOREM.—If $f_1(x), f_2(x), \dots$ is a succession of summable functions,* having a summable upper (lower) function, then the succession is absolutely semi-integrable above (below) provided

$$\int_E f_n(x) dx,$$

regarded as a function of the ensemble (n, E) , has no positive (negative) double limit, when n increases indefinitely and E approaches zero.

For, if G_k denote the set of points at which at least one of the functions $f_n(x)$ is greater than 2^k , the set G of all the points common to all the sets G_k (which, as is evident, are such that each contains the next set after it), consists of the points at which at least one of the functions f_n is $+\infty$, and of the points at which the upper function $u(x)$ is $+\infty$.

Since the functions f_n and u are summable, each of these sub-sets is of content zero, and therefore G is itself of content zero. Hence the content of G_k has, as k increases indefinitely, zero as limit. Thus we may denote the content of G_k by E , and we then have, by hypothesis,

$$\text{Llt}_{k=\infty, n=\infty} \int_E f_n(x) dx \leq 0. \quad (1)$$

Let g_n denote the function which $= f_n$ except at the points of G_k , where it is zero, and let v be the function got by the same rule from $u(x)$. Then v is evidently the upper function of the succession of functions $g_n(x)$, which is bounded above, so that, by the preceding theorem, it is absolutely semi-integrable above.

* These functions may be infinite, but of course only at a set of content zero, and the succession need not oscillate finitely.

Now, by the definition of g_n , the integral of g_n over any set S is the integral of f_n over the part of S not belonging to G_k . Hence, denoting the set of the common points of S and G_k by (S, E) , we have

$$\int_S f_n(x) dx = \int_S g_n(x) dx + \int_{(S, E)} f_n(x) dx,$$

and
$$\int_S u(x) dx = \int_S v(x) dx + \int_{(S, E)} u(x) dx.$$

Hence, since the g -succession is absolutely semi-integrable above,

$$\text{upper } \text{Lt}_{n=\infty} \left[\int_S f_n(x) dx - \int_{(S, E)} f_n(x) dx \right] \leq \int_S u(x) dx - \int_{(S, E)} u(x) dx.$$

By the fundamental property of an integral, the second integral on the right has zero as unique limit as k increases indefinitely, and in consequence, E and (S, E) diminish indefinitely. Also, by (1), all the double limits of the second integral on the left are negative or zero, so that the same is true of the repeated limits; hence the second integral on the left may be omitted without affecting the inequality which we then obtain, when we let k increase indefinitely,

$$\text{upper } \text{Lt}_{n=\infty} \int_S f_n(x) dx \leq \int_S u(x) dx.$$

which shews that the f -succession is absolutely semi-integrable above. This proves the first of the alternative theorems; similarly the other may be proved.

COR.—If both the upper and the lower function are summable, and $\int_E f_n(x) dx$ has the unique double limit zero, the f -succession is absolutely semi-integrable both above and below.

If, in addition, the upper and lower functions agree except at a set of content zero the f -succession may be integrated term-by-term over every set of points, that is,*

$$\text{Lt}_{n=\infty} \int_S f_n(x) dx \leq \int_S u(x) dx = \int_S l(x) dx.$$

* This would at the first glance seem to be in contradiction to a theorem of Vitali's, *loc. cit.*, p. 139, but this is only due to the fact that in the statement, not in the proof, of his theorem, Vitali has included the convergence of the series among his necessary and sufficient conditions for term-by-term integration.

The only case considered by Vitali is that of a sequence of functions, and he accordingly uses terms depending on this restriction. Thus the term "integrabile completamente per serie," used by him, implies that, whatever set we integrate over, the succession of integrals

24. THEOREM.—*If the upper (lower) function of a succession of summable functions is itself a summable function, and the succession is absolutely semi-integrable above (below), then*

$$\int_E f_n(x) dx$$

has, when n increases indefinitely and E decreases with zero as limit, no positive (negative) double limit.

In the first place it is obvious that, if the succession is bounded above, none of the double limits can be positive; for, if for all values of n , $f_n < 2^k$, the integral whose limits are considered is less than $E2^k$, which has zero as limit when E has zero as limit, however n may vary.

If, then, the f_n 's be unbounded above, and $g_n = f_n$, except at the points of the set G_k where it is zero, $\int_E g_n(x) dx$ will have no positive double limit. Here, as in a previous proof, G_k denotes the set of all the points at which at least one of the functions f_n is greater than 2^k , and, as before, the content of G_k has zero as limit, when k is indefinitely increased. Hence, by the fundamental property of an integral, so often used in the present paper, we can determine k so that the integral of $g_n(x)$

converges, and has as limit the integral of the unique limiting function of the sequence. This is a special case of the circumstances contemplated in the above corollary.

Vitali's condition is formally different, but actually the same, as that given above. He demands that the integrals of the sequence should be what he calls "assolutamente continui," that is to say, *pre-assigning any positive quantity e , he requires that it should be possible to determine a positive quantity E_0 , such that, provided only the content of the set E , over which we propose to integrate, is less than E_0 , $\int_E f_n(x) dx$ is less than e .*

Whatever succession of ensembles (E, n) , we choose, n increasing indefinitely and E approaching zero as limit, there will then be a stage from and after which all the sets considered have content less than E_0 , so that all the double limits of $\int_E f_n(x) dx$ are numerically less than e . Since this is true for all choices of e , the integral considered has zero as unique double limit. Thus *Vitali's condition being satisfied, my condition is so also.*

On the other hand, if my condition is satisfied, so that $\int_E f_n(x) dx$ has zero as unique double limit, we can assign e and then determine E'_0 and m , so that, for all integers n greater than m , and all sets E whose content is less than E'_0 the integral considered is less than e . But, by the fundamental property of an integral so often used in the present paper, we can find E_i so that, provided the content of the set E is less than E_i , $\int_E f_i(x) dx$ is less than e . Thus, if we choose E_0 to be the least of the quantities $E'_0, E_1, E_2, \dots, E_m$, $\int_E f_n(x) dx$ is, for all values of n , less than e . Thus *my condition being satisfied, Vitali's is so also.*

over the set G_k , or any of its sub-sets, is less than e , where e is any pre-assigned positive quantity. Thus

$$\int_{G_k} u(x) dx < e;$$

and therefore, since the succession is absolutely semi-integrable above,

$$\text{upper } \text{Lt}_{n=\infty} \int_{G_k} f_n(x) dx < e.$$

We can, in consequence, determine m , so that for all values of n greater than m ,

$$\int_{G_k} f_n(x) dx < e.$$

Now, if E_k denote the common part of G_k and the set E over which we propose to integrate, $f_n(x)$ is certainly positive at every point of G_k , and therefore

$$\int_{E_k} f_n(x) dx \leq \int_{G_k} f_n(x) dx \leq e,$$

$$\text{whence } \int_E f_n(x) dx = \int_E g_n(x) dx + \int_{E_k} f_n(x) dx < \int_E g_n(x) dx + e.$$

Since this inequality holds for all values of n greater than m , and for the fixed value of k chosen and E has been chosen arbitrarily, we have, remembering that all the double limits of the integral on the right are ≤ 0 ,

$$\text{Llt}_{n=\infty, E=0} \int_E f_n(x) dx < e.$$

Since this is true for all positive values of e , it shews that the integral on the left has no positive double limit, when n increases indefinitely, and E has zero as limit.

25. From the preceding theorem we have immediately the following :—

THEOREM.—*If a succession of functions is absolutely semi-integrable above (below), the succession of integrals oscillates uniformly and homogeneously above (below).**

$$\text{For } \int_a^{x+h} f_n(x) dx = \int_a^x f_n(x) dx + \int_x^{x+h} f_n(x) dx,$$

and, by the preceding theorem, the latter integral on the right has no

* In this case all the upper functions and all the lower functions of the succession of integrals or any of its sub-successions are upper (lower) semi-continuous functions. Cp. W. H. Young, "On Homogeneous Uniform Oscillation," 1910, *Proc. London Math. Soc.*, Vol. 8, pp. 353-364.

positive double limit, when n increases indefinitely and h has zero as limit. But the upper double limit of the integral on the left is the peak function $\Pi(x)$ of the succession of integrals, and the upper limit of the first integral on the right is the upper function $U(x)$ of its succession of integrals; thus

$$\Pi(x) \leq U(x).$$

But the peak function is never less than the upper function, therefore

$$\Pi(x) = U(x),$$

that is to say, the succession of integrals oscillates uniformly above. Moreover, the necessary and sufficient condition in order that a succession may be absolutely integrable above is a homogeneous condition, that is to say, it applies equally to the whole succession and to any of its sub-successions. Hence the above applies equally to the succession and to any of its sub-successions, so that any sub-succession of the succession of integrals oscillates uniformly above, that is, the succession of integrals oscillates uniformly and homogeneously above.

26. The necessary and sufficient condition that a succession of functions should be absolutely semi-integrable above (below), proved in the two Theorems of §§ 23, 24, viz., that $\int_E f_n(x)dx$ should have no positive (negative) double limit, enables us to see that *a succession of positive functions which is semi-integrable above is not necessarily absolutely semi-integrable above.*

For, if the condition above referred to holds when f_n is positive, it obviously holds for another succession of positive functions, g_1, g_2, \dots , where $g_n \leq f_n$. Hence the g -succession would be also semi-integrable above absolutely, and therefore certainly semi-integrable above ordinarily. We have, however, given an example which contradicts this statement, namely, that in § 21; the two successions in question are there called the $(f+g)$ -succession, and the f -succession. Of these the former is a succession of positive functions which is semi-integrable above, and the latter a smaller succession of positive functions which is not semi-integrable above.

Hence, also, *a succession of functions which is bounded below (above) and semi-integrable above (below), is not necessarily absolutely semi-integrable above (below).*

We have, however, the following theorem, already referred to in § 21.

THEOREM.—*If a succession of functions is bounded below (above) and*

integrable term-by-term, the succession is necessarily absolutely integrable, that is to say, the integrals of the upper and lower functions over any set of points are equal to one another and to the unique limit of the integrals of the constituent functions of the succession over the set of points considered.*

For a succession which is bounded below is, as we saw (§ 22), absolutely semi-integrable below. Hence, denoting by S any set of points, and by T the complementary set, we have, in this case,

$$\text{Lt}_{n=\infty} \int_S f_n(x) dx \geq \int_S l(x) dx,$$

$$\text{Lt}_{n=\infty} \int_T f_n(x) dx \geq \int_T l(x) dx.$$

But, by hypothesis,
$$\text{Lt}_{n=\infty} \int_a^b f_n(x) dx = \int_a^b l(x) dx,$$

which necessitates each of the two above inequalities being an equality, and so proves the theorem.

27. THEOREM.—*If a summable function $g(x)$ can be found, such that, for all values of n and x ,*

$$f_n(x) \leq g(x) \text{ [or, in the second case, } g(x) \geq f_n(x)\text{]},$$

the f -succession is semi-integrable above (below).

For, let $w_n(x)$ denote the function whose value at each point x is the upper bound of the values of $f_n, f_{n+1}, f_{n+2}, \dots$ there, then, it follows from the hypothesis that, for all values of n ,

$$f_n(x) \leq w_n(x) \leq g(x); \tag{1}$$

also the functions w_n form a monotone descending sequence, so that we may integrate term-by-term, and, since their limit is the upper function $u(x)$ of the f -succession,

$$\text{Lt}_{n=\infty} \int w_n(x) = \int u(x) dx.$$

But, by (1),

$$\int f_n(x) \leq \int w_n(x);$$

therefore

$$\text{Llt}_{n=\infty} \int f_n(x) dx \leq \int u(x) dx,$$

which proves the theorem.

* See § 18.

COR.—Under these circumstances the f -succession is also absolutely semi-integrable above (below).

In fact, if we replace the values of all the functions concerned at the complementary set of any chosen set S by zero, the above reasoning holds good of the new functions, and the result may be written

$$\text{Llt}_{n=\infty} \int_S f_n(x) dx \leq \int_S u(x) dx.$$

Alternative proof.—The above proof is independent of the considerations we have been adducing. It is evident, however, that the theorem is an immediate consequence of the fact that a succession of functions is certainly integrable above if its integrals are less by a positive monotone increasing function than a function which is an integral; for the necessary and sufficient condition for semi-integrability above is satisfied. The theorem is, in fact, a particular case of the following theorem.

28. THEOREM.—A succession of functions is absolutely semi-integrable above (below) if another succession which is absolutely semi-integrable above (below) can be found, whose constituent function is \geq (\leq) the corresponding constituent function of the given succession.

Here again the condition for absolute semi-integrability above (below) given in § 23 is obviously satisfied.

As in practice there may be a difficulty in determining whether a succession which is semi-integrable above is so absolutely, the following special case of this theorem rises in importance:—

COR.—A succession of functions f_1, f_2, \dots is absolutely semi-integrable above (below) if an integrable succession g_1, g_2, \dots can be found, which is bounded below (above), such that $g_n(x)$ is greater (less) than or equal to $f_n(x)$, for all values of n and x .

It will be remarked that the condition that the g -succession should be absolutely semi-integrable above (below) is essential in this reasoning. We cannot, therefore, omit the boundedness below of the g -succession in the corollary. The reasoning, moreover, by which we proved the theorem of § 27 does not apply here, since it does not follow because the g -succession is absolutely semi-integrable above (below) that there exists a summable function which is \geq (\leq) $g_n(x)$ for all values of n .

The limitation that the g -succession must be bounded below is not one which in practice is likely to cause much inconvenience. If, however, the

succession whose semi-integrability is under discussion is not absolutely semi-integrable, the test is necessarily inapplicable; as it, however, constitutes a necessary and sufficient condition for the absolute semi-integrability above of the f -succession, it must succeed, except in the rare case in question. It is, in fact, obvious, that a succession which is absolutely semi-integrable above, remains so when we replace all the values less than $-K$ by the value $-K$.

29. It remains to give a test applicable in the exceptional case. This we are only able to do when it is either known *a priori*, or ascertainable, that the points in the neighbourhood of which the succession is unbounded above (below) form a set of zero content. We then have the theorem given in § 17.

If it be further known that the set of points in question is not only of zero content, but countable, we have the theorem given in my paper on "Term-by-Term Integration of Oscillating Series." *It is sufficient to shew that the upper function of the succession of integrals is lower semi-continuous on the left and upper semi-continuous on the right to be sure that semi-integration above is allowable.* This is a test which is very convenient to apply in practice; it is even more convenient in many cases than the test by comparison, and it is applicable whether the given succession is as a matter of fact absolutely semi-integrable or not; it constitutes, moreover, a necessary as well as sufficient test.

30. The tests we have given have all been of a type which permitted of the separate consideration of semi-integration above and below, and the wide character of their applicability as well as the conciseness of the proofs is largely due to this characteristic. A test of a different character which deals simultaneously with semi-integrability above and below is the following, and is in certain cases very useful. It has been stated and proved by Lebesgue* for sequences in a slightly different form.

THEOREM.—*The succession*

$$f_1(x)g(x), f_2(x)g(x), \dots,$$

is absolutely semi-integrable both above and below, if the upper and lower functions of the succession are summable, if g^2 is summable, and if further

* H. Lebesgue, "Sur les intégrales singulières," p. 50.

$\int f_n^2 dx$ is, for all values of n and all sets over which we integrate, less than a finite constant, say B .

For, by Schwarz's lemma,*

$$\left(\int_E f_n g dx \right)^2 \leq \int_E f_n^2 dx \int_E g^2 dx \leq B \int_E g^2 dx.$$

This shews that the left-hand side has the unique double limit zero when n increases without limit, and E decreases to zero, which, by § 23, proves the theorem.

31. We have seen that the theory of semi-integrability above (below) is less simple than that of absolute semi-integrability above (below). In the only theorems that we have been able to give we have had to predicate that a certain set of points should in one form of test be of zero content, and, in another form, narrower but more easily applied, that this set should be countable. It is very remarkable, however, that, when we come to the theory of integrable sequences, the theory simplifies itself considerably. We have, in fact, the following necessary and sufficient test for the integrability of a sequence.

THEOREM.†—*The necessary and sufficient condition that a sequence*

$$f_1, f_2, \dots, f_n, \dots$$

should be integrable is that it should be possible to find two other sequences, each of which is integrable, and such that

$$h_n \leq f_n \leq g_n.$$

For the functions $g_n - h_n$ form a sequence except at most at a set of content zero, and each constituent function of this sequence, if we make a suitable convention as to its value at the points where g_n and h_n are infinite, is a positive function. Hence this sequence, since it is certainly integrable, is absolutely integrable. But

$$0 \leq f_n - h_n \leq g_n - h_n.$$

Hence the sequence of functions $f_n - h_n$ is absolutely integrable. But the h -sequence is, by hypothesis, integrable, therefore the sequence of func-

* Id., p. 37; see also W. H. Young, "On a New Method in the Theory of Integration."

† See *clow*, § 34.

tions $h_n + (f_n - h_n)$ is integrable, which proves that the condition is sufficient. That it is necessary is obvious, for we only have to put

$$g_n = x + f_n,$$

$$h_n = -x + f_n,$$

and the g -sequence and h -sequence will satisfy the requisites.

COR.—*The above theorem is equally true if for sequences we substitute successions which converge except at a set of content zero, as is clear from § 26.*

32. For the convenience of readers who may be more especially interested in tests for the integration term-by-term of converging series or sequences, we collect here the more striking of such tests.

Test 1. (Osgood-Lebesgue.)—*If the sequence of functions is in its entirety bounded both above and below, term-by-term integration is always allowable.*

Test 2.—*A monotone sequence of functions, each of which is summable, may always be integrated term-by-term.*

Test 3.—*If a sequence of functions is bounded below, then, provided another sequence can be found which is integrable term-by-term and is such that its constituent function is greater than or equal to the corresponding constituent of the given sequence, term-by-term integration of the original sequence is allowable.*

A similar test may be used interchanging “below” with “above,” and “greater” with “less.”

Test 4. (This is a special case of Test 3.)—*If a sequence of functions is bounded below (above), then, provided a summable function can be found which is greater (less) than or equal to every function of the sequence, term-by-term integration is allowable.*

Test 5.—*If two summable functions $g(x)$ and $h(x)$ can be found, such that the constituent function of the sequence lies at every point between $g(x)$ and $h(x)$, then term-by-term integration is allowable.*

Test 6.—*If two integrable sequences of functions $g_n(x)$ and $h_n(x)$ can*

be found, such that

$$g_n(x) \leq f_n(x) \leq h_n(x),$$

for all values of n and x , term-by-term integration of the f -sequence is allowable.

Test 7. (Osgood.)—If the points in the neighbourhood of which the sequence of functions is unbounded are at most countable, then the sequence is integrable term-by-term provided only the integrals of the sequence converge to a continuous limiting function.

It should be noticed that Tests 3, 4, 7 are, under the circumstances stated, necessary as well as sufficient. Test 6 is always necessary and sufficient. It should also be noticed that one test may be utilised in the application of another; thus Test 2 can be utilised in giving a special form to other tests, e.g., we have the following:—

Test 8.—If the constituent function of a sequence lies between the corresponding constituents of two monotone sequences of summable functions, whose limiting functions are summable, term-by-term integration is always allowable.

We note that the integrability of a monotone sequence is always absolute, and hence, also, that it follows that the given sequence is not merely integrable, but absolutely integrable.

In fact we have the following test for absolute integrability:—

Test 9.—If a g -sequence and an h -sequence can be found, each of which is absolutely integrable, and for all values of n and x ,

$$g_n(x) \leq f_n(x) \leq h_n(x),$$

then absolute term-by-term integration of the f -sequence is allowable.

The following test, though of a very special character, may in certain cases be applied with great ease.

Test 10. (Lebesgue.)—The sequence $f_1g, f_2g, \dots, f_ng, \dots$ is integrable term-by-term if both its limiting function and g^2 are summable functions, and if $\int f_n^2 dx$ over the interval considered is for all values of n less than a finite constant.

Another form of the same test is the following:—

The sequence in question is integrable term-by-term if the limiting function and g^2 are both summable, and $\int (f_n - f)^2 dx$ over the interval considered is for all values of n less than a finite constant.

The following theorems which can be utilised as tests are worth noting :—

THEOREM.—*If two sequences of positive functions f_1, f_2, \dots and g_1, g_2, \dots are integrable term-by-term, the sequence whose constituent function is $f_n g_n$ is integrable term-by-term.*

THEOREM.—*Under the same circumstances the sequence whose constituent function is h_n , where h_n is equal at each point to the greater of f_n and g_n , is also integrable term-by-term.*

The first of these theorems follows from the second by means of Test 3. It suffices therefore to prove the latter theorem.

We have merely to remark that the succession whose constituent is $(f_n + g_n)$ is certainly integrable; the required result follows then by Test 3.

THEOREM.—*If the sequence whose constituent is f_n^2 is integrable term-by-term, so is the f -sequence.*

THEOREM.—*If the sequence whose constituent is $|f_n|$ is integrable term-by-term, so is the f -sequence.*

Both these theorems follow from Test 6, in its two portions. To prove the second, put $g_n(x) = -|f_n(x)|$ and $h_n(x) = +|f_n(x)|$.

To prove the first, take $h_n(x) = f_n^2$, wherever f_n is greater than 1, and $= f_n$ at the remaining points, and $g_n = -f_n^2$, wherever f_n is less than -1 , and $= f_n$ at the remaining points.

33. So far we have tacitly assumed that the limits of integration are finite. It is plain that, when this is not the case, the reasoning of the paper will not always hold as it stands, and many of the results will require modification. We content ourselves here with shewing that the very important test, obtained in § 31, still holds, when the interval over which we integrate is an infinite interval. For this purpose it will be convenient first to prove the following inequality between the repeated limits of a function of two variables, which is monotone with respect to one of them :—

LEMMA.—*If $F(X, n)$ is a monotone increasing function of X ,*

$$\text{Lt}_{X=\infty} [\text{lower Lt}_{n=\infty} F(X, n)] \leq \text{lower Lt}_{n=\infty} [\text{Lt}_{X=\infty} F(X, n)].$$

For, since $F(X, n)$ increases with X ,

$$F(X, n) \leq \text{Lt}_{X=\infty} F(X, n);$$

therefore $\text{lower Lt}_{n=\infty} F(X, n) \leq \text{lower Lt}_{n=\infty} [\text{Lt}_{X=\infty} F(X, n)],$

whence the required result at once follows.*

THEOREM.—If f_1, f_2, \dots is a succession of positive functions, each of which is summable throughout the whole infinite interval $(0, \infty)$, then the succession is semi-integrable below throughout that interval, whether, or not, the succession is bounded above.

Denoting by $l(x)$ the lower function of the f -succession, we prove, as before, that

$$\int_0^X l(x) dx \leq \text{lower Lt}_{n=\infty} \int_0^X f_n(x) dx.$$

Now, since $f_n(x)$ is positive, the integral on the right is a monotone function of its upper limit; hence, by the lemma,

$$\text{Lt}_{X=\infty} \left[\text{lower Lt}_{n=\infty} \int_0^X f_n(x) dx \right] \leq \text{lower Lt}_{n=\infty} \left[\text{Lt}_{X=\infty} \int_0^X f_n(x) dx \right].$$

Therefore $\int_0^\infty l(x) dx \leq \text{lower Lt}_{n=\infty} \left[\int_0^\infty f_n(x) dx \right].$

34. From the above theorem we at once get the following:—

THEOREM.—If f_1, f_2, \dots be a sequence of positive functions, which is integrable throughout the whole infinite interval $(0, \infty)$, and g_1, g_2, \dots is any sequence of positive functions, such that $g_n \leq f_n$, the g -sequence is also integrable throughout the whole infinite interval.

In fact, denoting the excess of f_n over g_n by h_n , the h 's will, like the g 's, form a sequence, to which the preceding theorem applies, for plainly the g 's and the h 's will all be summable.

Adding then the two inequalities for the g 's and the h 's respectively,

* Similarly, of course, the same inequality holds if on both sides we change "lower" into "upper."

we must get an equality, which proves that both the inequalities are themselves equalities.

The reasoning of § 31 applies therefore as it stands to this case, for it depends precisely on the theorem for a finite interval which we have just shewn to hold good for an infinite interval. Hence the theorem stated in that article remains true when the interval under consideration is infinite.

It is also of interest to remark that the reasoning by which this result has been proved, like that of § 33, is unaffected if the integrals considered are non-absolutely convergent integrals. This follows from the fact that $h_n - g_n$ and $f_n - g_n$ are positive functions for all values of n .

35. The theorems we have obtained in the theory of oscillating series, may obviously be applied to obtain corresponding theorems in the theory of integrals. Thus we have the following important theorem in parametric integration :—

THEOREM.—*If $f(x, y)$ is for each fixed value of y continuous with respect to x , except for a set of values of x of zero content, and if $g(x, y)$, $h(x, y)$ be two other functions possessing the same property, then, without any hypothesis as to the boundedness of these functions, we can assert that $\int f(x, y) dx$ is a continuous function of y , if $\int g(x, y) dx$ and $\int h(x, y) dx$ are so, provided only*

$$g(x, y) \leq f(x, y) \leq h(x, y).$$

This theorem has a number of important consequences, analogous to the corresponding ones for series. In particular we have the following theorem, which I had already proved elsewhere* by another method :—

THEOREM.—*Under the same circumstances as to the continuity with respect to y of the function $f(x, y)$, $\int f(x, y) dx$ is a continuous function of y , provided $\int |f(x, y)| dx$ is so.*

That these theorems follow from the corresponding theorems for successions is obvious, if we reflect that, under the circumstances hypothesized,

$$\int \text{upper } \lim_{y=y_0} f(x, y) dx = \int \text{lower } \lim_{y=y_0} f(x, y) dx = \int f(x, y_0) dx.$$

* W. H. Young, "On Parametric Integration," *Monatshefte für Math. und Physik*, January, 1910, pp. 1-24.

36. Again we have the following theorem in the theory of differentiation under the sign of integration :—

THEOREM.—*Differentiation under the sign of integration is allowable for $\int f(x, y)dx$, provided two functions $g(x, y)$ and $h(x, y)$ can be found, for which differentiation under the sign of integration is allowable, one of these functions being $\leq f$, and the other $\geq f$, and the difference in each case being a monotone increasing function of y .*

For, since $f(x, y) - g(x, y)$ and $h(x, y) - f(x, y)$ are monotone increasing functions of y ,

$$g(x, y+k) - g(x, y) \leq f(x, y+k) - f(x, y) \leq h(x, y+k) - h(x, y).$$

The same is therefore true if we divide each member of this inequality by k . Hence, by the concluding test of § 33, the sequence whose general term is

$$\frac{f(x, y+k) - f(x, y)}{k},$$

if k is positive, and df/dy when k is zero, is integrable term-by-term, provided the same is true when we change f into g or into h . Now, by hypothesis,

$$\frac{d}{dy} \int g(x, y) dx = \int \frac{dg}{dy} dx,$$

$$\text{that is, } \text{Lt}_{k=0} \int \frac{g(x, y+k) - g(x, y)}{k} dx = \int \text{Lt}_{k=0} \frac{g(x, y+k) - g(x, y)}{k} dx,$$

so that the change of f into g renders the sequence integrable term-by-term. Similarly the change of f into h renders the sequence integrable. Thus

$$\text{Lt}_{k=0} \int \frac{f(x, y+k) - f(x, y)}{k} dx = \int \text{Lt}_{k=0} \frac{f(x, y+k) - f(x, y)}{k} dx,$$

$$\text{that is, } \frac{d}{dy} \int f(x, y) dx = \int \frac{df}{dy} dx,$$

which proves the theorem.

37. We have also the following theorem in change of order of integration in an infinite integral :—

THEOREM.—*If two functions $g(x, y)$ and $h(x, y)$ can be found, such that*

$$g(x, y) \leq f(x, y) \leq h(x, y),$$

then change of order of integration is allowable in the integrals

$$\int_0^{\infty} dx \int_0^{\infty} f(x, y) dy = \int_0^{\infty} dy \int_0^a f(x, y) dx,$$

provided it is so when the integrand is g and also when it is h , always supposing that change of order of integration is possible when the limits of integration are finite.

For, by hypothesis,

$$\begin{aligned} \int_0^a dx \int_0^{\infty} g(x, y) dy &= \int_0^a dx \operatorname{Lt}_{b=\infty} \int_0^b g(x, y) dy = \int_0^{\infty} dy \int_0^a g(x, y) dx \\ &= \operatorname{Lt}_{b=\infty} \int_0^b dy \int_0^a g(x, y) dx = \operatorname{Lt}_{b=\infty} \int_0^a dx \int_0^b g(x, y) dy. \end{aligned}$$

That is, denoting $\int_0^b g(x, y) dy$ by $G(x, b)$,

$$\int_0^a \operatorname{Lt}_{b=\infty} G(x, b) dx = \operatorname{Lt}_{b=\infty} \int_0^a G(x, b) dx.$$

In like manner, and with a like meaning to $H(x, b)$,

$$\int_0^a \operatorname{Lt}_{b=\infty} H(x, b) dx = \operatorname{Lt}_{b=\infty} \int_0^a H(x, b) dx.$$

Thus the G -sequence and the H -sequence are integrable term-by-term. Also, from the fact that $f - g$ is positive and less than $h - g$ which possesses repeated integrals, we have

$$\int_0^b dy \int_0^a f(x, y) dx = \int_0^a dx \int_0^b f(x, y) dy = \int_0^a F(x, b) dx,$$

where, since f lies between g and h , F lies between G and H . Therefore, since by the concluding test of § 32, the F -sequence is integrable term-by-term, we have

$$\begin{aligned} \int_0^{\infty} dy \int_0^a f(x, y) dx &= \operatorname{Lt}_{b=\infty} \int_0^a F(x, b) dx = \int_0^a \operatorname{Lt}_{b=\infty} F(x, b) dx \\ &= \int_0^a dx \int_0^{\infty} f(x, y) dy, \end{aligned}$$

which proves the theorem.

COR.—If, in addition,

$$\int_0^\infty dx \int_0^b g(x, y) dy = \int_0^b dy \int_0^\infty g(x, y) dx,$$

for b finite and also for b infinite, and the same is true when we change g into h , then

$$\int_0^\infty dx \int_0^\infty f(x, y) dy = \int_0^\infty dy \int_0^\infty f(x, y) dx.$$

For in this case, we are given that

$$\begin{aligned} \int_0^\infty dx \operatorname{Lt}_{b=\infty} \int_0^b g(x, y) dy &= \operatorname{Lt}_{b=\infty} \int_0^b dy \int_0^\infty g(x, y) dx \\ &= \operatorname{Lt}_{b=\infty} \int_0^\infty dx \int_0^b g(x, y) dy, \end{aligned}$$

that is,
$$\int_0^\infty dx \operatorname{Lt}_{b=\infty} G(x, b) = \operatorname{Lt}_{b=\infty} \int_0^\infty G(x, b) dx.$$

Hence, since in this equation we may change G into H , it follows by the test that we may change G into F , which gives

$$\int_0^\infty dx \int_0^\infty f(x, y) dy = \operatorname{Lt}_{b=\infty} \int_0^\infty dx \int_0^b f(x, y) dy,$$

which, by the theorem,

$$= \operatorname{Lt}_{b=\infty} \int_0^b dy \int_0^\infty f(x, y) dx = \int_0^\infty dy \int_0^\infty f(x, y) dx.$$

This proves the corollary.

38. We also have the following theorem in parametric integration as an immediate corollary from the concluding theorem of § 32.

THEOREM.—If $f(x, y)$ is a continuous function of y , and two other functions $g(x, y)$ and $h(x, y)$ can be found respectively \geq and $\leq f(x, y)$ and also continuous with respect to y , or more generally if the continuity of the three functions with respect to y holds except at a set of values of x of content zero, then the continuity of $\int_0^\infty g(x, y) dx$ and $\int_0^\infty h(x, y) dx$ regarded as functions of y , carry with them the continuity of $\int_0^\infty f(x, y) dx$.

Ex.— $\int_0^\infty \frac{\cos x}{a^2 + x^2} dx$ is a continuous function of a , except when $a = 0$.

Here the larger of the two auxiliary functions g and h is $1/(a^2+x^2)$ and the smaller is $-1/(a^2+x^2)$.

We terminate this brief account of applications of the general test at the end of § 32 with the following particular test for the integration over an infinite interval which follows immediately from the general test:—

The series $u_1+u_2+\dots$ is integrable term-by-term from 0 to ∞ , if a function $v(x)$ can be found, such that $|u_n/v| < a_n$, for all values of n , a_n being the general term of a convergent series of constants, and $v(x)$ being a function which is summable in the infinite interval in question.

This is given here as being a generalisation of a particular test of Osgood's, in which $v(x)$ is x^{-k} and the u 's are continuous functions. It is, as will be at once perceived, a very special case of our theorem.