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Pamphlets by A. Voss:—"Ueber die mit einer bilinearen Form vertauschbaren bilinearen Formen," 8vo; "Ueber die conjugirte Transformation einer bilinearen Form in sich selbst," 8vo; "Ueber einen Satz aus der Theorie der Determinanten," 8vo; "Ueber die cogredienten Transformationen einer bilinearen Form in sich selbst," 4to.

On the Stability of a Plane Plate under Thrusts in its own Plane, with Applications to the "Buckling" of the Sides of a Ship.

By G. H. BRYAN.

[*Read Dec. 11th*, 1890.]

Introduction.

1. The problems discussed in this paper are the analogues for a plane rectangular or circular plate of the well-known investigations of the stability of a thin wire or shaft, due in the first place to Euler, and since developed by Greenhill. I have employed the energy criterion of stability, the use of which I have already illustrated in this connexion in two papers published in the *Proceedings of the Cambridge Philosophical Society.**

The case of a plate supported on equidistant parallel ribs will be considered more fully, on account of the practical use of such structures in the construction of ships.

Suppose a plane elastic plate is submitted to edge tractions in its own plane which produce compression of its middle surface, and let every point of that surface receive a displacement normal to the plane, such displacements being chosen in accordance with the prescribed boundary conditions. If this displacement be everywhere of the first order of small quantities, the surface of the plate will thereby become extended by small quantities of the second order,

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* *Camb. Phil. Proc.*, Vol. VI., pp. 199, 286.

and the strains produced in the surface will also be thereby diminished by quantities of the second order. Multiplying these strain variations by the corresponding stresses, and integrating over the surface, we find the work done by the stresses. This work is the loss of potential energy consequent on stretching, or, more strictly, on diminution of the compression originally produced in the middle surface by the given edge tractions. If this loss is greater than the corresponding gain of potential energy dependent on the bending of the surface, the total energy will be greater in the plane form, which will therefore be a form of unstable equilibrium. The plate will then be liable to "buckle up," and corrugations will appear on its surface.

If equilibrium is "critical," the displaced form is also in equilibrium, and may be investigated by the method of variation.

General Theory of the Stability of a Plane Plate.

2. Take any rectangular axes of x and y in the plane of the plate, and let the stresses due to the edge tractions at a point (x, y) of the surface consist of a thrust T_1 (per unit length) parallel to x, a thrust T_2 parallel to y, and a shearing stress of magnitude M. Let the plate be displaced in such a manner that the point (x, y) receives a displacement w perpendicular to the plane of the plate. Let σ_1, σ_2 be the resulting extensions parallel to the axes of x, y at the point, ϖ the shear of the angle between them. If ds is an element of length, measured between any contiguous points on the deformed surface, then evidently

$$ds^2 = dx^2 + dy^2 + dw^2.$$

But, since w is a function of x and y,

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy,$$

whence, substituting,

$$ds^2 = \left\{1 + \left(\frac{\partial w}{\partial x}\right)^2\right\} dx^2 + 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx\, dy + \left\{1 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dy^2.$$

Also, $$ds^2 = (1+\sigma_1)^2 dx^2 + 2\varpi\, dx\, dy + (1+\sigma_2)^2 dy^2;$$

therefore, neglecting powers of w after the second,

$$2\sigma_1 = \left(\frac{\partial w}{\partial x}\right)^2, \quad 2\varpi = 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, \quad 2\sigma_2 = \left(\frac{\partial w}{\partial y}\right)^2.$$

The *loss* of potential energy due to extension, or the work done by the stresses, is equal to W, where

$$W = \iint (T_1\sigma_1 + M\varpi + T_2\sigma_2)\,dx\,dy$$

$$= \tfrac{1}{2}\iint \left\{ T_1\left(\frac{\partial w}{\partial x}\right)^2 + 2M\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + T_2\left(\frac{\partial w}{\partial y}\right)^2 \right\} dx\,dy \quad \text{...(1).}$$

Let $2h$ be the thickness of the plate, E Young's modulus, σ Poisson's ratio, and let β be given by

$$\beta = \tfrac{2}{3}h^3 \frac{E}{1-\sigma^2} \quad \text{.............................(2).}$$

Then, if V be the potential energy due to bending, we have

$$V = \tfrac{1}{2}\beta \iint \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\sigma)\left\{ \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} \right] dx\,dy$$

$$\text{..................(3).}$$

The condition of stability is that V must be greater than W for every possible type of displacement. If for any displacement $V < W$, the plane form will be unstable, and the plate will buckle up. The number of corrugations produced when this happens will be determined by the form of w as a function of x and y.

Supported Rectangular Plate.

3. Consider the case of a rectangular plate bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$, and supported at these edges so as to prevent lateral motion. Suppose also that the given edge tractions are uniform and perpendicular to the edges, being T_1 per unit length on the edges $x = 0$, $x = a$, acting parallel to the axis of x, and T_2 on the edges $y = 0$, $y = b$, parallel to the axis of y. Then, in the expression for W, $M = 0$, and T_1, T_2 are constant over the plate, having the values given at the edges.

Since the edges are supported so that the normal displacement w is zero round the boundary, it follows that w can be expanded by Fourier's theorem in the form

$$w = \sum\sum A_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \quad \text{.....................(4).}$$

Therefore

$$W = \tfrac{1}{2}T_1\int_0^b\int_0^a\left\{\sum\sum A_{mn}\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\right\}^2 dx\,dy$$

$$+\tfrac{1}{2}T_2\int_0^b\int_0^a\left\{\sum\sum A_{mn}\frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\right\}^2 dx\,dy$$

$$= \tfrac{1}{8}ab\pi^2\sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2 \quad\ldots\ldots\ldots\ldots(5).$$

In like manner, we find on integration

$$V = \tfrac{1}{8}ab\pi^4\beta\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}^2 \quad\ldots\ldots\ldots\ldots(6).$$

The plane form will therefore be stable, provided that, for all values of the constants A_{mn},

$$\pi^2\beta\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}^2 > \sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2,$$

or

$$\pi^2\beta > \frac{\sum\sum\left(T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}\right)A_{mn}^2}{\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}^2} \quad\ldots\ldots\ldots\ldots(7).$$

Suppose that the thrusts T_1, T_2 are given, then inequality (7) determines the limit to the value of β, and hence, the limiting thickness of the plate in order that it may just not buckle. To find this limit, we must choose the constants A_{mn} so that the right-hand side of (7) is a maximum. This will be the case when all but one of the constants are zero, and, if A_{mn} be this constant, the limiting value of β will be given by

$$\pi^2\beta = \frac{T_1\frac{m^2}{a^2}+T_2\frac{n^2}{b^2}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \quad\ldots\ldots\ldots\ldots(8),$$

the numbers m, n being so chosen as to make this expression for β the greatest possible.

The corresponding form of the displaced surface is given by

$$w = A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad\ldots\ldots\ldots\ldots(9).$$

w vanishes along the lines

$$x = a/m, \quad x = 2a/m, \quad x = 3a/m, \ \&c.,$$

and also along the lines

$$y = b/n, \quad y = 2b/n, \quad y = 3b/n, \ \&c.$$

Hence it readily follows that, if the thickness of the plate be less than that given by (8), the surface will buckle up, and the corrugations will divide it into mn rectangles, over which the displacement is alternately to one side and to the other.

We shall see hereafter that one of the numbers m, n must be unity, so that the corrugations can only divide the plate into a single row of rectangles. The methods of determining the number of corrugations are, however, better shown by first considering the following special cases.

4. Let $T_1 = T_2$. Put each of these $= T$.
Equation (8) becomes

$$\pi^2\beta = \frac{T}{\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}} \quad \text{.........................(10).}$$

β is greatest when m and n are least, that is, when $m = 1$, $n = 1$. Therefore the limit for β is given by

$$\pi^2\beta = \frac{T}{\dfrac{1}{a^2} + \dfrac{1}{b^2}}.$$

If the plate be too thin, its initial form on buckling will be found by putting $m = 1$, $n = 1$ in (8). We see that there will be a single corrugation, all points being displaced towards the same side.

5. Let $T_2 = 0$.
Equation (8) becomes

$$\pi^2\beta = \frac{T_1 \dfrac{m^2}{a^2}}{\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \quad \text{.......................(11).}$$

The right-hand side is greatest when $n = 1$, but the same is not necessarily true as regards m. The number (m) of corrugations parallel to the side a will depend on the length a. The greatest length a of a plate which will buckle into m corrugations will be that for which the limiting values of β, corresponding to m and $m+1$ corrugations, respectively, are equal. Therefore, putting $n=1$ in (11), we must have for this greatest length

$$\frac{\dfrac{m^2}{a^2}}{\left(\dfrac{m^2}{a^2}+\dfrac{1}{b^2}\right)^2} = \frac{\dfrac{(m+1)^2}{a^2}}{\left(\dfrac{(m+1)^2}{a^2}+\dfrac{1}{b^2}\right)^2}.$$

Put $a/b = k$, so that $a = kb$. The last equation gives

$$\frac{m^2}{\{m^2+k^2\}^2} = \frac{(m+1)^2}{\{(m+1)^2+k^2\}^2};$$

whence $\qquad m\{(m+1)^2+k^2\} = \pm(m+1)\{m^2+k^2\}.$

Since k^2 is positive, we must take the upper sign to the radical.

This gives $\qquad k^2 = m(m+1),$

whence $\qquad a = b\sqrt{\{m(m+1)\}}$ ....(12).

Similarly, the least length a for which the plate, if unstable, will buckle into m corrugations parallel to the side a, will be given by

$$a = b\sqrt{\{m(m-1)\}} \text{ ....(13).}$$

Hence there will be m corrugations parallel to the side a, provided that

a lies between the values $b\sqrt{\{m(m-1)\}}$ and $b\sqrt{\{m(m+1)\}}$.

Condition (12) may also be written

$$\frac{a}{m} : b = b : \frac{a}{m+1},$$

which shows that, whether the corrugations divide the plate into m or $m+1$ rectangles, the ratio of their length to their breadth will be the same in both cases, but the directions of the longer and shorter sides of the rectangles will be interchanged.

Hence it readily follows that the number of corrugations must always be such that the ratio of the longer to the shorter side of the rectangles differs as little as possible from unity, that is, their shape must be as nearly square as possible.

Hence it is evident that if the length of the plate become infinite, the corrugations will divide it exactly into squares.

This may also be shown as follows:—

Supposing the strip infinitely long, or $a = \infty$, let $\lambda = a/m$, so that λ is the length of a corrugation. Then (11) gives

$$\pi^2\beta = \frac{\dfrac{T_1}{\lambda^2}}{\left(\dfrac{1}{\lambda^2}+\dfrac{1}{b^2}\right)^2},$$

in which the right-hand side must be made a maximum by the variation of λ. By the ordinary methods we find that this is the case when $\lambda = b$.

6. As the collapse is of a different nature in the two cases above considered, we will now investigate the limits to the ratio of T_2 to T_1, in order that any form of the plate may buckle up in a series of corrugations. Let the length a be infinite, and let λ be the length of a corrugation. Putting $n = 1$, $a/m = \lambda$ in (8), the critical value of β is found from the relation

$$\pi^2\beta = \frac{T_1\dfrac{1}{\lambda^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{1}{\lambda^2}+\dfrac{1}{b^2}\right)^2} \quad \text{......................(14)},$$

by making β a maximum by the variation of λ.

Write $b/\lambda = \mu$, $T_2/T_1 = r$; then we must make the expression

$$\frac{\mu^2+r}{(\mu^2+1)^2}$$

a maximum by the variation of μ. Differentiating with respect to μ^2, the condition is

$$\frac{1}{(\mu^2+1)^2}-\frac{2(\mu^2+r)}{(\mu^2+1)^3}=0,$$

whence $$\mu^2+1 = 2(\mu^2+r),$$

or $$\mu^2 = 1-2r,$$

that is, $$b^2/\lambda^2 = 1-2T_2/T_1 \quad \text{......................(15)}.$$

In order that this may give a real value of λ, we must have

$$T_2 < \tfrac{1}{2}T_1 \quad\text{..............................(16)},$$

and if this is not the case, the value of β will be greatest when $\lambda = \infty$. If the inequality (16) is satisfied, equation (15) determines the length of the corrugations produced in the case of collapse. We notice that λ diminishes as the ratio of T_2 to T_1 diminishes. If T_2 becomes negative, so that the strip is acted on by lateral *tension* combined with the longitudinal thrust, the length λ of the wrinkles is less than b, and it diminishes as the lateral tension increases.

The last-mentioned property may be easily illustrated by wetting a sheet of paper in the middle, and then stretching it over two parallel rulers. The moisture causes the surface of the paper to expand and wrinkle, and if the rulers be pulled apart with increasing force, the wrinkles will become finer and closer.

We also see (as previously stated) that a rectangular plate cannot buckle up into a *network* of rectangles—*i.e.*, that m, n cannot both be different from unity. For this would require $T_2 < \tfrac{1}{2}T_1$ and $T_1 < \tfrac{1}{2}T_2$, which are incompatible.

7. If $T_2 < \tfrac{1}{2}T_1$, and we substitute for λ from (15) in (14), we find

$$\pi^2\beta = \frac{b^2}{4}\,\frac{T_1^2}{T_1 - T_2} \quad\text{........................(17)},$$

which determines the least thickness of an infinite strip of breadth b, supported at its edges in order that it may resist an end-thrust T_1, and a lateral thrust T_2.

In particular, if $T_2 = 0$, we must have

$$\pi^2\beta = \frac{b^2}{4}\,T_1 \quad\text{..............................(18)}.$$

If $T_2 > \tfrac{1}{2}T_1$, we must put $\lambda = \infty$ in (14), and we obtain

$$\pi^2\beta = b^2 T_2 \quad\text{..................................(19)},$$

which is independent of T_1, as it evidently should be, for the corresponding displaced surface is given by

$$w = A \sin\frac{\pi y}{b}.$$

Here w is independent of x; hence no work is done by the thrust T_1 during the displacement.

If the length of the strip be finite, the case will be different. The limit for β will always depend on the values both of T_1 and T_2.

We notice that, when $T_2 = \frac{1}{2}T_1$, the values of β given by (17) and (19) are equal, as they should be.

Comparing (19) with (18), we see that the strip is capable of resisting four times as much simple longitudinal thrust as of lateral thrust.

We also see that, even when the lateral thrust is the greatest which the strip can resist, it is still possible to subject it, in addition, to any amount of longitudinal thrust on its ends, not greater than double the lateral thrust, without diminishing the strength.

Applications to the Sides of a Ship.

8. The preceding results are applicable, without any modification whatever, to the case of an infinite plate supported on parallel ribs at distances b apart. This kind of structure is met with in the sides of ships, a fact which adds considerably to the practical interest of problems of this class.

We see that, in order to obtain the greatest strength, the ribs must be placed parallel with the direction of the greatest thrust, and that, to obtain the same strength, they may then be placed twice as far apart as if they were perpendicular to the thrust.

The strength of the plate will not then be increased or altered in any way by the addition also of a second set of ribs, perpendicular to the first, and at distances apart equal to the natural length of the corrugations. Thus, if there is no lateral thrust, a plate supported on parallel longitudinal ribs will not be strengthened by the addition of transverse ribs which divide it into squares. If the transverse ribs are at any other distance apart, however, the system will be strengthened. In this particular case, the most effective distance between the transverse ribs will be $b\sqrt{2}$, because [putting $m = 1$ in (12)] each rectangle formed by the framework will buckle with equal facility into one or two corrugations, and the length of these corrugations will then differ as much as possible from their natural length.

Putting $T_2 = 0$, $m = n = 1$, $a = b\sqrt{2}$ in (8), we find

$$\pi^2\beta = \tfrac{2}{9}b^2T_1,$$

and the same increase in strength could be obtained much more economically by diminishing the distance between the ribs from b to $b\sqrt{\frac{8}{9}}$, *i.e.*, $\frac{2}{3}\sqrt{2}\,.\,b$.

When, however, the structure is subject to thrusts of about equal intensity in all directions, the advantage of a network of ribs will be much greater. Moreover, the framework itself will be strengthened —a consideration not brought out in our present theoretical treatment, in which the framework is supposed perfectly rigid.

9. The *general* problem, to determine the number of corrugations produced in the buckling of a finite rectangular plate (sides a, b) under the influence of both thrusts (T_1, T_2, where $T_1 > 2T_2$), may now be considered. In the critical case when the plate will buckle with equal facility into m or $m+1$ rectangles in the direction of its length, we have

$$\frac{T_1\dfrac{m^2}{a^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{m^2}{a^2}+\dfrac{1}{b^2}\right)^2}=\frac{T_1\dfrac{(m+1)^2}{a^2}+T_2\dfrac{1}{b^2}}{\left(\dfrac{(m+1)^2}{a^2}+\dfrac{1}{b^2}\right)^2}.$$

Whence, on reduction, we find

$$\frac{T_1}{a^4}m^2(m+1)^2+\frac{T_2}{a^2b^2}\{2m(m+1)+1\}-\frac{T_1-2T_2}{b^4}=0\text{......}(20).$$

Solving as a quadratic in a^2/b^2, we find

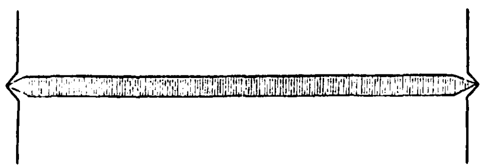
$$\frac{a^2}{b^2}=\frac{\{2m(m+1)+1\}T_2\pm\sqrt{\{4m^2(m+1)^2(T_1-T_2)^2+(2m+1)^2T_2^2\}}}{T_1-2T_2}\text{..................}(21).$$

In accordance with previous results, this will give a real value for the ratio of a to b, provided that $T_1 > 2T_2$, and that the radical is taken with the positive sign.

10. A few words on the mode of supporting the boundaries may not be out of place. In his *Theory of Sound*, Vol. I., § 225, Lord Rayleigh gives the following suggestion as a means of realizing this kind of support:—"We may consider the plate to be held in its place by friction against the walls of a cylinder circumscribed closely round it."

This method is strictly applicable to the problems of the present paper, where the normal reaction of the cylinder is supposed to produce thrusts in the plane of the plate. But it is *not* applicable to the problem considered by Lord Rayleigh, viz., the determination of the frequencies of vibration. Obviously, the effect of the thrusts will be generally to lower the pitch of all the vibrations, and if the thrusts be increased till the plane form is in critical equilibrium, one of the frequencies will then vanish.

The annexed diagram shows a better means of support. The boundaries, being brought to sharp edges, are made to fit in grooves of a rather wider angle.



The sides of a ship are rivetted to the ribs, and this implies more constraint than that afforded by mere support. This constraint more closely resembles "clamping."

Application of Variational Method.—Clamped Circular Plate.

11. In applying the energy method to the rectangular plate, we supposed w expanded by Fourier's series, and the success of our method depended on the fact that the expression for the total energy contained no products of the coefficients. In other problems it may not be easy to discover the form of the functions in which we must expand; we therefore use another method, more analogous to that adopted by Greenhill in his paper on the "Stability of Shafting."* If equilibrium in the plane form is *critical*, there must be another form of equilibrium indefinitely near to it. To find this, we apply the method of variation of energy, and the boundary conditions then lead to a relation which must be satisfied in the critical case. This equation determines the criteria of stability.

12. The method of variation may be illustrated by its application to the critical equilibrium of a circular plate acted on by a normal

* *Proc. Inst. Mech. Eng.*, 1883.

thrust T (per unit length) round the circumference. The slightly displaced form being also in equilibrium, the displacement is found by making

$$\delta V - \delta W = 0$$

for all variations of w.

We take as our example the simplest case, in which the boundary of the plate is clamped so as to allow of no variation of position or direction. Let the radius of the plate be a.

Putting

$$T_1 = T_2 = T \quad \text{and} \quad M = 0$$

in (1), we have

$$W = \tfrac{1}{2} T \iint \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} dx\,dy\,;$$

therefore

$$\delta W = T \iint \left\{ \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} \right\} dx\,dy$$

$$= T \int \delta w \frac{\partial w}{\partial r}\, ds - T \iint \delta w\, \nabla^2 w\, dx\,dy.$$

Also,

$$\delta V = \beta \iint \delta w\, \nabla^4 w\, dx\,dy + \text{(line integrals taken round the boundary)}.$$

Since the plate is clamped, we have, round the boundary,

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \text{.......................... (22)};$$

thus, the line integrals all vanish, and the differential equation for w is

$$\beta \nabla^4 w + T \nabla^2 w = 0.$$

Writing $\kappa^2 = T/\beta$, this becomes

$$\nabla^2 (\nabla^2 + \kappa^2)\, w = 0 \quad \text{.......................... (23)}.$$

If this be transferred to polar coordinates, a solution which is not infinite at the centre is

$$w = \cos n\theta \left\{ A r^n + B J_n(\kappa r) \right\} \quad \text{................. (24)}.$$

The boundary conditions (22) give, when $r = a$,

$$\left.\begin{aligned} Aa^n + BJ_n(\kappa a) &= 0 \\ nAa^{n-1} + \kappa BJ'_n(\kappa a) &= 0 \end{aligned}\right\} \quad\text{.....................(25).}$$

Eliminating the ratio $A : B$, we have

$$\begin{aligned} 0 &= nJ_n(\kappa a) - \kappa a J'_n(\kappa a) \\ &= \kappa a J_{n+1}(\kappa a), \end{aligned}$$

by the well-known relations between Bessel's Functions. Therefore, for critical equilibrium corresponding to the displacement of the form (24), we must have

$$J_{n+1}(\kappa a) = 0 \quad\text{.............................(26),}$$

and we must choose n so that the value of κ given by this equation is the least possible.

If the plate be free at the centre, we may take

$$n = 0,$$

and the least solution of the equation

$$J_1(\kappa a) = 0$$

is known to be

$$\kappa a = 3{\cdot}832\ldots.$$

Therefore $$T = 14{\cdot}684 \ldots \times \beta/a^2.$$

If the plate be supported at the centre, the solution (24) corresponding to $$n = 0$$

cannot make w vanish, both when

$$r = 0,$$

and when $$r = a.$$

We must therefore take $$n = 1,$$

and we have $$J_2(\kappa a) = 0,$$

whence $$\kappa a = 5{\cdot}135\ldots,$$

and $$T = 26{\cdot}368 \ldots \times \beta/a^2.$$

If the boundary be supported or free, the solution of the differential equation for w is still of the form given by (24), but the boundary conditions will lead to a far more complicated equation for κ.

A very good illustration of the buckling of a circular plate is frequently afforded by the lid of a circular canister, in which the thrust is due to the tension of the rim. The "dint" in such a lid can be readily pushed from one side to the other, but it is impossible to keep the surface flat, as that position is unstable.

The same principle is also illustrated in the "castanets," in which a "clicking" sound is produced by pushing a disc of metal from one side to the other of the unstable plane form.

13. In all the cases discussed in this paper, the stresses in the surface are proportional to β; and, therefore, to the cube of the thickness of the plate. Since these stresses are distributed over the thickness of the plate, the strains they produce are proportional to the square of the thickness. If, therefore, the plate be thin, these strains will be small, and there will be no rupture of the material accompanying the buckling. This accords with the general results obtained in my paper "On the Stability of Elastic Systems."*

In a future paper, I hope to deal with further applications of the variational method, with special reference to the stability of a rectangular plate or strip in certain cases when the shear M does not vanish, and when the boundary conditions are different to those assumed in the present communication.

On the Application to Matrices of any Order of the Quaternion Symbols S and V. By Henry Taber, Docent in Clark University, Worcester, Mass. U.S.A.

[*Read Dec.* 11*th*, 1890.]

1. *Properties of the Symbols S and V.*

The conception of scalar and vector parts of a quaternion, or matrix of the second order, may be extended to matrices of any order.† Regarded as a matrix, the scalar of any quaternion is one half the sum of its latent roots; following this analogy, I shall define the scalar of any matrix m of order ω as the ω^{th} part of the sum of

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* *Camb. Phil. Proc.*, Vol. VI., p. 204.

† See paper by author on the "Theory of Matrices," *Amer. Journ. Math.*, Vol. XII.