SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: THE LATTICE-POINTS OF A RIGHT-ANGLED TRIANGLE

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received March 30th, 1920.—Read April 22nd, 1920.]

1. *Introduction.*

1.1. The problem considered in this paper may be stated as follows.

Suppose that ω and ω' are two positive numbers whose ratio $\theta = \omega/\omega'$ is irrational; and denote by Δ the triangle whose sides are the coordinate axes and the line

$$\omega x + \omega' y = \eta > 0, \tag{1.11}$$

and by $N(\eta)$ the number of lattice-points* which lie inside Δ. *How accurate an approximation can we find for $N(\eta)$ when η is large? And how does the accuracy of the approximation depend upon the arithmetic character of θ?* We call this problem *Problem A*.

Such "lattice-point" problems are, in general, very difficult. It is enough to recall the two most famous of them, the *problem of the circle* (the problem of Gauss and Sierpinski), and the *problem of the rectangular hyperbola* (Dirichlet's divisor problem), both of which have been the subject of numerous researches during the last ten years. The particular problem which we consider here has not, so far as we know, been stated quite in this form before. It is however easily brought into connection with another problem which has attracted a certain amount of attention, and which has been considered, from varying points of view, by Lerch,† by Weyl,‡ and by ourselves.§ This problem, which we shall call

* A lattice-point (*Gitterpunkt*) is a point whose coordinates x and y are both integral.

† M. Lerch, *l'Intermédiaire des Mathématiciens*, Vol. 11 (1904), pp. 145–146 (Question 1547).

‡ H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins", *Math. Annalen*, Vol. 77 (1916), pp. 313–352.

§ G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Proceedings of the fifth international congress of mathematicians*, Cambridge, 1912, Vol 1, pp. 223–229.

Problem B, is as follows. Suppose that, as usual, $[x]$ denotes the integral part of x, and that

$$(1.12) \qquad \{x\} = x-[x]-\tfrac{1}{2}.$$

Then *what is the most that can be said as to the order of magnitude of*

$$(1.13) \qquad s(\theta, n) = \sum_{\nu=1}^{n} \{\nu\theta\}$$

when n is large?

1.2. We begin, in § 2, by proving the formula which establishes the connection between Problems A and B, and shows that the first problem is a generalised and more symmetrical form of the second. We prove in fact that

$$(1.21) \qquad N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + S(\eta),$$

where $S(\eta)$ is a sum very similar to the sum 1.13.

It is trivial that

$$(1.211) \qquad N(\eta) = \frac{\eta^2}{2\omega\omega'} + O(\eta),$$

the area of the triangle, together with an error of the order of the perimeter. The second and third terms of (1.21) occur naturally when we consider, instead of Δ, the similar and similarly situated triangle whose vertex is at (1, 1) instead of the origin; for the area of this triangle is

$$\frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{1}{2}.$$

But no closer approximation than (1.211) is in any way trivial; and, when θ is rational, $S(\eta)$ is effectively of order η, so that a universal formula, professing to be more precise than (1.211), would necessarily be false.

In § 3 we deduce transformation formulæ for N and S, which are generalisations of a formula given without proof by Lerch, and which enable us to study these sums by means of the expression of θ as a simple continued fraction. In § 4 we prove (a) that

$$(1.22) \qquad S(\eta) = o(\eta)$$

for any irrational θ, and (b) that (1.22) is the most that is true for every such irrational. Incidentally we obtain the corresponding results concerning Problem B: the first of them at any rate is in this case familiar.

In § 5 we consider more closely cases in which the rate of increase of the quotients in the continued fraction is comparatively slow, and in particular the case in which they are bounded; and we prove that in this case

$$S(\eta) = O(\log \eta), \tag{1.23}$$

and that this result too is a best possible result of its kind. There are naturally analogous results for Problem B; that corresponding to (1.23) was stated as a new theorem in our communication to the Cambridge congress, but had, as was pointed out to us by Prof. Landau, been given already by Lerch.

Up to this point our argument is entirely elementary, and both methods and results are of a kind to be found in our previous papers on Diophantine approximation or in those of other writers. We have therefore aimed at the maximum of compression and have omitted a good deal of elementary algebraical calculation. The concluding section (§ 6) is more novel. In it we prove that *if θ is algebraic then*

$$S(\eta) = O(\eta^{a}), \tag{1.24}$$

where $a < 1$. This result is unlike any which we have been able to prove before, and is obtained by entirely different methods, based on the properties of the analytic function

$$\zeta_2(s, a, \omega, \omega') = \sum_{m,\, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s}. \tag{1.25}$$

This function will be recognised as a degenerate form of the "Double Zeta-function" introduced into analysis by Dr. Barnes.*

2. *Reduction of Problem* A.

2.1. We write

$$\frac{\eta}{\omega} = \left[\frac{\eta}{\omega}\right] + f, \quad \frac{\eta}{\omega'} = \left[\frac{\eta}{\omega'}\right] + f', \tag{2.11}$$

where
$$0 \leqslant f < 1, \quad 0 \leqslant f' < 1.$$

* E. W. Barnes, "A memoir on the Double-Gamma-function", *Phil. Trans. Roy. Soc.*, (A), Vol. 196 (1901), pp. 265–387; see in particular pp. 314–349. For a study of some of the properties of the degenerate function (for which the ratio ω/ω' is real) see G. H. Hardy, "On double Fourier series, and in particular those which represent the double Zeta-function with real and incommensurable parameters", *Quarterly Journal*, Vol. 37 (1906), pp. 53–79.

Suppose first that there is no lattice-point on the line (1 . 11), or AB of the figure. Then the number of lattice-points inside OAB is

(2 . 12) $$N(\eta) = \sum_{\mu=1}^{\eta/\omega}\left[\frac{\eta-\mu\omega}{\omega'}\right] = \left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right]+\sum_{\mu=1}^{\eta/\omega}[f'-\mu\theta],$$

Now $[-x] = -[x]-1+\epsilon_x$, where ϵ_x is 1 or 0 according as x is or is not an integer; and $\mu\theta-f'$ cannot be an integer, since then $\eta-\mu\omega$ would be an integral multiple of ω' and there would be a lattice-point on AB. Thus

(2 . 13) $$[f'-\mu\theta] = -[\mu\theta-f']-1 = -(\mu\theta-f')+\{\mu\theta-f'\}-\tfrac{1}{2}.$$

Substituting into (2 . 12), and using (2 . 11), we obtain, after a little reduction

(2 . 14) $$N(\eta) = \frac{\eta^2}{2\omega\omega'}-\frac{\eta}{2\omega}-\frac{\eta}{2\omega'}+\phi+S(\eta),$$

where

(2 . 141) $$\phi = \tfrac{1}{2}f+\tfrac{1}{2}\theta f(1-f)$$

and

(2 . 142) $$S(\eta) = \sum_{\mu=1}^{\eta/\omega}\{\mu\theta-f'\}.$$

Since ϕ is bounded, the problem is reduced, substantially, to the discussion of $S(\eta)$.

The preceding argument requires a trifling modification when there is a lattice-point on AB; there cannot be more than one, since θ is irrational. In this case the sum (2 . 12) gives $N(\eta)+1$ instead of $N(\eta)$. There is one value of μ for which $\mu\theta-f'$ is integral, and for this μ the $-\frac{1}{2}$ in (2 . 13) is changed into $\frac{1}{2}$. The net result is to leave the final formulæ unchanged.

3. *The Transformation Formulæ.*

3 . 1. In order to obtain a formula for the transformation of $N(\eta)$ or of $S(\eta)$, we employ the familiar device of adding together the number of lattice-points of the triangles OAB, $O'A'B'$ of the figure.

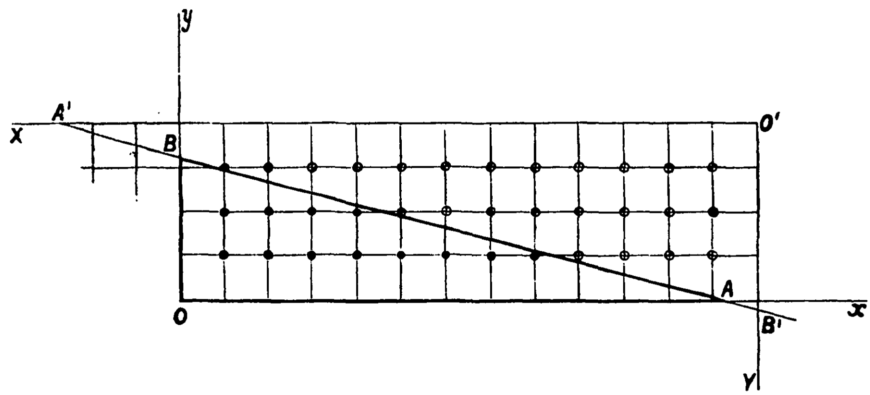
If we take new axes $O'X$, $O'Y$, as shown in the figure, it is plain that

$$x+Y = \left[\frac{\eta}{\omega}\right]+1, \qquad X+y = \left[\frac{\eta}{\omega'}\right]+1;$$

and the equation of AB, referred to the new axes, is

(3.11) $$\omega' X+\omega Y = \eta+\omega(1-f)+\omega'(1-f') = H,$$

say. Repeating the arguments of § 2, we find, for the number $N'(H)$ of



lattice-points of $O'A'B'$,

(3.12) $$N'(H) = \frac{H^2}{2\omega\omega'}-\frac{H}{2\omega}-\frac{H}{2\omega'}+\Phi+S'(H),$$

where

(3.121) $$\Phi = \tfrac{1}{2}F'+\frac{F'(1-F')}{2\theta}$$

and

(3.122) $$S'(H) = \sum_{\nu=1}^{H/\omega'}\left\{\frac{\nu}{\theta}-F\right\},$$

F and F' being defined by

(3.123) $$\frac{H}{\omega} = \left[\frac{H}{\omega}\right]+F,\quad \frac{H}{\omega'} = \left[\frac{H}{\omega'}\right]+F',\quad 0\leqslant F<1,\quad 0\leqslant F'<1.$$

3.2. We suppose now that $\omega<\omega'$, $\theta<1$. A glance at the figure shows that

$$\left[\frac{H}{\omega'}\right] = \left[\frac{\eta}{\omega'}\right]+1.$$

Substituting for H in terms of η, from (3.11), we find at once that

(3.21) $$F' = \theta(1-f).$$

The same argument shows that

(3.22) $$F = \frac{1-f'}{\theta}-p,$$

where p is an integer; it happens that the value of p is not material to the argument.

It is also clear from the figure that

$$(3.23) \qquad N(\eta)+N'(H) = \left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right]-\epsilon,$$

where ϵ is zero unless there is a lattice point on AB, and then unity. Substituting for $N(\eta)$ and $N'(H)$ from (2.14) and (3.12), using (2.11), (3.11), and (3.21), and reducing, we obtain, finally,

$$(3.24) \qquad S+S'+\epsilon = -\tfrac{1}{2}+\tfrac{1}{2}(f+f')-\tfrac{1}{2}\theta f(1-f)+\frac{f'(1-f')}{2\theta}.$$

3.3. It is important, in view of Problem B, to show that this formula includes a formula given by Lerch.* Suppose then in particular that $\omega' = 1$, $\omega = \theta < 1$, and write

$$(3.31) \qquad s = \sum_{1}^{n}\{\mu\theta\}, \quad s' = \sum_{1}^{m}\left\{\frac{\nu}{\theta}\right\},$$

where m is the integral part of $n\theta$.

Starting with an arbitrary positive integral n, we write $n\theta = M+\delta$, where M is an integer and $0 < \delta < 1$, and take

$$\eta = M+1 = n\theta+1-\delta.$$

Then

$$f' = 0, \quad F \equiv \frac{1}{\theta} \pmod 1,$$

by (2.11) and (3.22); and there is no lattice point on AB, so that $\epsilon = 0$.

Suppose now that q is a positive integer and

$$q < \frac{1-\delta}{\theta} < q+1.†$$

Then

$$\frac{\eta}{\theta} = n+\frac{1-\delta}{f} = n+q+f, \quad f = \frac{1-\delta}{\theta}-q.$$

Also $H = \eta+1+\theta(1-f)$ lies between $M+2$ and $M+3$. Hence

$$(3.32) \qquad S' = \sum_{\nu=1}^{M+2}\left\{\frac{\nu-1}{\theta}\right\} = -\tfrac{1}{2}+\left\{\frac{M+1}{\theta}\right\}+s';$$

* M. Lerch, *loc. cit.*

† It is easy to see that $(1-\delta)/\theta$ cannot be integral.

and

(3.321) $$\left\{\frac{M+1}{\theta}\right\} = \left\{\frac{\eta}{\theta}\right\} = \left\{\frac{1-\delta}{\theta}\right\} = \frac{1-\delta}{\theta} - q - \tfrac{1}{2}.$$

Also

(3.33) $$S = \sum_{\mu=1}^{[\eta/\theta]} \{\mu\theta\} = \sum_{1}^{n+q} \{\mu\theta\} = s + \sum_{r=1}^{q} \{(n+r)\theta\} = s + S_0,$$

say. And $(n+1)\theta, \ldots, (n+q)\theta$ have all the integral part M, since $q\theta < 1-\delta < (q+1)\theta$. Hence

(3.34) $$S_0 = \sum_{r=1}^{q} (n\theta + r\theta - M - \tfrac{1}{2}) = \sum_{r=1}^{q} (r\theta + \delta - \tfrac{1}{2}) = \tfrac{1}{2}q(q+1)\theta + q(\delta - \tfrac{1}{2}).$$

Substituting from (3.32), (3.321), (3.33), and (3.34) into (3.24), and reducing, it will be found that

(3.35) $$s + s' = \tfrac{1}{2}\delta - \frac{\delta(1-\delta)}{2\theta},$$

which is the formula of Lerch.

4. *Results concerning an arbitrary irrational* θ.

4.1. THEOREM **A1.**—*If* $\theta = \omega/\omega'$ *is irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + o(\eta).$$

We may clearly suppose that $\theta < 1$. Suppose that

(4.11) $$\theta = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

(4.12) $$\theta = \frac{1}{a_1+\theta_1}, \quad \theta_1 = \frac{1}{a_2+\theta_2}, \quad \ldots .$$

We have, from (3.24),

(4.13) $$S + S' = O(1/\theta),$$

the constant of the O being independent of both η and θ.

We write $\eta = \omega\xi$, so that

$$\frac{H}{\omega'} = \xi\theta + \theta(1-f) + 1 - f' = \xi\theta + O(1),$$

and we write f_1 and μ_1 in S' instead of F' and ν. Then

$$S' = \sum_{\mu_1=1}^{H/\omega'} \left\{\frac{\mu_1}{\theta} - f_1\right\} = O(1) + \sum_{\mu_1=1}^{\xi\theta} \{\mu_1\theta_1 - f_1\} = O(1) + S_1,$$

say; so that

(4.14) $$S = O(1/\theta) - S_1.$$

Similarly, we have

$$S_1 = O(1/\theta_1) - S_2, \quad S_2 = O(1/\theta_2) - S_3, \quad \ldots,$$

where $S_2, S_3, \ldots$ are sums of the types

$$S_2 = \sum_{\mu_2=1}^{\xi\theta\theta_1} \{\mu_2\theta_2 - f_2\}, \quad S_3 = \sum_{\mu_3=1}^{\xi\theta\theta_1\theta_2} \{\mu_3\theta_3 - f_3\}, \quad \ldots,$$

so that $$S_2 = O(\xi\theta\theta_1), \quad S_3 = O(\xi\theta\theta_1\theta_2), \quad \ldots.$$

It follows that

(4.151) $$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_{\nu-1}}\right) + O(\xi\theta\theta_1 \ldots \theta_{\nu-1})$$

and

(4.152) $$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_\nu}\right) + O(\xi\theta\theta_1 \ldots \theta_{\nu-1}\theta_\nu).$$

We shall require both of these equations.

4.2. We choose ν so that

(4.21) $$\xi\theta\theta_1 \ldots \theta_{\nu-1}\theta_\nu < 1 \leqslant \xi\theta\theta_1 \ldots \theta_{\nu-1}.$$

It may be verified at once* that $\theta_s\theta_{s+1} < \frac{1}{2}$ for every s. Hence on the one hand

(4.22) $$\theta\theta_1 \ldots \theta_{\nu-1} = O(2^{-\frac{1}{2}\nu}),$$

and on the other

(4.23) $$\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}} = O\left(\nu \operatorname{Max} \frac{1}{\theta_s}\right) = O\left(\frac{\nu 2^{-\frac{1}{2}\nu}}{\theta\theta_1 \ldots \theta_{\nu-1}}\right) = O(\nu 2^{-\frac{1}{2}\nu}\xi).$$

From (4.151), (4.22), and (4.23), we obtain

(4.24) $$S = O(\nu 2^{-\frac{1}{2}\nu}\xi) + O(2^{-\frac{1}{2}\nu}\xi) = o(\xi),$$

since ν tends to infinity with ξ; and the theorem follows from (2.14) and (4.24).

* See our paper "Some problems of Diophantine approximation (II)" [*Acta Mathematica*, Vol. 37 (1914), pp. 193–238 (p. 212)].

4 . 3. To Theorem **A1** corresponds, for Problem B, the well known theorem:

THEOREM **B1.**—*If θ is irrational, then*

$$s(\theta, n) = \sum_{\mu=1}^{n} \{\mu\theta\} = o(n).$$

The proof of this theorem is included in that of Theorem **A1.** We have only to take $\eta = k\omega'$, where k is an integer, so that $f' = 0$, and to write $\xi = \eta/\omega = k/\theta$, $n = [\xi]$.

4 . 4. THEOREM **A2.**—*If $\psi(\eta)$ is any function of η which tends steadily to infinity with η, then there is an irrational θ such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > \frac{\eta}{\psi(\eta)}, \qquad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -\frac{\eta}{\psi(\eta)}$$

is satisfied for a sequence of indefinitely increasing values of η.

Thus Theorem **A1** is the best possible theorem of its kind.

Making the transformations indicated in 4 . 3, we see at once that it is enough to prove

THEOREM **B2.**—*If $\psi(n)$ is any function of n which tends steadily to infinity with n, then there is an irrational θ such that each of the inequalities*

$$s(\theta, n) > \frac{n}{\psi(n)}, \qquad s(\theta, n) < -\frac{n}{\psi(n)}$$

is satisfied for an infinity of values of n.

To prove this we use Lerch's formula (3 . 35). Writing

(4 . 41) $$n_1 = [n\theta] = n\theta - \delta,\ n_2 = n_1\theta_1 - \delta_1,\ \ldots,\ n_{r+1} = n_r\theta_r - \delta_r,$$

(4 . 42) $$\phi_s = \tfrac{1}{2}\delta_s - \frac{\delta_s(1-\delta_s)}{2\theta_s},$$

we have

(4 . 43) $$s(\theta, n) = \phi_0 - s\left(\frac{1}{\theta}, n_1\right) = \phi_0 - s(\theta_1, n_1) = \phi_0 - \phi_1 + s(\theta_2, n_2)$$

$$= \ldots = \phi_0 - \phi_1 + \ldots + (-1)^r\phi_r + s(\theta_{r+1}, n_{r+1}).$$

We suppose a_{r+1} even, and exceedingly large in comparison with the preceding quotients $a_1, a_2, \ldots, a_r$, and take $n_r = \tfrac{1}{2}a_{r+1}$. Then $n_{r+1} = 0$ and

δ_r is practically $\frac{1}{2}$, so that $\frac{1}{2}\delta_r(1-\delta_r)$ is certainly greater than $\frac{1}{9}$. Having fixed n_r, we can determine $n_{r-1}, n_{r-2}, \ldots, n_1, n$ from the equations (4.41); and

$$n \leqslant \frac{2n_1}{\theta} \leqslant \frac{2^2 n_2}{\theta\theta_1} \cdots \leqslant \frac{2^r n_r}{\theta\theta_1 \ldots \theta_{r-1}} = \frac{2^{r-1} a_{r+1}}{\theta\theta_1 \ldots \theta_{r-1}}.$$

It is then plain that, if a_{r+1} is sufficiently large in comparison with the preceding partial quotients, $s(\theta, n)$ will have the sign of $(-1)^r$, and

$$(4.41) \qquad |s(\theta, n)| > \tfrac{1}{2}|\phi_r| > \frac{1}{20\theta_r} > \frac{a_{r+1}}{20} > \frac{n}{\psi(n)}.$$

And, by choosing a θ for which sufficiently violent increments in the order of magnitude of the quotients occur at an infinity of stages in the continued fraction, we can secure the truth of (4.41) for an infinity of values of n.

5. *Results concerning special classes of irrationals.*

5.1. **Theorem A3.**—*If the quotients a_n in the continued fraction for $\theta = \omega/\omega'$ are bounded, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\log \eta).$$

Theorem B3.—*Under the same condition,*

$$s(\theta, n) = O(\log n).$$

To prove Theorem **A3**, we return to the analysis of 4.1 and 4.2, but use (4.152) instead of (4.151). In this case we have plainly

$$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_\nu}\right) = O(\nu).$$

Since $$2^{\frac{1}{2}\nu} = O\left(\frac{1}{\theta\theta_1 \ldots \theta_{\nu-1}}\right) = O\left(\frac{1}{\theta\theta_1 \ldots \theta_\nu}\right) = O(\xi),$$

we have $\nu = O(\log \xi) = O(\log \eta)$; and the theorem is proved. Theorem **B3** follows *a fortiori*: this is the theorem which, as we explained in 1.2, was claimed as a new theorem in our communication to the Cambridge congress, but is really due to Lerch.

It will easily be verified that, if we assume

$$a_n = O(n^\rho) \quad (\rho > 0),$$

we obtain an error term of the order

$$S = O\{(\log \eta)^{\rho+1}\};$$

if we assume $a_n = O(e^{\rho n})$, where ρ lies below a certain limit, we obtain

$$S = O(\eta^{\sigma}) \quad (\sigma < 1).*$$

As so little is known concerning the order of magnitude of the quotients in the continued fractions which express irrationals of particular types, it is hardly worth while to go into further detail.

5.2. THEOREM **A4**. — *There are values of $\theta = \omega/\omega'$, with bounded quotients, such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > K \log \eta, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -K \log \eta,$$

where K is a positive constant, is satisfied for a sequence of indefinitely increasing values of η.

THEOREM **B4**.—*There are values of θ, with bounded quotients, such that each of the inequalities*

$$s(\theta, n) > K \log n, \quad s(\theta, n) < -K \log n$$

is satisfied for an infinity of values of n.

Thus Theorems **A3** and **B3** are also best possible theorems of their kind. To prove this, it is plainly enough to prove Theorem **B4**: and this we shall do by considering the simplest irrational of all, viz.

$$\theta = \frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots.$$

We write

$$\Theta = \frac{1}{\theta} = \frac{\sqrt{5}+1}{2},$$

and take the convergents to θ to be

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{1}, \quad \frac{p_2}{q_2} = \frac{1}{2}, \quad \ldots.$$

Then it is easily verified that

$$q_s = \frac{1}{\sqrt{5}}(\Theta^{s+1} + (-1)^s \theta^{s+1}), \qquad p_s = q_{s-1}.$$

---

* Compare p. 214 of our memoir in the *Acta Mathematica* referred to above (p. 22).

5.3. We first take $n = q_s$ in the formula (3.31). We find without difficulty that

$$[q_s\theta] = q_{s-1}, \quad \delta = q_s\theta - [q_s\theta] = \theta^{s+1},$$

if s is even, and $\quad [q_s\theta] = q_{s-1} - 1, \quad \delta = 1 - \theta^{s+1},$

if s is odd; and that in either case

$$\sigma_s = \sum_{\nu=1}^{q_s} \{\nu\theta\} \tag{5.31}$$

satisfies the equation

$$\sigma_s + \sigma_{s-1} = \tfrac{1}{2}\left(\theta^{2s+1} + (-1)^{s+1}\theta^{s+2}\right). \tag{5.32}$$

Using this recurrence equation to express σ_s in terms of

$$\sigma_0 = \{\theta\} = \tfrac{1}{2}\sqrt{5} - 1,$$

we find, after reduction, that

$$\sigma_s = \frac{\theta^{2s+2}}{2\sqrt{5}} - \tfrac{1}{2}(-1)^{s+1}\theta^{s+1} + (-1)^{s+1}\frac{\theta}{\sqrt{5}}. \tag{5.33}$$

Suppose now that

$$s(\theta, n) = \sum_{1}^{n} \{\nu\theta\} \quad (q_s \leqslant n < q_{s+1}). \tag{5.34}$$

We can express n in one and only one way in the form

$$n = q_s + q_{s_1} + q_{s_2} + \dots + q_{s_k} = q_s + Q_1,$$

where $s, s_1, s_2, \dots$ are descending integers differing by at least 2: and

$$s(\theta, n) = \sigma_s + \sum_{\mu=1}^{Q_1} \{(q_s + \mu)\theta\}.$$

Now $q_s\theta$ differs from an integer by less than does any $\mu\theta$. Hence

$$[(q_s + \mu)\theta] = q_{s-1} + [\mu\theta]$$

and $\quad \{(q_s + \mu)\theta\} = q_s\theta - q_{s-1} + \mu\theta - [\mu\theta] - \tfrac{1}{2} = (-1)\,\theta^{s+1} + \{\mu\theta$

$$s(\theta, n) = \sigma_s + (-1)^s\theta^{s+1}Q_1 + s_{Q_1}.$$

We now write

$$Q_1 = q_{s_1} + q_{s_2} + \dots + q_{s_k} = q_{s_1} + Q_2, \qquad Q_2 = q_{s_2} + Q_3.$$

and so on, and repeat the argument. We thus obtain

$$\begin{aligned} s(\theta, n) = {} & \sigma_s + \sigma_{s_1} + \sigma_{s_2} + \dots + \sigma_{s_k} \\ & + (-1)^s\theta^{s+1}Q_1 + (-1)^{s_1}\theta^{s_1+1}Q_2 + \dots + (-1)^{s_{k-1}}\theta^{s_{k-1}+1}Q_{s_k}. \end{aligned} \tag{5.35}$$

5 . 4. If in (5 . 35) we substitute the values of the σ's given by (5 . 33), the first two terms of (5 . 33) will plainly give a contribution bounded for all values of s, so that

$$(5.41)\quad \sigma_s+\sigma_{s_1}+\ldots+\sigma_{s_k} = -\frac{\theta}{\sqrt{5}}\left((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\right)+O(1).$$

Again

$$(5.42)\qquad Q_1 = \sum_{r=1}^{k} q_{s_r} = \frac{1}{\sqrt{5}}\sum_{r=1}^{k}\left(\Theta^{s_r+1}+(-1)^{s_r}\theta^{s_r+1}\right),$$

and the sum of the second terms is numerically less than k, and *a fortiori* than s. The sum of the contributions of all such terms to (5 . 35) is therefore less in absolute value than

$$s\theta^{s+1}+s_1\theta^{s_1+1}+\ldots = O(1).$$

These terms, then, may be disregarded. Making this simplification, and substituting from (5 . 41) and (5 . 42) into (5 . 35), we obtain, finally,

$$(5.43)\quad s(\theta, n) = O(1)-\frac{\theta}{\sqrt{5}}\left((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\right)$$
$$+\frac{(-1)^s}{\sqrt{5}}(\theta^{s-s_1}+\theta^{s-s_2}+\ldots+\theta^{s-s_k})$$
$$+\frac{(-1)^{s_1}}{\sqrt{5}}(\theta^{s_1-s_2}+\theta^{s_1-s_3}+\ldots+\theta^{s_1-s_k})$$
$$+\ldots+\frac{(-1)^{s_{k-1}}}{\sqrt{5}}\theta^{s_{k-1}-s_k}.$$

5 . 5. This formula enables us to study the behaviour of $s(\theta, n)$ for different forms of n, and in particular to prove our theorem. Let us take, for example,

$$s = 4k+4,\quad s_1 = 4k,\quad s_2 = 4k-4,\quad \ldots,\quad s_k = 4.$$

Then the right-hand side of (5 . 43) becomes

$$-\frac{s\theta}{4\sqrt{5}}+\frac{1}{\sqrt{5}}\left(\frac{\theta^4-\theta^s+\theta^4-\theta^{s-4}+\ldots+\theta^4-\theta^8}{1-\theta^4}\right)+O(1) = Cs+O(1),$$

where
$$C = \frac{1}{4\sqrt{5}}\left(\frac{\theta^4}{1-\theta^4}-\theta\right) = -\frac{1}{20} \neq 0:$$

and $s(\theta, n)$ is negative and greater than a constant multiple of s. Similarly, if we were to take

$$s = 4k+3,\ s_1 = 4k-1,\ \ldots,\ s_k = 3,$$

we should find $s(\theta, n)$ to be positive and greater than a constant multiple of s. Since s is greater than a constant multiple of $\log n$, this completes the proof of Theorems **A4** and **B4**.

5.6. We should perhaps, before passing to more transcendental investigations, add a word concerning the case, so far excluded, of a *rational* θ. It is easy to see that, when θ is rational, no such results as we have proved in the irrational case are true: $s(\theta, n)$ is effectively of order n, and the oscillatory part of $N(\eta)$ of order η. Thus, to take a simple case, the series $\Sigma \{\frac{2}{3}\mu\}$ is

$$\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\dots,$$

and

$$s(\tfrac{2}{3}, n) \sim -\tfrac{1}{6}n.$$

In general, for a fixed rational $\theta = p/q$, we have $s(\theta, n) \sim A_q n$, where $A_q \to 0$ when $q \to \infty$.

6. *Transcendental methods: results true for all algebraical values of* θ.

6.1. The substance of our concluding section lies somewhat deeper. Our goal is to prove

THEOREM **A5.**—*If* $\theta = \omega/\omega'$ *is an algebraic irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^a),$$

where $a < 1$.

THEOREM **B5.**—*Under the same conditions*

$$s(\theta, n) = O(n^a) \quad (a < 1).$$

We require some preliminary lemmas concerning the function

$$(6.11) \qquad \zeta_2(s, a, \omega, \omega') = \sum_{m, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s},$$

where a, ω, and ω' are positive, and $s = \sigma + it$. This function is a degenerate form of the double Zeta-function of Dr. E. W. Barnes. Barnes considers only the case in which (as in the theory of elliptic functions) the ratio $\theta = \omega/\omega'$ is *complex*. The series (6.11) defines the function in the first instance for $\sigma > 2$.

6.21. LEMMA *a.*—*The function* $\zeta_2(s, a, \omega, \omega')$ *is an analytic function of* s, *regular all over the plane except for simple poles at the points* $s = 2$

and $s=1$, *where it behaves like*

$$\frac{1}{\omega\omega'}\,\frac{1}{s-2},\qquad \frac{\omega+\omega'-2a}{2\omega\omega'}\,\frac{1}{s-1}$$

respectively.

This is proved by Barnes when θ is complex, and his proof, depending on the formula

$$(6.211)\qquad \zeta_2(s,a,\omega,\omega')=\frac{i\Gamma(1-s)}{2\pi}\int\frac{e^{-au}(-u)^{s-1}}{(1-e^{-\omega u})(1-e^{-\omega' u})}\,du,$$

is equally applicable in the case considered here. We should observe that $(-u)^{s-1}=e^{(s-1)\log(-u)}$, where $\log(-u)$ has its principal value, that the contour of integration is the same as in the well-known Riemann-Hankel formulæ for the ordinary Gamma and Zeta functions, and that the formula is valid for all values of s except positive integral values.

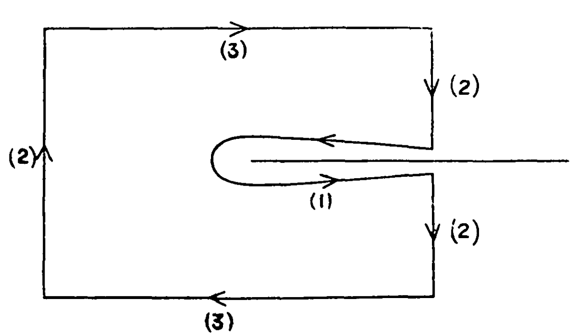
6.22. LEMMA β.—*Suppose that* $0<a\leqslant\omega+\omega'$, *and that* $\theta=\omega/\omega'$ *is an algebraic irrational. Then there is a* K *such that*

$$(6.221)\qquad \frac{\zeta_2(s,a,\omega,\omega')}{(2\pi)^{s-1}\Gamma(1-s)}=\frac{1}{\omega^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega}(\tfrac{1}{2}\omega'-a)+\tfrac{1}{2}(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega'\pi}{\omega}}$$

$$+\frac{1}{\omega'^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega'}(\tfrac{1}{2}\omega-a)+\tfrac{1}{2}(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega\pi}{\omega'}}$$

for $\sigma<-K$.

To prove this formula we start from the integral (6.211) and integrate



round the contour shown in the figure. We suppose, as plainly we may, that the horizontal lines (3) pass at a distance greater than a constant δ from any pole of the subject of integration, and that the loop (1) passes between the origin and the poles $\pm 2\pi i/\omega$, $\pm 2\pi i/\omega'$ nearest the origin. This being so, it is easy to see that the contributions of the rectilinear parts of the contour tend to zero when the sides of the rectangle move away to infinity, and that

$$\zeta_2 = \Gamma(1-s) \lim \Sigma R,$$

where R is a residue of the integrand. A simple calculation shows that the residues yield the two series required. If $\theta = \omega/\omega'$ is algebraic, we have

$$\left| \sin \frac{m\omega'\pi}{\omega} \right| > m^{-c}, \quad \left| \sin \frac{m\omega\pi}{\omega'} \right| > m^{-c},$$

where c is a constant depending on the degree of the algebraic equation which defines θ. It follows that the two series of the lemma are absolutely convergent if σ is negative and sufficiently large.* We shall suppose in what follows that the series are absolutely convergent for $\sigma \not< -K$. The formula (6.221) may of course hold in a wider region than this.

6.23. LEMMA γ.—*If* $|t| \to \infty$ *then*

$$\zeta_2(s, a, \omega, \omega') = O(e^{\epsilon|t|}),$$

for every positive ϵ, *and uniformly throughout any finite interval of values of* σ.

Suppose that $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. We may suppose the contour of integration in (6.211) deformed in such a manner that

$$|\phi| = |\arg(-u)| \leqslant \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$$

at every point of it, and $$|\phi| = \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$$

at all distant points. We have then

$$|(-u)^{s-1}| < A|u|^A e^{|\phi t|} < A|u|^A e^{(\frac{1}{2}\pi+\frac{1}{2}\epsilon)|t|},$$

where A is a number depending on σ_1 and σ_2,

$$|\Gamma(1-s)| = O(e^{-\frac{1}{2}\pi|t|}|t|^{\frac{1}{2}-\sigma}) = O(e^{-(\frac{1}{2}\pi-\frac{1}{2}\epsilon)|t|}),$$

$$\zeta_2 = O\left(e^{\epsilon|t|} \int \frac{|e^{-au}||du|}{|1-e^{-\omega u}||1-e^{-\omega' u}|}\right) = O(e^{\epsilon|t|}).$$

---

* It is hardly necessary to give fuller details of the proof, as the substance of the lemma is contained in the paper of Hardy referred to in the footnote to p. 17.

6.24. Lemma γ is required only in order to prove a somewhat deeper lemma, viz.:

LEMMA δ.*—*The function* $\zeta_2(s, a, \omega, \omega')$ *is of finite order in any half-plane* $\sigma > \sigma_0$, *and its* μ-*function* $\mu(\sigma)$ *satisfies the relations*

(6.241) $$\mu(\sigma) = 0 \quad (\sigma \geqslant 2),$$

(6.242) $$\mu(\sigma) \leqslant \frac{(\frac{1}{2}+K)(2-\sigma)}{2+K} \quad (-K \leqslant \sigma \leqslant 2),$$

(6.243) $$\mu(\sigma) \leqslant \tfrac{1}{2}-\sigma \quad (\sigma \leqslant -K).$$

Of these relations, (6.241) is obvious, since the series (6.11) is absolutely convergent for $\sigma > 2$; and (6.243) follows from (6.221), since we have

$$(2\pi)^{s-1}\Gamma(1-s)\sin\left\{\frac{2m\pi}{\omega}(\tfrac{1}{2}\omega'-a)+\tfrac{1}{2}(1-s)\pi\right\} = O\{e^{\frac{1}{2}\pi|t|}|\Gamma(1-s)|\}$$
$$= O(|t|^{\frac{1}{2}-\sigma})$$

uniformly in m, and, of course, a similar result in which ω and ω' are interchanged. Finally, (6.242) follows from (6.241), (6.243), and the well-known theorem of Lindelöf.† Lemma γ is used only to show that the conditions of Lindelöf's theorem are satisfied.

6.25. Our last lemma is of a different character. We write

(6.251) $$a+m\omega+n\omega' = l_p,$$

the numbers l_p (no two of which are equal, since θ is irrational) being arranged in order of magnitude. We suppose that ξ is not equal to any l_p, and we put
$$W(\xi) = \sum_{l_p<\xi} 1.$$

LEMMA ϵ.—*Suppose that* $c > 2$, $T > 1$, *and* $\xi = \sqrt{(l_q l_{q+1})}$. *Then there exists a number* H, *independent of* T *and* ξ, *such that*

$$\left| W(\xi) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta_2(s)\frac{\xi^s}{s}\,ds \right| < H\frac{\xi^c}{T}.$$

---

* For explanations concerning the "μ-function" of a function $f(s)$, defined initially by a Dirichlet's series, see G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series," *Cambridge Mathematical Tracts*, no. 18, 1915, pp. 14-18.

† Theorem 14 of the tract referred to above.

We have

$$(6.252)\qquad W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds = W-\sum_p \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}.$$

Since

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}\begin{array}{l} = 1 \quad (l_p<\xi) \\ = 0 \quad (l_p>\xi)\end{array},$$

the right-hand side of (6 . 252) may be written in the form

$$\sum_p\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c-iT}+\frac{1}{2\pi i}\int_{c+iT}^{c+i\infty}\right)\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}=\sum_p U_p,$$

say. Now*

$$|U_p|\leqslant\frac{2}{T}\,\frac{(\xi/l_p)^c}{|\log(\xi/l_p)|}.$$

Hence

$$(6.253)\qquad \left|W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds\right|\leqslant\frac{2\xi^c}{T}\sum_p\frac{l_p^{-c}}{|\log(\xi/l_p)|}.$$

If we write $l_p=e^{-\lambda_p}$, $\xi=e^{\rho}$, the series becomes

$$(6.254)\qquad \sum\frac{e^{-c\lambda_p}}{|\rho-\lambda_p|}.$$

and

$$\rho=\tfrac{1}{2}(\lambda_q+\lambda_{q+1}).$$

Now Bohr,† generalising a result of Landau,‡ has shown that the series (6 . 254) is bounded, provided only that

(C) *there is a number l, positive or zero, such that*

$$\frac{1}{\lambda_{p+1}-\lambda_p}=O(e^{(l+\delta)\lambda_p})$$

for every positive δ;

and it is easy to verify that the condition (C) is satisfied by our series $\sum l^{-s}=\sum e^{-s\lambda_p}$. For

$$l_{p+1}-l_p=a+m'\omega+n'\omega'-a-m\omega-n\omega'=h\omega+k\omega'=\omega'(k+h\theta),$$

say, and so, since θ is algebraic and $l_{p+1}<l_p+H$,

$$l_{p+1}-l_p>(|h|+2)^{-H}>H(|m|+|m'|+2)^{-H}>Hl_{p+1}^{-H}>Hl_p^{-H}\ (p>p_0),$$

$$\lambda_{p+1}-\lambda_p=\log\left(1+\frac{l_{p+1}-l_p}{l_p}\right)>Hl_p^{-H};$$

---

* Landau, *Handbuch*, § 86.

† H. Bohr, "Einige Bemerkungen zum Konvergenzproblem der Dirichletscher Reihen", *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), pp. 1–16.

‡ *Handbuch*, § 235.

H, wherever it occurs, denoting a positive constant, not of course the same at different occurrences. Thus Bohr's condition is satisfied, and Lemma ϵ follows from (6 . 253).

6 . 3. We can now prove our theorems. We take $T = \xi^{\gamma}$, where $0 < \gamma < 2$. We choose arbitrary positive numbers δ and ϵ, and take $c = 2+\delta$.

We then apply Cauchy's theorem to the integral

$$\frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds,$$

taken round the rectangle

$$(c-iT,\ c+iT,\ -K+iT,\ -K-iT),$$

the sides of which, taken in order, we denote by (1), (2), (3), and (4). Using Lemma α, we obtain

$$(6.31)\quad \frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds = \int_{(1)} + \int_{(2)} + \int_{(3)} + \int_{(4)} = \frac{\xi^2}{2\omega\omega'} + \frac{\omega+\omega'-2a}{2\omega\omega'}\,\xi + \zeta_2(0).$$

Now

$$(6 . 32)\qquad \int_{(1)} = W(\xi) + O\left(\frac{\xi^c}{T}\right) = W(\xi) + O(\xi^{2+\delta-\gamma}),$$

by Lemma ϵ; and

$$(6 . 33)\quad \int_{(3)} = O\left(\xi^{-K}\int_{-T}^{T} |t|^{K-\frac{1}{2}+\epsilon}\,dt\right) = O\left(\frac{T^{K+\frac{1}{2}+\epsilon}}{\xi^K}\right) = O(\xi^{-K+(K+\frac{1}{2})\gamma+\epsilon}),$$

by Lemma δ. It remains to estimate the contributions of the horizontal sides; and it is clear, from Lemma δ, that the contribution of either is of the form

$$O(\max_{-K\leqslant\sigma\leqslant c} \qquad {}^{\sigma)-1+\epsilon}) = O(\operatorname{Max} \xi^{\eta}),$$

where

$$\eta = \sigma + \left\{\frac{(\frac{1}{2}+K)(2-\sigma)}{2+K} - 1\right\}\gamma + \epsilon \quad (-K \leqslant \sigma \leqslant 2),$$

$$\eta = \sigma - \gamma + \epsilon \quad (2 \leqslant \sigma \leqslant c).$$

It is clear that η cannot exceed the greater of its values for $\sigma = -K$ and $\sigma = c$, viz.

$$-K + (K-\tfrac{1}{2})\gamma + \epsilon, \qquad 2+\delta-\gamma+\epsilon.$$

The possible error-term arising from the first of these values may be absorbed into that already present in (6 . 33). That corresponding to the second, as well as that in (6.32), may be absorbed in a single term $O(\xi^{2+\delta-\gamma+\epsilon})$. We have therefore, on collecting our results,

$$(6.34)\qquad W(\xi) = \frac{\xi^2}{2\omega\omega'} + \frac{\omega+\omega'-2a}{2\omega\omega'}\xi + O\{\xi^{-K+(K+\frac{1}{2})\gamma+\epsilon}\} + O(\xi^{2+\delta-\gamma+\epsilon}).$$

We have still γ at our disposal. Taking

$$-K+(K+\tfrac{1}{2})\gamma = 2+\delta-\gamma,$$

we obtain

$$\gamma = \frac{2+\delta+K}{\frac{3}{2}+K}$$

(which is, as we supposed, positive and less than 2), and

$$2+\delta-\gamma = \frac{(2+\delta)(\frac{1}{2}+K)-K}{\frac{3}{2}+K}.$$

This is equal to $(1+K)/(\frac{3}{2}+K) < 1$ when $\delta = 0$, and is therefore less than unity if δ is sufficiently small. We have therefore

$$(6.35)\qquad W(\xi) = \frac{\xi^2}{2\omega\omega'} + \frac{\omega+\omega'-2a}{2\omega\omega'}\xi + O(\xi^{\alpha}),$$

where $\alpha < 1$. In order to obtain Theorem **A5**, it is only necessary to attribute to a the particular value $\omega+\omega'$ and to replace ξ by η, since $W(\xi)$ then becomes $N(\eta)$.

Our argument naturally yields a definite value for α. But it becomes clear, when we consider the particular case of a *quadratic* θ, that the value so obtained is, in the light of Theorem **A2**, not the best value possible. We are therefore content to show that α is in any case less than unity.

Additional Note (*March* 13*th*, 1921).

We have developed the transcendental method of § 6 considerably since this paper was first communicated to the Society.

Suppose that $k \geqslant 0$ and

$$W_k(\xi) = \sum_{l_p<\xi} (\xi - l_p)^k.$$

Then

$$W_k(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_2(s)\frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)}\xi^{s+k}\,ds$$

if $c > 2$. We transform this equation by (1) moving back the path of integration to the line $\sigma = -q < 0$, with the appropriate corrections for the residues, (2) substituting for $\zeta_2(s)$ from (6.221), and (3) integrating term by term. This process can be justified if $\theta = \omega/\omega'$ is algebraic and k and q are chosen appropriately, and we obtain an expression for $W_k(\xi)$ in the form of an absolutely convergent series.

We then make use of a lemma which is of some interest in itself, viz.: *if there are constants* $h \geqslant 1$ *and* $H > 0$ *such that*

$$(1) \qquad n^h \,|\sin n\theta\pi| > H$$

for all positive integral values of n, *then the series*

$$\Sigma \frac{1}{n^{h+\epsilon}\,|\sin n\theta\pi|}$$

is convergent for every positive ϵ.

Using this lemma and our series for $W_k(\xi)$, we are able to show that *if* (1) *is true for all positive integral values of* n, *then*

$$(2) \qquad N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^{a+\epsilon}),$$

where $a = (h-1)/h$, *for every positive* ϵ. This is included in Theorem **A3** if $h = 1$; but is in all other cases considerably more precise than anything proved in the paper.

In (2) the index $a = (h-1)/h$ of the power of η is the best possible one. For we can also show that *if*

$$(3) \qquad n^h \,|\sin n\theta\pi| < H$$

for an infinity of values of n, *then each of the inequalities*

$$(4) \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} > A\eta^a, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} < -A\eta^a,$$

where A *is a positive constant depending on* h *and* H, *is true for a sequence of indefinitely increasing values of* η.

We are further able to obtain an "explicit formula" for $N(\eta)$; viz.

$$(5) \quad N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{\omega^2 + \omega'^2 + 3\omega\omega'}{12\omega\omega'}$$
$$- \frac{1}{2\pi} \Sigma \left(\frac{\cos \dfrac{2\mu\pi}{\omega}(\eta - \tfrac{1}{2}\omega')}{\mu \sin \dfrac{\mu\omega'\pi}{\omega}} + \frac{\cos \dfrac{2\nu\pi}{\omega'}(\eta - \tfrac{1}{2}\omega)}{\nu \sin \dfrac{\nu\omega\pi}{\omega'}} \right).$$

Here $\theta = \omega/\omega'$ is irrational and algebraic, and the series is to be interpreted as meaning

$$\lim \sum_{\mu<\omega R,\ \nu<\omega' R}$$

when $R \to \infty$ in an appropriate manner.

The most difficult of the remaining problems is that of determining whether there is *any* θ for which the error-term in $N(\eta)$, or the sum $s(\theta, n)$ is *bounded*. The answer is in the negative. We can prove, in fact, that *there exists an* $A > 0$ *such that, for every irrational* θ,

$$|s(\theta, n)| > A \log n$$

for an infinity of values of n. Further, *given* K, *there exists a* $B = B(K) > 0$ *such that, for every* θ *for which* $a_n < K$, *the inequalities*

$$s(\theta, n) > B \log n, \qquad s(\theta, n) < -B \log n,$$

are satisfied each for an infinity of values of n.

The corresponding Cesàro means behave rather differently. It is possible to find θ's for which the first Cesàro mean $\sigma(\theta, n)$ of $s(\theta, n)$ is bounded, and others for which $\sigma(\theta, n)/\log n$ tends to a limit other than zero.

We may take this opportunity of correcting a misstatement in our communication to the Cambridge Congress referred to on p. 15. It was stated there that

$$\sum_{\nu=1}^{n} \{\nu\theta\}^2 = \tfrac{1}{12}n + O(1)$$

for *every* irrational θ. This is untrue; but the equation holds for very general classes of values of θ, and in particular for any θ whose partial quotients are bounded.