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Concerning the ellipsoidal variables SZ Tauri and S Antliae. By *Harlow Shapley*.

1. In a recent Bulletin of the Laws Observatory of the University of Missouri I have given a detailed discussion of the light variations of RU Camelopardalis, and have shown that the curve is satisfactorily represented on the hypothesis that the changes in brightness are due to the axial rotation of a single ellipsoidal body. In the same paper brief mention is made of the similar variables, SZ Tauri and S Antliae. In the present communication I shall discuss more fully the light curves of these two stars, and also compare the light changes of a rotating ellipsoid with the somewhat similar variations of the short period stars of the ζ Geminorum type. For the details of the theory and method of treatment the reader is referred to the paper mentioned above and to the articles cited there relative to the orbits of eclipsing binaries. I am greatly indebted to Professor *Russell* for the remarks at the end of the present paper on the possible densities and polar flattening of these ellipsoidal variables, as well as for many other suggestions concerning the general problem.

2. SZ Tauri [1]). The variability of this star was discovered from four of the plates of the Göttingen Aktinometrie, and after ample confirmation through a special investigation, was announced by *Schwarzschild* in A. N. 4425 under the provisional designation of 41.1910 Tauri. *Hertzsprung* called attention to the similarity of the proper motion of the variable to that of the Hyades stream, and the star was considered of especial interest because of the possibility that its parallax is known. Accordingly the star was followed photographically for a year at Potsdam by *Münch* and *Hertzsprung*, and the observations are discussed by *Schwarzschild* in A. N. 4532. The formula for the minima is

$$\text{J. D. } 2418724.12 + 3^{d}1484 \cdot E, \text{ Gr. M. T.};$$

and the photographic range of variation is from $7^{m}17$ to $7^{m}73$. The curve is practically symmetrical.

For my study of the light curve I have followed *Schwarzschild* in making use only of the measures of the 77 Potsdam plates. The magnitudes estimated from the Göttingen plates are too few in number and too unevenly distributed to aid in the determination of the light variation. The normal magnitudes from the Potsdam observations are plotted in Fig. 2. They are given again in Table I where the first four columns contain the observational material. The phases have been changed so that they refer to the epoch of minimum, which is 2.140 days after the zero epoch used at Potsdam.

Table I. Normal magnitudes for SZ Tauri.

No.	Phase	Magn.	Wt.	$\cos^2\theta$	$1-L$	Darkened O − C	Darkened O − C	L^2	Uniform O − C
1	$-1^{d}464$	$7^{m}214$	6	0.012	+0.035	$+0^{L}035$	$+0^{m}034$	0.931	$-0^{L^2}062$
2	−1.215	7.235	10	0.122	+0.062	+0.014	+0.006	0.879	−0.040
3	−0.920	7.356	4	0.369	+0.154	+0.006	+0.008	0.716	−0.036
4	−0.696	7.427	3	0.590	+0.207	−0.021	−0.040	0.629	+0.024
5	−0.426	7.656	6	0.830	+0.358	+0.026	+0.043	0.412	−0.035
6	−0.195	7.682	10	0.963	+0.373	−0.012	−0.021	0.393	+0.039
7	+0.049	7.739	4	0.998	+0.405	+0.006	+0.011	0.354	+0.022
8	+0.388	7.629	6	0.857	+0.342	−0.001	−0.002	0.433	+0.006
9	+0.623	7.524	4	0.660	+0.275	+0.011	+0.016	0.526	−0.030
10	+0.833	7.378	6	0.454	+0.170	−0.011	−0.015	0.688	−0.008
11	+1.024	7.262	10	0.271	+0.077	−0.031	−0.038	0.852	+0.033
12	+1.451	7.142	8	0.015	−0.030	−0.030	−0.039	1.061	+0.071

In order to test the application to SZ Tauri of the hypothesis of a rotating ellipsoid, I have computed $\cos^2\theta$ from the phases (θ is the angle of rotation, counted from minimum); and taking maximum light at $7^{m}175$ as unity, I have determined the relative light intensity for each of the normal points. These values are entered in columns 5 and 6 of Table I.

Assuming first that the apparent disk of the ellipsoidal body (or bodies) is completely darkened at the limb, we have the relation [2]) $L = L_0\,(1 - {}^4/_5\,\varepsilon^2 \sin^2 i \cos^2\theta)$ where L_0 is a constant, ε is the eccentricity of the equatorial section, and i is the inclination of the plane of rotation to the plane tangent to the celestial sphere. In Fig. 1 L is plotted against $\cos^2\theta$. The evident linear relation between the two quantities indicates satisfactorily that the hypothesis of a single rotating ellipsoid is sufficient to explain the entire light variations of the star. The effective ellipticity, $\varepsilon^2 \sin^2 i$, is read directly from the line and is found to be 0.47 ± 0.006;

[1]) 1900.0: RA. $= 4^{h}31^{m}4$, Decl. $= +18^{\circ}20'$. Spectrum F8, (Harv. Ann. 56.192.)

[2]) See the development of this formula, Aph. J. 36.399 ff. Terms involving fourth and higher powers of ε are omitted here.

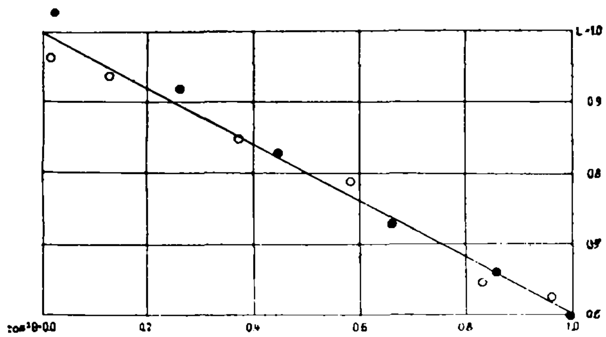

Fig. 1. SZ Tauri. Ellipticity determination; darkened disk.

the average deviation of a normal point, each given equal weight, is $0^L.017$. With the adopted value of $\varepsilon^2 \sin^2 i$ the ellipticity-magnitude curve, which is drawn in Fig. 2, was computed. The average deviation of a normal point from the curve is $\pm 0^m.023$ — a representation of the observations practically identical with that derived by *Schwarzschild* from an expansion in *Fourier*'s Series. The residuals from the straight line and from the magnitude curve are given in Table I in columns 7 and 8, respectively.

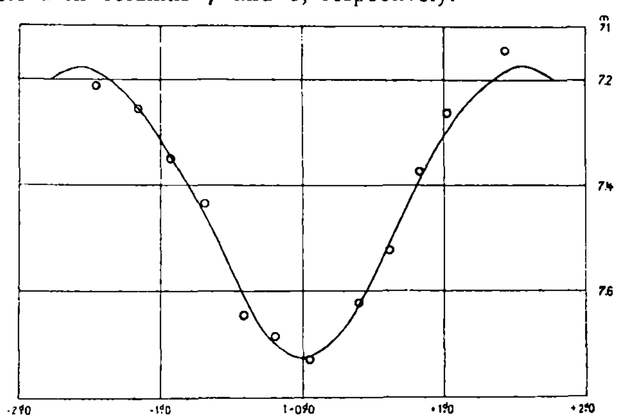

Fig. 2. SZ Tauri. Ellipticity-magnitude; darkened disk.

If we assume that the elliptical disk is uniformly luminous, we must determine the effective ellipticity from the relation

$$L^2 = L_0^2 (1 - \varepsilon^2 \sin^2 i \cos^2 \theta) .$$

The value of $\varepsilon^2 \sin^2 i$ determined from the plot of L^2 and $\cos^2 \theta$ is 0.67. The representation of the observations is again excellent, the average deviation of a point from the straight line being ± 0.034 in terms of L^2. The last two columns of the table give the values of L^2 and the residuals.

It is evident from these results that a large uncertainty in the actual value of the ellipticity must inevitable exist because of our ignorance of the actual degree of darkening on stellar disks. The values above probably represent the limiting values of $\varepsilon^2 \sin^2 i$. For RU Camelopardalis (op. cit.) the assumption of a uniformly luminous disk satisfied the observations better than the assumption of complete darkening, but for SZ Tauri there is no essential difference. It is probable that any intermediate degree of darkening would also represent the observations. And further, it should be noted, there remains the possibility of interpreting the light variation as a result of the rotation of a spherical or spheroidal star one side of which is considerably brighter than the other [1]. If the light intensity on such a rotating body diminished uniformly from a point of maximum brightness on one side to a minimum point diametrically opposite, the light curve would be identical in form with the »darkened« curve obtained above. The period of rotation of this unevenly luminous star would be equal to that of the light changes, while the period of the rotating ellipsoid is twice as long. The two cases could be distinguished by the behavior of the spectral lines, or more simply, perhaps, by the relative visual and photographic ranges of variation. For the ellipsoid the character of the light over the whole surface would be the same and the ranges should be sensibly equal.

3. S Antliae [2]. The observations of this star have been affected by serious difficulties and uncertainties owing to its low altitude for northern observatories [3]. However, a fairly good curve was obtained at Harvard in 1896–98 by Professor *E. C. Pickering* with the Meridian photometer [4], and a short series of accurate observations have been made by *Wendell* with a sliding-prism polarizing photometer in order to determine the relative depths of successive minima [5]. For the present discussion I use the 207 Meridian photometer observations which are given in Table XIV of Harv. Ann. 46. Normal groups have been formed and are given in Table II.

Table II. Normal magnitudes of S Antliae.

No.	Phase	Magn.	No. Obsns.	Uniform O — C
1	$0^d.005$	$6^m.39$	8	$-0^m.02$
2	0.013	6.45	10	+0.02
3	0.030	6.51	11	+0.02
4	0.045	6.56	8	+0.04
5	0.055	6.50	6	−0.05
6	0.070	6.62	9	+0.02
7	0.085	6.67	7	+0.02
8	0.101	6.67	5	−0.02
9	0.121	6.77	14	+0.05
10	0.135	6.74	8	+0.02
11	0.144	6.68	11	−0.03
12	0.156	6.62	8	−0.06
13	0.165	6.64	9	−0.02
14	0.175	6.63	10	0.00
15	0.195	6.55	10	−0.01
16	0.222	6.48	9	+0.01
17	0.251	6.45	14	+0.05
18	0.268	6.41	12	+0.03
19	0.283	6.36	8	−0.01
20	0.292	6.35	8	−0.02
21	0.305	6.35	8	−0.02
22	0.316	6.36	14	−0.03

[1] *H. N. Russell.* On the light variations of asteroids and satellites. Aph. J. 24, 1–18, 1906.
[2] 1900.0: RA. = $9^h 27^m.9$, Decl. = $-28° 11'$. Spectrum F. (Harv. Ann. 56.192).
[3] *Luizet*, A. N. 3955; *Sperra*, Astr. J. 18.38. [4] Harv. Ann. 46.149.
[5] Harv. Ann. 69.44; Harv. Circ. 41, A. N. 3561. No difference was found and Prof. *Pickering* accordingly changed the classification of the variable from »suspected Algol« to »short period«.

The points are plotted in Fig. 3 and show that the light variation is continuous, with an amplitude of $0^m.35$. In *Pickering*'s curve the ascending branch appears a little more gradual than the descending; but the reverse effect was found by *Luizet*. In both cases the departure from symmetry is small and in all probability is not real. At maximum the star is of magnitude 6.37. The interval between minima (one-half the rotation period) is 0.32417 days, according to *Luizet*.

The ellipticity of S Antliae was determined in the same manner as for SZ Tauri. On the »uniform« hypothesis, $\varepsilon^2 \sin^2 i = 0.475$; and for a disk completely darkened at the limb, $\varepsilon^2 \sin^2 i = 0.33$. The representation of the observations is shown in Fig. 3, where the theoretical »uniform« ellipticity-magnitude curve is drawn. The residuals from this curve are given in Table II. The average deviation of a normal point is $\pm 0^m.03$.

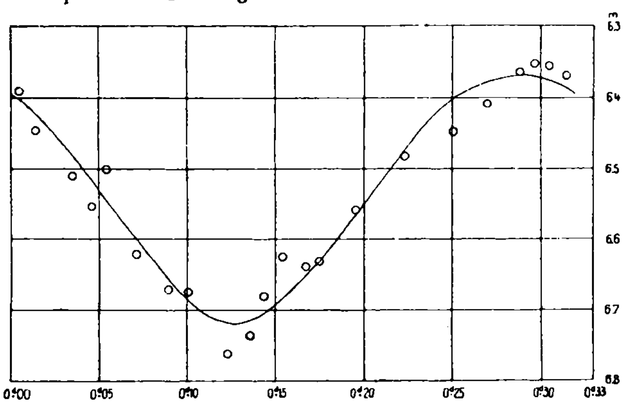

Fig. 3. S Antliae. Ellipticity-magnitude curve; uniform disk.

4. Two non-elliptical short period variables. As an illustration of the method of determining whether or not a light curve can be ascribed to a rotating ellipsoid, I give below two examples of symmetrical light curves that fail in the ellipticity test.

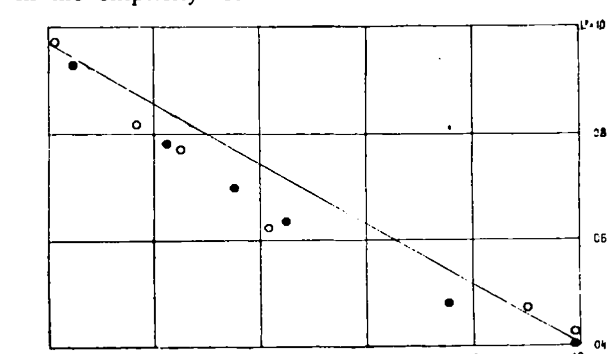

Fig. 4. SU Cassiopeiae. Ellipticity test; uniform.

The light variations of SU Cassiopeiae have been accurately determined at the Yerkes Observatory by *J. A. Parkhurst*, using the extrafocal method [1]). The curve as drawn by him is slightly asymmetrical, but I find that it can be considered absolutely symmetrical without misrepresenting the observations. The normal points, as formed by *Parkhurst*, are given in Fig. 4, where I have plotted the square of the light intensity against $\cos^2\theta$. It is obvious that the theory of a rotating ellipsoid will not satisfy the observations.

The measures of the light of ζ Geminorum by *Pickering* with the Meridian photometer (Harv. Ann. 46.136) have been grouped in fives in order of phase, and the resulting points plotted in Fig. 5. The curve is sensibly symmetrical. The negative result of the »uniform« ellipticity test is indicated in the same manner as before.

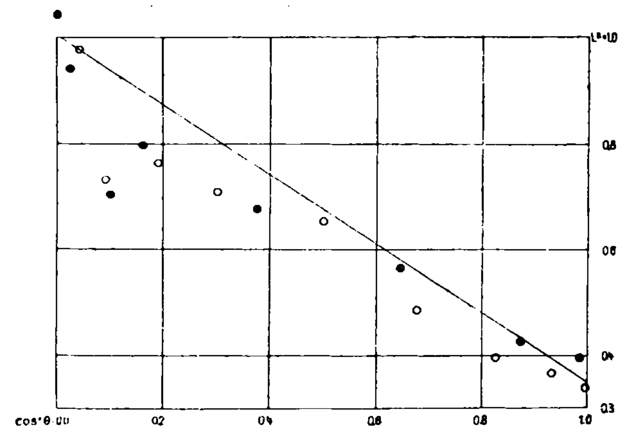

Fig. 5. ζ Geminorum. Ellipticity test; uniform.

The behavior of the plotted points in these two cases is typical of all the true Cepheid variables whose light curves approach symmetry. A value of the ellipticity that would represent maximum and minimum light would not conform to the rest of the curve; one that would satisfy the lower half of the curves would be impossible for the maxima. The solutions in each case for the »darkened« ellipticity curve gave similar results. In the case of ζ Geminorum we know that the star is not an ellipsoid (with period double that of the light changes), for it is a well known spectroscopic binary — in fact a typical Cepheid system without the usual asymmetrical curve [2]). The same is suspected of SU Cassiopeiae [1]). It is not impossible, however, that there may be true Cepheid variables whose light variations can be satisfactorily fitted with an ellipticity curve — Polaris, for example.

When the light curves of other short period variables of the ζ Geminorum type are accurately determined, it is very likely that further additions can be made to the list of ellipsoidal variables. It is probable, for instance, that the variable W Serpentis, suspected at Harvard of being an Algol variable [3]), and by *Zinner* of belonging to the ζ Geminorum type [4]), is really a rotating ellipsoid, or at least a highly elliptical binary with shallow eclipses. One criterion for the ellipsoidal variable is, of course, that the range of variation be less than a magnitude, for otherwise the elongation of the star, even for maximum inclination, becomes unbelievably great.

5. Concerning the Density and Figure of Ellipsoidal Variables. It is of interest to attempt to estimate the probable figure and density of these stars on the as-

[1]) Aph. J. 28.278.
[2]) Orbit by *W. W. Campbell*, Aph. J. 13.93, 1901.
[3]) Harv. Circ. 127, A. N. 4181.
[4]) A. N. 4558.

sumption that they are actual ellipsoids of equilibrium. Our best guide is *G. H. Darwin*'s work on masses of homogeneous, incompressible fluid. The actual stars are undoubtedly denser toward the center, which tends toward a smaller ellipticity, and they are composed of elastic and expansible material, which, as *Jeans* has shown [1]), may cause ellipticity or even instability to set in for a much smaller rate of rotation than would be the case for a mass of incompressible fluid of equal mean density. The results given below may perhaps give a fair approximation to the relative magnitude of the third axis of the ellipsoids, but should be considered as only an indication of the possible order of magnitude of their densities.

From *Darwin*'s paper [2]) we may take the following values for the dimensions and angular velocity of the *Jacobi*an ellipsoids of homogeneous incompressible fluid, the unit of length being the radius of the sphere of equal volume, and the unit of density being so chosen that the *Gauß*ian constant is unity. The fifth of these computed ellipsoids represents the condition just beyond which instability occurs and the pear-shaped figure begins to form.

Table III. *Jacobi*'s Ellipsoids of Equilibrium.

No.	a	b	c	$\omega^2/4\pi\varrho$	b/a	c/a
1	1.197	1.197	0.698	0.0936	1.000	0.583
2	1.279	1.123	0.696	0.093	0.879	0.544
3	1.383	1.045	0.692	0.0906	0.756	0.501
4	1.601	0.924	0.677	0.0830	0.577	0.423
5	1.899	0.811	0.649	0.0705	0.427	0.342
6	2.346	0.702	0.607	0.0536	0.299	0.259
7	3.136	0.586	0.545	0.0334	0.187	0.174

In Table III the axes of the ellipsoids are designated by a, b, c, in descending order of magnitude; the angular velocity is denoted by ω, and the density by ϱ. For the three ellipsoidal stars, the quantity $b/a = (1-\varepsilon^2)^{1/2}$ would be obtained from the light curves if the inclination were known. We will take $i = 90°$, which is probably near the truth, at least for the two stars of greatest range, and this will give the minimum value of the ellipticity. Plotting the values of b/a in the above table against c/a, and against $\omega^2/4\pi\varrho$, we can read from the curves the values of the angular velocity and relative polar axis for each of the variables. The density is then readily computed from the known period of rotation. The results are tabulated below.

	RU Camelopardalis		SZ Tauri		S Antliae	
	Uniform	Darkened	Uniform	Darken.	Uniform	Darken.
ε^2	0.82	0.62	0.67	0.47	0.475	0.33
b/a	0.424	0.617	0.574	0.728	0.724	0.819
c/a	0.341	0.443	0.421	0.490	0.489	0.524
$\omega^2/4\pi\varrho$	0.0703	0.0853	0.0828	0.0897	0.0897	0.0922
ϱ	0.00003	0.00003	0.0014	0.0013	0.119	0.116

The densities are expressed in terms of the mean solar density. It should be noted that the uniform solution for RU Camelopardalis gives dimensions practically identical to those of the homogeneous ellipsoid that is near to instability and to the transformation into the pear-shaped figure.

Princeton University Observatory, 1913 March 1.

Harlow Shapley.

[1]) Philosophical Transactions, A, 201.157; A, 199.1.
[2]) Roy. Soc. Proc. 41.319, 1887.

Die Veränderlichkeit des Polarsterns. Von *Ant. Pannekoek.*

Als ich in den Jahren 1889 und 1890 Beobachtungen der Helligkeit mehrerer Sterne zweiter und dritter Größe nach *Argelanders* Methode anstellte, zeigte sich bei einigen Sternen ein starker Verdacht der Veränderlichkeit, u. a. bei α Ursae minoris. Die Schätzungen zeigten eine Periode von etwas weniger als 4 Tagen; da aber die Amplitude äußerst gering war, gelang es mir nicht, einen Wert für die Periode zu finden, der den Beobachtungen genügte, und damit die Veränderlichkeit zweifelsfrei festzustellen. Als dann nachher *Campbell* eine Veränderlichkeit der visuellen Geschwindigkeit in einer Periode von 3.968 Tagen feststellte, und sich bei einer Untersuchung im Jahre 1906 zeigte, daß der Polarstern den c-Charakter des Spektrums und die geringe Dichtigkeit mit den kurzperiodischen Veränderlichen gemeinsam hat, wurde die Realität einer Veränderlichkeit im hohen Maße wahrscheinlich. *Hertzsprung* hat dann neulich auf photographischem Wege eine Veränderlichkeit mit einer Amplitude von 0.17 Größenklassen nachgewiesen und *Stebbins* hat sie mit dem Selenphotometer bestätigt.

Die Beobachtungen, die ich in den Jahren 1890 bis 1899 mit freiem Auge angestellt habe, zerfallen in zwei getrennte Reihen, Vergleichungen mit α Persei, β und γ Andromedae in der zweiten und Vergleichungen mit ε und η Ursae majoris in der ersten Hälfte jedes Jahres. Werden sie alle mit dem Periodenwert 3^d968 auf eine einzige Periode, 3. bis 7. August 1894, zurückgebracht und dann nach der Phase geordnet, so ergeben sich folgende Mittelwerte:

Erste Reihe.

Vergl.-St.: α Persei = 6.3, β Andr. = 3.8, γ Andr. = 3.1.

Epoche	Helligk.	Anz.	B—R
Aug. 3.12	3.72	(18)	+0.03
3.42	3.94	(16)	+0.11
3.73	4.11	(21)	+0.08
3.96	3.91	(17)	−0.29
4.22	3.94	(20)	−0.42
4.54	4.68	(17)	+0.18
4.76	4.86	(16)	+0.33
4.94	4.76	(16)	+0.24
5.18	4.11	(17)	−0.34
5.52	4.45	(14)	+0.19
5.76	4.11	(20)	+0.02
5.94	3.67	(18)	−0.29
6.23	3.67	(16)	−0.11
6.46	3.78	(14)	+0.10
6.73	3.79	(19)	+0.16

Zweite Reihe.

Vergl.-St.: ε Ursae maj. = 2.4, η Ursae maj. = 0.0.

Epoche	Helligk.	Anz.	B—R
Aug. 3.21	0.59	(18)	−0.03
3.48	0.68	(16)	−0.08
3.65	0.59	(13)	−0.28
3.92	1.13	(18)	+0.07
4.25	1.39	(20)	+0.10
4.46	1.40	(22)	0.00
4.65	1.69	(24)	+0.22
4.94	1.55	(14)	+0.05
5.22	1.20	(17)	−0.23
5.46	1.27	(18)	−0.03
5.72	1.02	(20)	−0.11
6.22	0.74	(19)	−0.04
6.48	0.86	(16)	+0.22
6.86	0.67	(16)	+0.11

Die Veränderlichkeit des Polarsterns tritt in beiden