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III. *On the Theory of the Oscillations in Three Coupled Electric Circuits.* By L. C. JACKSON, B.Sc.*

INTRODUCTION.

THE problem of two coupled electric circuits has been the subject of a considerable number of investigations both from the experimental and theoretical standpoint. The question has been attacked from the theoretical point of view by J. v. Geitler, A. Overbeck, V. Bjerknes, P. Drude, F. Kiebitz, and others. The corresponding problem of three coupled electric circuits does not seem to have received any very great amount of attention. Apart from the work of B. Mackù† and E. Bellini‡, the literature of the subject is very scanty.

In view of the growing importance of the use of systems containing three coupled circuits in wireless telegraphy and for the determination of various physical constants§, it seems desirable that the question should receive further investigation.

The purpose of the present paper is to obtain a solution of the problem by a simple method not involving any considerable amount of complicated mathematical analysis.

NOTATION.

The following notation will be used throughout the present work :—

L_1, L_2, L_3 = Self-inductances of circuits 1, 2, and 3 respectively.

M_{12}, M_{23}, M_{31} = Mutual inductances between circuits 1 and 2, 2 and 3, 3 and 1 respectively.

C_1, C_2, C_3 = Capacities in circuits 1, 2, and 3.

R_1, R_2, R_3 = Resistances of circuits 1, 2, and 3.

e_1, e_2, e_3 = Charges on condensers in circuits 1, 2, and 3.

i_1, i_2, i_3 = Currents in circuits 1, 2, and 3.

α, β, γ = Coefficients of coupling.

l, m, n = Frequencies of the simple oscillations in circuits if entirely separate from each other.

$p, \omega_1, \omega_2, \omega_3$ = Frequencies of oscillation in coupled circuits.

q, r, s = Damping factors.

θ, ϕ, ψ = Phase angles.

* Communicated by the Author.

† *Jahrbuch der Drahtlosen Telegraphie*, iv. pp. 188-195 (Dec. 1910).

‡ 'Electrician,' lxxxv. p. 78 (July 16, 1920).

§ Cf. J. H. Vincent, *Proc. Phys. Soc.*, vol. xxxii. pp. 84-90 (Feb. 1919); and L. Pungs and G. Preuner, *Phys. Zeits.* xx. pp. 543-545 (Dec. 1, 1919).

THE DIFFERENTIAL EQUATIONS AND THEIR SOLUTION.

Let us now consider the oscillations possible in three coupled electric circuits in which the inductances, capacities, and resistances are $L_1, L_2, L_3, C_1, C_2, C_3, R_1, R_2, R_3$ respectively, and which are inductively coupled through the mutual inductances M_{12}, M_{23}, M_{31} .

The equations of motion of the circuits may be written :

$$\begin{aligned}\frac{e_1}{C_1} &= i_1 R_1 + L_1 \frac{di_1}{dt} - M_{12} \frac{di_2}{dt} - M_{13} \frac{di_3}{dt}, \\ \frac{e_2}{C_2} &= i_2 R_2 + L_2 \frac{di_2}{dt} - M_{23} \frac{di_3}{dt} - M_{21} \frac{di_1}{dt}, \\ \frac{e_3}{C_3} &= i_3 R_3 + L_3 \frac{di_3}{dt} - M_{31} \frac{di_1}{dt} - M_{32} \frac{di_2}{dt},\end{aligned}$$

in which e_1, e_2, e_3 are the charges on the three condensers in circuits 1, 2, and 3 respectively at any instant ; i_1, i_2, i_3 are the corresponding currents, and also

$$M_{12} = M_{21}, \quad M_{23} = M_{32}, \quad M_{13} = M_{31}.$$

Since the current i is the time rate of decrease of charge e , we have

$$\frac{de_1}{dt} = -i_1, \quad \frac{de_2}{dt} = -i_2, \quad \frac{de_3}{dt} = -i_3. \quad \dots \quad (1)$$

Using equations (1), the above equations of motion may be written in the form :

$$\left. \begin{aligned} e_1 + R_1 C_1 \frac{de_1}{dt} + L_1 C_1 \frac{d^2 e_1}{dt^2} - M_{12} C_1 \frac{d^2 e_2}{dt^2} \\ \qquad \qquad \qquad - M_{13} C_1 \frac{d^2 e_3}{dt^2} = 0, \\ e_2 + R_2 C_2 \frac{de_2}{dt} + L_2 C_2 \frac{d^2 e_2}{dt^2} - M_{23} C_2 \frac{d^2 e_3}{dt^2} \\ \qquad \qquad \qquad - M_{21} C_2 \frac{d^2 e_1}{dt^2} = 0, \\ e_3 + R_3 C_3 \frac{de_3}{dt} + L_3 C_3 \frac{d^2 e_3}{dt^2} - M_{31} C_3 \frac{d^2 e_1}{dt^2} \\ \qquad \qquad \qquad - M_{32} C_3 \frac{d^2 e_2}{dt^2} = 0. \end{aligned} \right\} \quad \dots \quad (2)$$

Following the usual practice, the coefficients of coupling may be written :

$$\left. \begin{array}{lll} \text{For coupling between circuits 1 and 2, } \alpha^2 = \frac{M_{12}^2}{L_1 L_2}, \\ \text{,, ,, ,, 2 and 3, } \beta^2 = \frac{M_{23}^2}{L_2 L_3}, \\ \text{,, ,, ,, 3 and 1, } \gamma^2 = \frac{M_{31}^2}{L_3 L_1}. \end{array} \right\} \quad (3)$$

Let us assume a solution of (2) in the form

$$\left. \begin{array}{l} e_1 = A e^{ipt}, \\ e_2 = B e^{ipt}, \\ e_3 = C e^{ipt}. \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting these values in (2), we obtain

$$\begin{aligned} e_1(1 + R_1 C_1 p i - L_1 C_1 p^2) + e_2 M_{12} C_1 p^2 &+ e_3 M_{13} C_1 p^2 = 0, \\ e_1 M_{21} C_2 p^2 &+ e_2(1 + R_2 C_2 p i - L_2 C_2 p^2) &+ e_3 M_{23} C_2 p^2 = 0, \\ e_1 M_{31} C_3 p^2 &+ e_2 M_{32} C_3 p^2 &+ e_3(1 + R_3 C_3 p i - L_3 C_3 p^2) = 0. \end{aligned}$$

Eliminating e_1 , e_2 , and e_3 from these equations by means of the determinant

$$\begin{vmatrix} (1 + R_1 C_1 p i - L_1 C_1 p^2), & M_{12} C_1 p^2, & M_{13} C_1 p^2, \\ M_{21} C_2 p^2, & (1 + R_2 C_2 p i - L_2 C_2 p^2), & M_{23} C_2 p^2, \\ M_{31} C_3 p^2, & M_{32} C_3 p^2, & (1 + R_3 C_3 p i - L_3 C_3 p^2), \end{vmatrix} = 0.$$

we obtain an equation of the sixth degree in p , viz. :

$$\begin{aligned} & p^6 [2 M_{12} M_{23} M_{31} - L_1 L_2 L_3 + L_1 M_{23}^2 + L_2 M_{31}^2 + L_3 M_{21}^2] \\ & + p^5 i [R_1 L_2 L_3 + L_1 R_2 L_3 + L_1 L_2 R_3 - R_1 M_{23}^2 - R_2 M_{31}^2 - R_3 M_{12}^2] \\ & + p^4 \left[\frac{L_1 L_2 - M_{12}^2}{C_3} + \frac{L_2 L_3 - M_{23}^2}{C_1} + \frac{L_3 L_1 - M_{31}^2}{C_2} + R_1 L_2 R_3 + R_1 R_2 L_3 + L_1 R_2 R_3 \right] \\ & - p^3 i \left[\frac{R_1 L_2 + L_1 R_2}{C_3} + \frac{R_2 L_3 + L_2 R_3}{C_1} + \frac{R_3 L_1 + L_3 R_1}{C_2} + R_1 R_2 R_3 \right] \\ & - p^2 \left[\frac{L_1}{C_2 C_3} + \frac{L_2}{C_1 C_3} + \frac{L_3}{C_1 C_2} + \frac{R_1 R_2}{C_3} + \frac{R_2 R_3}{C_1} + \frac{R_3 R_1}{C_2} \right] \\ & + p i \left[\frac{R_1}{C_2 C_3} + \frac{R_2}{C_1 C_3} + \frac{R_3}{C_1 C_2} \right] + \frac{1}{C_1 C_2 C_3} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

This is a complete equation of the sixth degree, of which, for our present purpose, we can assume a solution in the form

$$p = iq \pm \omega_1 \quad \text{or} \quad ir \pm \omega_2 \quad \text{or} \quad is \pm \omega,$$

in which q , r , and s are small quantities whose squares are negligible.

We thus obtain the equivalent equation

$$(p - iq - \omega_1)(p - iq + \omega_1)(p - ir - \omega_2)(p - ir + \omega_2)(p - is - \omega_3)(p - is + \omega_3) = 0$$

or, on expanding and omitting the negligible quantities,

$$\begin{aligned} & p^6 - 2p^5i(q+r+s) - p^4(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ & + 2p^3i\{\omega_1^2(r+s) + \omega_2^2(s+q) + \omega_3^2(q+r)\} \\ & + p(\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2) \\ & - 2pi(\omega_1^2\omega_2^2r + \omega_2^2\omega_3^2q + \omega_3^2\omega_1^2s) - \omega_1^2\omega_2^2\omega_3^2 = 0. \end{aligned} \quad (6)$$

Comparing coefficients in (5) and (6), we obtain

$$\left. \begin{aligned} 1 &= 2M_{12}M_{23}M_{31} - L_1L_2L_3 + L_1M_{23}^2 + L_2M_{31}^2 + L_3M_{12}^2, \quad \dots \quad (i.) \\ -2(q+r+s) &= R_1L_2L_3 + L_1R_2L_3 + L_1L_2R_3 - R_1M_{23}^2 - R_2M_{31}^2 - R_3M_{12}^2, \quad (ii.) \\ -(\omega_1^2 + \omega_2^2 + \omega_3^2) &= \frac{L_1L_2 - M_{12}^2}{C_3} + \frac{L_2L_3 - M_{23}^2}{C_1} + \frac{L_3L_1 - M_{31}^2}{C_2} + L_1R_2R_3 \\ &\quad + R_1L_2R_3 + R_1R_2L_3, \quad \dots \quad (iii.) \\ -2\{\omega_1^2(r+s) + \omega_2^2(s+q) + \omega_3^2(q+r)\} &= \frac{R_1L_2 + L_1R_2}{C_3} + \frac{R_2L_3 + L_2R_3}{C_1} + \frac{R_3L_1 + L_3R_1}{C_2} + R_1R_2R_3, \quad (iv.) \\ -(\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2) &= \frac{L_1}{C_2C_3} + \frac{L_2}{C_1C_3} + \frac{L_3}{C_1C_2} + \frac{R_1R_2}{C_3} + \frac{R_2R_3}{C_1} + \frac{R_3R_1}{C_2}, \quad \dots \quad (v.) \\ -2(\omega_1^2\omega_2^2s + \omega_2^2\omega_3^2q + \omega_3^2\omega_1^2r) &= \frac{R_1}{C_2C_3} + \frac{R_2}{C_1C_3} + \frac{R_3}{C_1C_2}, \quad \dots \quad (vi.) \\ -\omega_1^2\omega_2^2\omega_3^2 &= \frac{1}{C_1C_2C_3}, \quad \dots \quad (vii.) \end{aligned} \right\} \quad (7)$$

From equations (7) i., iii., v., vii. we obtain the following equation for ω_2 :—

$$\begin{aligned} & \omega_2^6(2M_{12}M_{23}M_{31} - L_1L_2L_3 + L_1M_{23}^2 + L_2M_{31}^2 + L_3M_{12}^2) \\ & + \omega_2^4\left(\frac{L_1L_2 - M_{12}^2}{C_3} + \frac{L_2L_3 - M_{23}^2}{C_1} + \frac{L_3L_1 - M_{31}^2}{C_2} + L_1R_2R_3 + R_1L_2R_3 + R_1R_2L_3\right) \\ & - \omega_2^2\left(\frac{L_1}{C_2C_3} + \frac{L_2}{C_1C_3} + \frac{L_3}{C_1C_2} + \frac{R_1R_2}{C_3} + \frac{R_2R_3}{C_1} + \frac{R_3R_1}{C_2}\right) \\ & + \frac{1}{C_1C_2C_3} = 0. \quad \dots \quad (8) \end{aligned}$$

If now we make use of the abbreviations given below, equation (8) can be expressed in a slightly more compact form, containing only the couplings and the constants of the separate circuits as coefficients :

$$\begin{aligned} l^2 &= \frac{1}{L_1C_1}, & m^2 &= \frac{1}{L_2C_2}, & n^2 &= \frac{1}{L_3C_3}, \\ k_1 &= \frac{R_1}{L_1}, & k_2 &= \frac{R_2}{L_2}, & k_3 &= \frac{R_3}{L_3}. \end{aligned}$$

Remembering the values of α , β , and γ given in equation (3), we can write equation (8) in the form :

$$\begin{aligned} & \omega_2^6(1 - \alpha^2 - \beta^2 - \gamma^2 - 2\alpha\beta\gamma) \\ & - \omega_2^4\{(1 - \alpha^2)n^2 + (1 - \beta^2)l^2 + (1 - \gamma^2)m^2 + k_1k_2 + k_2k_3 + k_3k_1\} \\ & + \omega_2^2(l^2m^2 + m^2n^2 + n^2l^2 + k_1k_2n^2 + k_2k_3l^2 + k_3k_1m^2) \\ & - l^2m^2n^2 = 0. \quad \dots \quad (9) \end{aligned}$$

This is a cubic equation in ω_2^2 , of which the three (real) roots may be termed ω_1^2 , ω_2^2 , and ω_3^2 . The formal solution of equation (9) presents considerable difficulties. In any actual case in which the numerical values of the various coefficients are known, the roots may be obtained simply by graphical methods. As will be shown later, equation (9) becomes more tractable if certain simplifications are applied. Solutions are thus obtained for certain cases rather less general than that represented by equation (9).

Returning to equation (7), we find that the damping coefficients q , r , and s have the following values :

$$q = \left[\frac{R_1}{C_2 C_3} + \frac{R_2}{C_1 C_3} + \frac{R_3}{C_1 C_2} \right. \\ \left. - \omega_1^2 \left(\frac{R_1 L_2 + L_1 R_2}{C_3} + \frac{R_2 L_3 + L_2 R_3}{C_1} + \frac{R_3 L_1 + L_3 R_1}{C_2} + R_1 R_2 R_3 \right) \right. \\ \left. - \omega_1^4 (R_1 L_2 L_3 + L_1 R_2 L_3 + L_1 L_2 R_3 - R_1 M_{23}^2 - R_2 M_{13}^2 - R_3 M_{12}^2) \right] \\ \div 2(\omega_1^2 \omega_2^2 - \omega_2^2 \omega_3^2 + \omega_3^2 \omega_1^2 - \omega_1^4),$$

$$r = \left[\frac{R_1}{C_2 C_3} + \frac{R_2}{C_1 C_3} + \frac{R_3}{C_1 C_2} \right. \\ \left. - \omega_2^2 \left(\frac{R_1 L_2 + L_1 R_2}{C_3} + \frac{R_2 L_3 + L_2 R_3}{C_1} + \frac{R_3 L_1 + L_3 R_1}{C_2} + R_1 R_2 R_3 \right) \right. \\ \left. - \omega_2^4 (R_1 L_2 L_3 + L_1 R_2 L_3 + L_1 L_2 R_3 - R_1 M_{23}^2 - R_2 M_{31}^2 - R_3 M_{12}^2) \right] \\ \div 2(\omega_1^2 \omega_2^2 + \omega_2^2 \omega_3^2 - \omega_3^2 \omega_1^2 - \omega_2^4),$$

$$s = \left[\frac{R_1}{C_2 C_3} + \frac{R_2}{C_1 C_3} + \frac{R_3}{C_1 C_2} \right. \\ \left. - \omega_3^2 \left(\frac{R_1 L_2 + L_1 R_2}{C_3} + \frac{R_2 L_3 + L_2 R_3}{C_1} + \frac{R_3 L_1 + L_3 R_1}{C_2} + R_1 R_2 R_3 \right) \right. \\ \left. - \omega_3^4 (R_1 L_2 L_3 + L_1 R_2 L_3 + L_1 L_2 R_3 - R_1 M_{23}^2 - R_2 M_{13}^2 - R_3 M_{12}^2) \right] \\ \div 2(-\omega_1^2 \omega_2^2 + \omega_2^2 \omega_3^2 + \omega_3^2 \omega_1^2 - \omega_3^4).$$

By means of the previously used abbreviations, the values of q , r , and s can be written in the following form :

$$\left. \begin{aligned} q &= [k_1 m^2 n^2 + k_2 n^2 l^2 + k_3 l^2 m^2 - \omega_1^2 \{n^2(k_1 + k_2) + l^2(k_2 + k_3) + m^2(k_3 + k_1) \\ &\quad + k_1 k_2 k_3\} - \omega_1^4 (k_1 + k_2 + k_3 - k_1 \beta^2 - k_2 \gamma^2 - k_3 \alpha^2)] \\ &\quad \div 2L_1 L_2 L_3 (\omega_1^2 \omega_2^2 - \omega_2^2 \omega_3^2 + \omega_3^2 \omega_1^2 - \omega_1^4), \\ r &= [k_1 m^2 n^2 + k_2 n^2 l^2 + k_3 l^2 m^2 - \omega_2^2 \{n^2(k_1 + k_2) + l^2(k_2 + k_3) + m^2(k_3 + k_1) \\ &\quad + k_1 k_2 k_3\} - \omega_2^4 (k_1 + k_2 + k_3 - k_1 \beta^2 - k_2 \gamma^2 - k_3 \alpha^2)] \\ &\quad \div 2L_1 L_2 L_3 (\omega_1^2 \omega_2^2 + \omega_2^2 \omega_3^2 - \omega_3^2 \omega_1^2 - \omega_2^4), \\ s &= [k_1 m^2 n^2 + k_2 n^2 l^2 + k_3 l^2 m^2 - \omega_3^2 \{n^2(k_1 + k_2) + l^2(k_2 + k_3) + m^2(k_3 + k_1) \\ &\quad + k_1 k_2 k_3\} - \omega_3^4 (k_1 + k_2 + k_3 - k_1 \beta^2 - k_2 \gamma^2 - k_3 \alpha^2)] \\ &\quad \div 2L_1 L_2 L_3 (-\omega_1^2 \omega_2^2 + \omega_2^2 \omega_3^2 + \omega_3^2 \omega_1^2 - \omega_3^4). \end{aligned} \right\} (10)$$

Then the general solution of equations (2) may be written in the form :

$$\left. \begin{aligned} e_1 &= a_1 e^{-qt} \cos(\omega_1 t + \theta_1) + a_2 e^{-rt} \cos(\omega_2 t + \theta_2) + a_3 e^{-st} \cos(\omega_3 t + \theta_3), \\ e_2 &= b_1 e^{-qt} \cos(\omega_1 t + \phi_1) + b_2 e^{-rt} \cos(\omega_2 t + \phi_2) + b_3 e^{-st} \cos(\omega_3 t + \phi_3), \\ e_3 &= c_1 e^{-qt} \cos(\omega_1 t + \psi_1) + c_2 e^{-rt} \cos(\omega_2 t + \psi_2) + c_3 e^{-st} \cos(\omega_3 t + \psi_3), \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} i_1 &= a_1 e^{-qt} \{ q \cos(\omega_1 t + \theta_1) + \omega_1 \sin(\omega_1 t + \theta_1) \} \\ &\quad + a_2 e^{-rt} \{ r \cos(\omega_2 t + \theta_2) + \omega_2 \sin(\omega_2 t + \theta_2) \} \\ &\quad + a_3 e^{-st} \{ s \cos(\omega_3 t + \theta_3) + \omega_3 \sin(\omega_3 t + \theta_3) \}, \\ i_2 &= b_1 e^{-qt} \{ q \cos(\omega_1 t + \phi_1) + \omega_1 \sin(\omega_1 t + \phi_1) \} \\ &\quad + b_2 e^{-rt} \{ r \cos(\omega_2 t + \phi_2) + \omega_2 \sin(\omega_2 t + \phi_2) \} \\ &\quad + b_3 e^{-st} \{ s \cos(\omega_3 t + \phi_3) + \omega_3 \sin(\omega_3 t + \phi_3) \}, \\ i_3 &= c_1 e^{-qt} \{ q \cos(\omega_1 t + \psi_1) + \omega_1 \sin(\omega_1 t + \psi_1) \} \\ &\quad + c_2 e^{-rt} \{ r \cos(\omega_2 t + \psi_2) + \omega_2 \sin(\omega_2 t + \psi_2) \} \\ &\quad + c_3 e^{-st} \{ s \cos(\omega_3 t + \psi_3) + \omega_3 \sin(\omega_3 t + \psi_3) \}, \end{aligned} \right\} \quad (12)$$

in which $a, b, c, \theta, \phi, \psi$ are arbitrary constants and the values of the other quantities are given by (9) and (10).

From equations (11) and (12) we see that in each of the coupled circuits the electromotive force and the current each consist of three damped harmonic oscillations, of which the dampings and periods can be calculated as above in terms of the couplings and the constants of the separate circuits. We also see that the same three oscillations occur in each of the three circuits, only the amplitudes and phase-angles being different.

An attempt to visualize the resultant of three simultaneous harmonic oscillations presents a more difficult problem than the familiar case of two simultaneous harmonic oscillations. Any means by which the significance of equations such as (11) and (12) can be grasped immediately without having recourse to the somewhat tedious process of drawing graphs of the oscillations under various conditions would be welcome. Reference may therefore be made to the recent work of Barton and Browning*, "Triple Pendulums with Mutual Interaction and the Analogous Electric Circuits.—I.," in which the traces of actual oscillations of the analogous case of three mechanical vibrators are exhibited. These bring out in a very clear manner the somewhat unexpected nature of the resultant vibration.

* Phil. Mag. Nov. 1920.

INITIAL CONDITIONS.

Let us now consider the form of the general solution for certain initial conditions of starting the oscillations in the coupled systems, and so evaluate the eighteen constants of integration a, b, c, θ, ϕ , and ψ , each with subscripts ^{1, 2, 3}, which are not altogether arbitrary, but are such that they satisfy equations (2) for any value of the time.

Suppose that, at the instant $t=0$, the circuits were without charge, but that a current i_0 is suddenly started in the first circuit.

We may then write

$$e_1 = e_2 = e_3 = 0, \quad i_1 = i_0, \quad i_2 = i_3 = 0 \quad \text{for } t = 0.$$

Introducing these conditions into equations (11) and (12), we obtain

$$\left. \begin{aligned} 0 &= a_1 \cos \theta_1 + a_2 \cos \theta_2 + a_3 \cos \theta_3, \\ 0 &= b_1 \cos \phi_1 + b_2 \cos \phi_2 + b_3 \cos \phi_3, \\ 0 &= c_1 \cos \psi_1 + c_2 \cos \psi_2 + c_3 \cos \psi_3, \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} i_0 &= a_1(q \cos \theta_1 + \omega_1 \sin \theta_1) + a_2(r \cos \theta_2 + \omega_2 \sin \theta_2) \\ &\quad + a_3(s \cos \theta_3 + \omega_3 \sin \theta_3), \\ 0 &= b_1(q \cos \phi_1 + \omega_1 \sin \phi_1) + b_2(r \cos \phi_2 + \omega_2 \sin \phi_2) \\ &\quad + b_3(s \cos \phi_3 + \omega_3 \sin \phi_3), \\ 0 &= c_1(q \cos \psi_1 + \omega_1 \sin \psi_1) + c_2(r \cos \psi_2 + \omega_2 \sin \psi_2) \\ &\quad + c_3(s \cos \psi_3 + \omega_3 \sin \psi_3). \end{aligned} \right\} \quad (14)$$

(13) and (14) are satisfied by $\theta = \phi = \psi = \frac{\pi}{2}$.

Inserting these values in (14), we have

$$\left. \begin{aligned} i_0 &= a_1 \omega_1 + a_2 \omega_2 + a_3 \omega_3, \\ 0 &= b_1 \omega_1 + b_2 \omega_2 + b_3 \omega_3, \\ 0 &= c_1 \omega_1 + c_2 \omega_2 + c_3 \omega_3. \end{aligned} \right\} \quad (15)$$

In order to be able to evaluate a, b , and c in terms of the constants of the circuits, we must obtain some other relation between them. The values of a_1, b_1, c_1 , etc., bear the same relation to one another as the values of A, B , and C in equation (4), and we can easily determine the ratios of A to B and A to C as below, thus providing the necessary supplementary relations required to evaluate a, b , and c .

It may be noted here that the fact that a_1, b_1, c_1 , etc., are related to each other in the same way as A, B , and C gives at once the information that $\theta_1 = \phi_1 = \psi_1, \theta_2 = \phi_2 = \psi_2, \theta_3 = \phi_3 = \psi_3$, agreeing with the conditions deduced from (13) and (14).

To determine the relations between A, B, and C, insert equation (4) in equation (2) and obtain

$$\begin{aligned} A(1 + R_1 C_1 p i - L_1 C_1 p^2) + B M_{12} C_1 p^2 &+ C M_{13} C_1 p^2 = 0, \\ A M_{21} C_2 p^2 &+ B(1 + R_2 C_2 p i - L_2 C_2 p^2) \\ &+ C M_{23} C_2 p^2 = 0, \\ A M_{31} C_3 p^2 &+ B M_{32} C_3 p^2 \\ &+ C(1 + R_3 C_3 p i - L_3 C_3 p^2) = 0. \end{aligned}$$

From these equations we find

$$\begin{aligned} \frac{B}{A} &= \frac{p^4(M_{12}M_{23}C_1C_2 - M_{12}L_3C_1C_3 - M_{31}M_{23}C_1C_2 + M_{31}L_2C_1C_2) + p^3i(M_{12}R_3C_1C_3 - M_{31}R_2C_1C_2) + p^2(M_{12}C_1 - M_{13}C_1)}{p^4(L_1M_{23}C_1C_2 - L_1L_3C_1C_3 + M_{13}M_{21}C_1C_2 + M_{31}^2C_1C_3) - p^3i(R_1L_3C_1C_3 - L_1R_3C_1C_3 + M_{23}R_1C_1C_2) + p^2(L_1C_1 + L_3C_3 + R_1R_3C_1C_3 - M_{23}C_2) - pi(R_3C_3 + R_1C_1) - 1} \\ \frac{C}{A} &= \frac{p^4(M_{12}M_{23}C_1C_2 - M_{21}L_3C_1C_3 - M_{31}M_{23}C_1C_3 + M_{13}L_2C_1C_2) + p^3i(M_{12}R_3C_1C_3 - M_{31}R_2C_1C_2) + p^2(M_{12}C_1 - M_{13}C_1)}{p^4(L_1M_{23}C_1C_3 - L_1L_2C_1C_2 + M_{12}M_{13}C_1C_3 + M_{21}^2C_1C_2) - p^3i(R_1L_2C_1C_2 - L_1R_2C_1C_2 + M_{23}R_1C_1C_3) + p^2(L_2C_2 + L_1C_1 + R_1R_2C_1C_2 - M_{23}C_3) - pi(R_1C_1 + R_2C_2) - 1} \end{aligned}$$

Let us rewrite the above relations in the form

$$\left. \begin{aligned} B &= EA, \\ C &= FA, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in which each of E and F have three values, which we will term $E_1, E_2, E_3, F_1, F_2, F_3$, corresponding to the three values $\omega_1, \omega_2, \omega_3$ of p .

Then for the initial conditions considered above we find that the constants of integration have the following values :

$$\left. \begin{aligned} a_1 &= \frac{i_0}{\omega_1} \left(1 - \frac{E_3F_1 - E_1F_3 + E_1F_2 - E_2F_1}{E_2F_3 - E_3F_2 + E_3F_1 - E_1F_3 + E_1F_2 - E_2F_1} \right), \\ a_2 &= \frac{i_0}{\omega_2} \left(\frac{E_3F_1 - E_1F_3}{E_2F_3 - E_3F_2 + E_3F_1 - E_1F_3 + E_1F_2 - E_2F_1} \right), \\ a_3 &= \frac{i_0}{\omega_3} \left(\frac{E_1F_2 - E_2F_1}{E_2F_3 - E_3F_2 + E_3F_1 - E_1F_3 + E_1F_2 - E_2F_1} \right), \\ b_1 &= E_1a_1, & b_2 &= E_2a_2, & b_3 &= E_3a_3, \\ c_1 &= F_1a_1, & c_2 &= F_2a_2, & c_3 &= F_3a_3. \end{aligned} \right\} \quad (17)$$

SPECIAL CASES.

Let us now apply the theory developed above to various special cases of the configurations of the coupled systems.

Case I.

We will now obtain the solution of equation (9) for the following case. Let the oscillations be undamped, this case resulting when the resistances in the circuits are negligibly small either actually or virtually through the use of some means such as the thermionic valve; also let circuits 2 and 3 be identical in all respects, and let the couplings between circuits 1 and 2 and 1 and 3 be equal.

For this special case we may write

$$R_1 = R_2 = R_3 = 0,$$

which gives

$$q = r = s = k_1 = k_2 = k_3 = 0,$$

$$m = n \quad \text{and} \quad \alpha = \gamma.$$

The solution of equation (9) can be more easily found if instead of using the equation itself we use the partially expanded form of the determinant from which it is obtained.

Eliminating the damping from the determinant and writing ω for p , we obtain

$$\omega^2 - l^2\{(\omega^2 - m^2)^2 - \beta^2\omega^4\} - 2\alpha^2\omega^4\{\beta\omega^2 + (\omega^2 - m^2)\} = 0,$$

which can be re-written

$$\begin{aligned} &(\omega^2(1 + \beta) - m^2)\{\omega^4(1 - \beta - 2\alpha^2) \\ &\quad - \omega^2(l^2(1 - \beta) + m^2) + l^2m^2\} = 0. \end{aligned}$$

From this we see that the three roots are

$$\left. \begin{aligned} \omega_1^2 &= \frac{l^2(1 - \beta) + m^2 + \sqrt{(l^2(1 - \beta) + m^2)^2 - 4(1 - \beta - 2\alpha^2)l^2m^2}}{2(1 - \beta - 2\alpha^2)}, \\ \omega_2^2 &= \frac{l^2(1 - \beta) + m^2 - \sqrt{(l^2(1 - \beta) + m^2)^2 - 4(1 - \beta - 2\alpha^2)l^2m^2}}{2(1 - \beta - 2\alpha^2)}, \\ \omega_3^2 &= \frac{m^2}{1 + \beta}. \end{aligned} \right\} \quad (18)$$

For the oscillations to occur these roots must be real, the condition for this being, since $1 + \beta$ cannot be negative,

$$(l^2(1 - \beta) + m^2)^2 - 4(1 - \beta - 2\alpha^2)l^2m^2 \geq 0.$$

The solutions for the cases in which $l=m$, $\beta=\gamma$, and $l=n$, $\alpha=\beta$ can be obtained from (18) by a cyclical interchange of the letters.

Eliminating the contribution of the third circuit from (18) we find the roots become

$$\omega_1^2 = \frac{l^2 + m^2 + \sqrt{(l^2 - m^2)^2 - 4\alpha^2 l^2 m^2}}{2(1 - \alpha^2)},$$

$$\omega_2^2 = \frac{l^2 + m^2 - \sqrt{(l^2 - m^2)^2 - 4\alpha^2 l^2 m^2}}{2(1 - \alpha^2)},$$

the well-known result for the case of two undamped coupled circuits.

Using equation (17) in the abbreviated form

$$\alpha_1 = \frac{i_0}{\omega_1} \left(1 - \frac{f+g}{f+g+h} \right),$$

$$\alpha_2 = \frac{i_0}{\omega_2} \left(\frac{f}{f+g+h} \right),$$

$$\alpha_3 = \frac{i_0}{\omega_3} \left(\frac{g}{f+g+h} \right),$$

we can now write the form which the general solution assumes for the special case here considered and for the initial conditions of the previous section :

$$\left. \begin{aligned} e_1 &= -i_0 \left\{ \frac{1}{\omega_1} \left(1 - \frac{f+g}{f+g+h} \right) \sin \omega_1 t + \frac{1}{\omega_2} \left(\frac{f}{f+g+h} \right) \sin \omega_2 t \right. \\ &\quad \left. + \frac{1}{\omega_3} \left(\frac{g}{f+g+h} \right) \sin \omega_3 t \right\}, \\ e_2 &= -i_0 \left\{ \frac{E_1}{\omega_1} \left(1 - \frac{f+g}{f+g+h} \right) \sin \omega_1 t + \frac{E_2}{\omega_2} \left(\frac{f}{f+g+h} \right) \sin \omega_2 t \right. \\ &\quad \left. + \frac{E_3}{\omega_3} \left(\frac{g}{f+g+h} \right) \sin \omega_3 t \right\}, \\ e_3 &= -i_0 \left\{ \frac{F_1}{\omega_1} \left(1 - \frac{f}{f+g+h} \right) \sin \omega_1 t + \frac{F_2}{\omega_2} \left(\frac{f}{f+g+h} \right) \sin \omega_2 t \right. \\ &\quad \left. + \frac{F_3}{\omega_3} \left(\frac{g}{f+g+h} \right) \sin \omega_3 t \right\}. \end{aligned} \right\} \quad (19)$$

Case II.

In the case in which all three circuits are identical and undamped and the coupling coefficients are all equal, equation (9) reduces to

$$\{\omega^2(1 + \alpha) - l^2\}^2 \{\omega^2(1 - 2\alpha) - l^2\} = 0,$$

thus giving as the three roots

$$\left. \begin{aligned} \omega_1^2 &= \frac{l^2}{1+\alpha}, \\ \omega_2^2 &= \frac{l^2}{1+\alpha}, \\ \omega_3^2 &= \frac{l}{1-2\alpha}. \end{aligned} \right\} \dots \dots \dots (20)$$

It will thus be seen that in this case there are only *two* periods of oscillation proper to the coupled system.

Case III.

As a check on the theory, let us now consider the case in which the couplings are put equal to zero, and also, to further simplify matters, let the circuits be undamped.

Then equation (9) gives the following values for the frequencies :

$$\omega_1^2 = l^2, \quad \omega_2^2 = m^2, \quad \omega_3^2 = n^2, \quad \dots \dots (21)$$

and equations (18) give

$$a_1 = \frac{i_0}{\omega_1}, \quad a_2 = a_3 = 0, \quad E = F = 0.$$

Thus the case reduces to a single oscillation in the first circuit given by

$$e_1 = -\frac{i_0}{l} \sin lt, \quad \dots \dots \dots (22)$$

as was to be expected, since the circuits are now entirely separate.

SUMMARY.

In the present paper the mathematical theory of the oscillations in three coupled electric circuits is developed by a simple method. It is shown that the problem involves the solution of a certain cubic equation involving the squares of the frequencies of oscillation proper to the system. This equation is obtained, and is then solved for certain special cases.

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