For the parabola, a and b are infinite, while $b^2/a = 2a'$, and $r^2 = 4a'z'$.

Thus

$$\left.\begin{aligned} \delta\phi &= r^s \cos s\phi \\ \delta r &= s r^{s+1} \sin s\phi \\ \delta z' &= -2\,(s+1)\, a' r^s \sin s\phi \end{aligned}\right\} \quad \text{..................(66).}$$

A Geometrical Representation of a System of two Binary Cubics and their Associated Forms.

By W. R. Westropp Roberts, M.A.

[*Read November* 10*th*, 1881.]

1. The object of this paper is to invest with a certain geometrical meaning the algebraic forms arising in a system of two binary cubics; that is, to construct geometrically points which shall represent the linear, quadratic, cubic, and quartic covariants of the system, and to express the vanishing of invariants by geometrical relations connecting such points. We may consider any binary quantic as derived from a system of three surfaces by assuming

$$X = \phi_1(x_1 x_2),\quad Y = \phi_2(x_1 x_2),\quad Z = \phi_3(x_1 x_2),\quad W = \phi_4(x_1 x_2),$$

equations which in themselves imply, by elimination of x_1 and x_2, two fixed relations between X, Y, Z, W, denoting a fixed curve in space, while the given binary quantic equated to zero, enables us to obtain a third such relation. The transformation here employed is one in which ϕ_1, ϕ_2, ϕ_3, ϕ_4, are cubic functions of x_1 and x_2. By linear transformation, this substitution is reducible to

$$X = x_1^3,\quad Y = x_1^2 x_2,\quad Z = x_1 x_2^2,\quad W = x_2^3,$$

the fixed curve in this case being evidently a twisted cubic. The equation of an osculating plane of the curve, the parameter of the corresponding point of which is $x_1 : x_2$, being (Salmon's *Geometry of Three Dimensions*, Art. 368)

$$Xx_2^3 - 3Yx_1^2 x_2^2 + 3Zx_1^2 x_2 - x_1^3 W = 0,$$

the parameters answering to osculating planes through any point O, the coordinates of which are X', Y', Z', W', are given by the equation

$$x_2^3 X' - 3x_2^2 x_1 Y' + 3x_1 x_2^2 Z' - x_1^3 W' = 0,$$

the points of contact lying in the plane

$$XW' - 3YZ' + 3ZY' - WX' = 0.$$

But this plane passes through O, the given point. To any plane there corresponds a point O, the point of intersection of the osculating planes at the points where it meets the curve, a point which plays an important part in the following investigation. It follows readily, from the last equation, that if a point O lie in the plane corresponding to a point O', then O' lies in the plane corresponding to O, and the line joining O and O' possesses a certain invariant relation to the curve. Also, the locus of corresponding points of planes passing through a given line is a right line, which may be called the corresponding line to the given one, their relation being reciprocal.

(2.) Let us consider the binary cubic

$$f = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3 = a_x^3 = b_x^3 = \&c.,$$

adopting Clebsch's notation. By our transformation, the binary cubic is transformed into a plane F, the equation of which is

$$a_0 X + 3a_1 Y + 3a_2 Z + a_3 W = 0.$$

Now it can be easily shown that the corresponding point of this plane is given by the equations

$$X = a_3, \quad Y = -a_2, \quad Z = a_1, \quad W = -a_0.$$

We shall call it the point O.

(3.) It is known that through any point O in space can be drawn one chord meeting the curve in two points. Let us now determine these points, being given the point O.

The coordinates of the line joining the points on the curve, the parameters of which are $x_1 : x_2$, $y_1 : y_2$, respectively, are easily found to be

$$a = R_0 R_2, \quad f = 4R_1^2 - R_0 R_2,$$
$$b = 2R_1 R_2, \quad g = -2R_1 R_0,$$
$$c = R_2^2, \quad h = R_0^2,$$

where $\quad R_0 = x_2 y_2, \quad -2R_1 = x_1 y_2 + y_1 x_2, \quad R_2 = x_1 y_1.$

Now take two equations of the chord, viz.,

$$aX + bY + cZ = 0,$$
$$hY - gZ + aW = 0,$$

and these furnish us with

$$R_0 X + 2R_1 Y + R_2 Z = 0,$$
$$R_0 Y + 2R_1 Z + R_2 W = 0.$$

If we now suppose $X = a_3$, $Y = -a_2$, $Z = a_1$, $W = -a_0$, the equation determining the parameters of the two points, in which the chord through O meets the curve, is

$$(a_0 a_2 - a_1^2) x_1^2 + (a_0 a_3 - a_1 a_2) x_1 x_2 + (a_1 a_3 - a_2^2) x_2^2 = 0.$$

But this is the equation of the Hessian of f, or

$$\Delta_x^2 = \Delta_0 x_1^2 + 2\Delta_1 x_1 x_2 + \Delta_2 x_2^2 = 0.$$

Thus, then, to the Hessian of f correspond the two points in which the chord through O meets the curve.

(4.) I shall now show that the plane F passes through the line of intersection of the osculating planes at the two Hessian points on the curve. To prove this, let us find the equation of the plane through O and this line.

Let X', Y', Z', W'; X'', Y'', Z'', W'' be the coordinates of the two Hessian points respectively, then the equation of such a plane must be of the form

$$\lambda\,(XW'-3YZ'+3ZY'-WX')-\mu\,(XW''-3YZ''+3ZY''-WX'')=0,$$

where $\qquad \lambda = a_0 X''+3a_1 Y''+3a_2 Z''+a_3 W'',$

$$\mu = a_0 X'+3a_1 Y'+3a_2 Z'+a_3 W'.$$

Remembering that $\qquad x_1 y_1 = a_1 a_3 - a_2^2,$

$$-(x_1 y_2 + y_1 x_2) = a_0 a_3 - a_1 a_2,$$

$$x_2 y_2 = a_0 a_2 - a_1^2,$$

we find, dividing by a factor $(x_1 y_2 - y_1 x_2)$, that the equation becomes

$$R\,(a_0 X+3a_1 Y+3a_2 Z+a_3 W)=0,$$

R being the discriminant of the binary cubic.

Hence the plane F passes through the line of intersection of the osculating planes at the Hessian points. If, then, the chord through O meet the curve in real points, the plane F must meet it in two imaginary points, since the binary cubic is then the difference of two cubes, and has but one real factor.

(5.) Let us now investigate the plane

$$\lambda\,(XW'-3YZ'+3ZY'-WX')+\mu\,(XW''-3YZ''+3ZY''-WX'')=0,$$

and we find, after a few obvious reductions, that it becomes

$$R\,\{(a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3)\,X + 3\,(a_0 a_1 a_3 + a_1^2 a_2 - 2a_0 a_2^2)\,Y$$
$$+3\,(2a_1^2 a_3 - a_0 a_2 a_3 - a_1 a_2^2)\,Z + (3a_1 a_2 a_3 - a_0 a_3^2 - 2a_2^3)\,W\} = 0;$$

but this is the cubic covariant plane Q.

We shall denote its corresponding point by ω.

We see, then, that the planes F and Q are harmonically conjugate with regard to the osculating planes at the Hessian points, and that, since the Hessian of the binary Q_x^3 has the same factors as Δ_x^2, the corresponding point ω of the plane Q lies on the chord through O, and is the harmonic conjugate of O with regard to the Hessian points.

Hence to the cubic covariant Q_x^3 corresponds the plane through the intersection of the osculating planes at the Hessian points and the harmonic conjugate on the chord through O of the same point with regard to the Hessian points. Again, if the point O lie at one side of the developable generated by tangent lines to the curve from which a real chord can be drawn, two of the roots of the binary cubic are imaginary; if the point O lies on the developable, two roots are equal; and if at the other side from which a real chord cannot be drawn, all the roots are real.

(6.) We can now discuss the system of two binary cubics and their associated forms, and shall adopt the notation of Clebsch in our investigation. Let, then, f and ϕ denote the two cubics, Δ and ∇ their Hessians, Q and K their cubic covariants, R and P their discriminants.*

For convenience, I give a list of the invariants and covariants of a system of two cubics, of which there are, according to Clebsch and Gordan, 28 forms in all, and which I discuss geometrically.

Prof. Cayley has, however, drawn my attention to the fact that two of the linear covariants ($xa^4\alpha^3$ and $xa^3\alpha^4$) of Clebsch and Gordan, have been shown to be non-fundamental by Prof. Sylvester. See Sylvester's "Tables of the Generating Functions," *American Journal of Mathematics*, t. ii. (1879).

Table of Covariants of a System of Two Cubics.

7 Invariants—

$a\alpha \; ; \; a^4 , \; a^3\alpha , \; a^2\alpha^2 , \; a\alpha^3 , \; \alpha^4 \; : \; a^3\alpha^3 .$

$(a\alpha)^3; \; (\Delta\Delta')^2, (\Delta\Theta)^2, (\Delta\nabla)^2, (\nabla\Theta)^2, (\nabla\nabla')^2 : \; (\Theta\Delta)(\Theta\nabla)(\Delta\nabla).$

6 Linear Covariants—

$xa^2\alpha , \; xa\alpha^2 \; ; \; xa^4\alpha , \; xa^3\alpha^2 , \; xa^2\alpha^3 , \; xa\alpha^4 .$

$(\Delta\alpha)^2\alpha_x, \; (\nabla a)^2 a_x; \; (\alpha\Delta)^2(\alpha\Delta')\Delta'_x, \; (Q\nabla)^2 Q_x, \; (K\Delta)^2 K_x, \; (a\nabla)^2(a\nabla')\nabla'_x.$

6 Quadratic Covariants—

$x^2a^2, \; x^2a\alpha, \; x^2\alpha^2, \; x^2a^3\alpha , \; x^2a^2\alpha^2 , \; x^2a\alpha^3 .$

$\Delta_x^2, \; \Theta_x^2, \; \nabla_x^2, \; (\Delta\Theta)\Delta_x\Theta_x, \; (\Delta\nabla)\Delta_x\nabla_x, \; (\nabla\Theta)\nabla_x\Theta_x.$

6 Cubic Covariants—

$x^3a, \; x^3\alpha ; \; x^3a^3, \; x^3a^2\alpha , \; x^3a\alpha^2 , \; x^3\alpha^3.$

$a_x^3, \; \alpha_x^3 ; \; Q_x^3, \; (\Delta\alpha)\Delta_x\alpha_x^2, \; (\nabla a)\nabla_x a_x^2, \; K_x^3 .$

1 Quartic Covariant—

$x^4a\alpha .$

$(a\alpha)a_x^2\alpha_x^2 .$

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* The reader is referred to Clebsch's "Theorie der Binären Algebraischen Formen, § 61.—Vollständiges System zweier Cubischen Formen."

In this table I have identified, at the suggestion of Prof. Cayley, the covariants given by Sylvester with the notation of Clebsch and Gordan.

We have then

$$f \equiv a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3 = a_x^3 = b_x^3 = \&c.,$$

$$\phi = \alpha_0x_1^3 + 3\alpha_1x_1^2x_2 + 3\alpha_2x_1x_2^2 + \alpha_3x_2^3 = \alpha_x^3 = \beta_x^3 = \&c.$$

There is one quartic form $(a\alpha)\, a_x^2\alpha_x^2$ which we shall first discuss. The coordinates of the lines of intersection of the planes F and Φ are

$$a = a_0\alpha_3 - a_3\alpha_0, \qquad f = 9\,(a_1\alpha_2 - a_2\alpha_1),$$
$$b = 3\,(a_1\alpha_3 - \alpha_3 a_1), \qquad g = 3\,(a_2\alpha_0 - a_0\alpha_2),$$
$$c = 3\,(a_2\alpha_3 - \alpha_3 a_2), \qquad h = 3\,(a_0\alpha_1 - a_1\alpha_0)\,;$$

while the coordinates of the corresponding line, or the line joining the the points O and O', are

$$a = a_1\alpha_2 - a_2\alpha_1, \qquad f = a_0\alpha_3 - a_3\alpha_0,$$
$$b = a_1\alpha_3 - a_3\alpha_1, \qquad g = \alpha_2 a_0 - \alpha_0 a_2,$$
$$c = a_2\alpha_3 - a_3\alpha_2, \qquad h = a_0\alpha_1 - a_1\alpha_0.$$

Now the coordinates of a tangent line to the curve are

$$a = x_1^2x_2^2, \qquad f = 3x_1^2x_2^2,$$
$$b = -2x_1^3x_2, \qquad g = 2x_1x_2^3,$$
$$c = x_1^4, \qquad h = x_2^4.$$

Forming, then, the well-known condition that the line (F, Φ) may meet a tangent line, we find

$$x_1^4(a_0\alpha_1 - \alpha_0a_1) + 2x_1^3x_2(a_0\alpha_2 - a_2\alpha_0) + x_1^2x_2^2\{a_0\alpha_3 - a_3\alpha_0 + 3(a_1\alpha_2 - a_2\alpha_1)\}$$
$$+ 2x_1x_2^3(a_1\alpha_3 - \alpha_1a_3) + x_2^4(a_2\alpha_3 - \alpha_3a_2) = 0,$$

or

$$(a\alpha)\, a_x^2\alpha_x^2 = 0.$$

Hence to the Jacobian of f and ϕ correspond the four points on the curve, the tangents at which meet the line F, Φ. From the values above given of the coordinates of the line OO', we see that the four tangents also meet this line. The Jacobian $(a\alpha)\, a_x^2\alpha_x^2$ is thus geometrically shown to be a combinantive covariant, since it depends only on the position of the line (F, Φ).

(7.) In addition to the forms f and ϕ, and their covariants Q and K, there are two forms of the third degree in the variables

$$(a\nabla)\, a_x^2\nabla x, \qquad (a\Delta)\, a_x^2\Delta_x,$$

which we now discuss. Since the cubic covariant is the evectant of the discriminant, it follows easily that the cubic covariant plane Q is the polar plane of O with regard to the developable. By taking the polar plane of a point on the line OO', we find it to be of the form

$$\lambda^3 Q + \lambda^2 \mu q + a\mu^2 k + \mu^3 K = 0.$$

Hence we have two new planes q and k, giving rise to two cubic covariants in the binary system, the leading terms of which are

$$q_0 = a_0 \frac{dQ_0}{da_0} + a_1 \frac{dQ_0}{da_1} + a_2 \frac{dQ_0}{da_2} + a_3 \frac{dQ_0}{da_3},$$

$$k_0 = a_0 \frac{dK_0}{d\alpha_0} + a_1 \frac{dK_0}{d\alpha_1} + a_2 \frac{dK_0}{d\alpha_2} + a_3 \frac{dK_0}{d\alpha_3},$$

where
$$Q_0 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$

$$K_0 = \alpha_0^2 \alpha_3 - 3\alpha_0 \alpha_1 \alpha_2 + 2\alpha_1^3.$$

Now it is easy to show that

$$q_0 = 3\{\alpha_0(a_0 a_3 - a_1 a_2) - 2\alpha_1(a_0 a_2 - a_1^2)\} + a_0\{a_0\alpha_3 - \alpha_0 a_3 - 3(a_1\alpha_2 - \alpha_1 a_2)\}$$

$$= 6(\alpha_0 \Delta_1 - \alpha_1 \Delta_0) + (a\alpha)^3 a_0;$$

hence
$$q_x = 6(\alpha\Delta)\, \alpha^2 \Delta_x + (a\alpha)^3 f,$$

and in the same way $k_x = 6(a\nabla)\, a_x \nabla_x - (a\alpha)^3 \phi.$

We have now expressed q_x^3 and k_x^3 in terms of Clebsch's forms, and can represent them as follows. The covariant q_x^3 is transformed into a plane which is the polar plane of O' with regard to the polar quadric of O.

In like manner, k_x^3 is transformed into a plane which is the polar plane of O with regard to the polar quadric of O'. These theorems are immediate algebraic consequences of the method of generation of q and k.

(8.) We now discuss Clebsch's two linear covariants p_x and π_x. If through a point X, Y, Z, W, and two points on the curve the parameters of which are determined by the equation

$$R_0 x_1^2 + 2R_1 x_1 x_2 + R_2 x_2^2 = R_x^2 = 0,$$

we draw a plane, it will meet the curve in a third point determined by the equation

$$x_1(R_0 Y + 2R_1 Z + R_2 W) + x_2(R_0 X + 2R_1 Y + R_2 Z) = 0,$$

and if X, Y, Z, W be the corresponding point of a plane resulting by transformation from a binary cubic form ρ_x^3, the above equation reduces to $(R\rho)^2 \rho_x = 0$. Now let us suppose $R_x^2 = \Delta_x^2$, and $\rho_x^3 = \phi$, we find

$$(\Delta\alpha)^2 \alpha_x = 0, \text{ or } p_x = 0.$$

Hence the linear covariant p_x is represented by the point in which a plane through O', and containing the chord through O, meets the curve, a similar construction representing π_x; where $\pi_x = (\nabla a)^2 a_x$.

(9.) We now turn to the quadratic covariants.

The equation determining the parameters of the points in which a chord through any point on the line OO' meets the curve, is

$$\lambda^2 \Delta + 2\lambda\mu\Theta + \mu^2 \nabla = 0.$$

The forms Δ and ∇ have been discussed before, and it remains to attach a geometrical meaning to $\Theta_x^2 \equiv (aa)^2 a_x a_x$. Now, from any given point on the curve can be drawn two chords to meet the line OO', the above equation in $\lambda : \mu$ determining the points of meeting on OO'; if, then, the coefficient of $\lambda\mu$ vanishes, the chords drawn from the given point meet the line OO' in two points harmonically conjugate with regard to O and O'.

Hence Θ is represented geometrically by the two points on the curve, the chords from which to meet the line OO' divide it harmonically.

(10.) Let us now determine the tangent lines to the curve which meet a line through X, Y, Z, W, and a point on the curve the parameter of which is $x_1' : x_2'$.

The coordinates of a line through X, Y, Z, W, and a point $x_1' : x_2'$ on the curve, are

$$a = x_1'^2 x_2'^2 Z - x_1' x_2'^2 Y, \qquad f = x_1'^3 W - X x_2'^3,$$
$$b = x_1' x_2'^2 X - x_1'^3 Z, \qquad g = x_1'^2 x_2' W - X x_2'^3,$$
$$c = x_1'^3 Y - x_1'^2 x_2' X, \qquad h = x_1 x_2'^2 W - Z x_2'^3.$$

Forming the condition that this line may meet a tangent line, the coordinates of which are given by (6), and dividing by a factor $(x_1 x_2' - x_2 x_1')^2$, we find

$$x_1^2 (W x_1' - Z x_2') + 2x_1 x_2 (Y x_2' - Z x_1') + (Y x_1' - X x_2') x_2^2 = 0.$$

Suppose now the point (X, Y, Z, W) to be the point O, and the point on the curve the point p_x or $(\Delta a)^2 a_x$, the above equation becomes

$$x_1^2 (a_0 x_1' + a_1 x_2') + 2x_1 x_2 (a_1 x_1' + a_2 x_2') + x_2^2 (a_2 x_1' + a_3 x_2') = 0,$$

or
$$(ap)\, a_x^2 = 0.$$

We can easily express this in terms of Clebsch's forms as follows:

$$\Theta_x^2 = (aa)^2 a_x a_x,$$

hence
$$2(\Theta\Delta)\,\Theta_x = (aa)^2 \{a_x (a\Delta) + a_x (a\Delta)\},$$

or
$$2(\Theta\Delta)\,\Theta_x \Delta_x = (aa)(a\Delta)(aa)\,\Delta_x a_x + (aa)^2 (a\Delta)\, a_x \Delta_x$$
$$= a_x (aa)(a\Delta)\{a_x (\Delta a) + a_x (a\Delta)\} + (aa)^2 (a\Delta)\, a_x \Delta^x$$
$$= -a_x (aa)(\Delta a)^2 \text{ (See Clebsch, § 34).}$$

Hence $$(pa)\, a_x^2 = 2\,(\Theta\Delta)\,\Theta_x\,\Delta_x.$$

The three remaining quadratic covariants are constructed as follows: Through O and the point p on the curve draw a line; then the two points on the curve, the tangents at which meet this line, represent the quadratic covariant $(\Theta\Delta)\,\Theta_x\,\Delta_x$. In like manner, the covariant $(\Theta\nabla)\,\Theta_x\,\nabla_x$ is represented, while the form $(\Delta\nabla)\,\Delta_x\nabla_x$ is represented by the points the tangents at which meet the line $O\pi$ or the line $O'p$.

(11.) We shall now discuss the six remaining linear covariants. We saw that the linear covariant obtained by drawing a plane through the corresponding point of a plane resulting by transformation from a binary form ρ_x^3, and two points on the curve given by the equation $R_x^2 = 0$, was $(R\rho)^2\,\rho_x$. If we substitute K for ρ and Δ for R, we obtain $(\Delta K)^2\,K_x$, which is Clebsch's form.

To represent the covariant $(\Delta K)^2\,K_x$, draw a plane through ω', the corresponding point of the plane K, and containing the chord through O: this plane will meet the curve in the required point. By a similar construction the covariant $(\nabla Q)^2\,Q_x$ is represented.

(12.) We now show how to represent the forms

$$(\pi\nabla)\,\nabla_x, \text{ and } (p\Delta)\,\Delta_x.$$

We have shown how to represent the two points corresponding to the form $(\Theta\Delta)\,\Theta_x\,\Delta_x$, which we shall call for the moment R_x^2, and if in the form $(R\rho)^2\rho_x$ we put a for ρ, we have $(Ra)^2\,a_x$.

Now	$R_x^2 = (\Theta\Delta)\,\Theta_x\,\Delta_x,$
therefore	$(Ra)^2 = (\Theta\Delta)(\Theta a)(\Delta a)$;
but	$(a\Delta)\,a_y\,a_x\,\Delta_y = (a\Delta)a_y^2\,\Delta_x,$
hence	$(Ra)^2\,a_x = (a\Delta)(a\Theta)^2\,\Delta_x.$
Now	$p_x = -2\,(\Theta a)^2\,a_x,$
therefore	$(p\Delta)\,\Delta_x = -2\,(\Theta a)^2\,(a\Delta)\,\Delta_x,$
or	$(Ra)^2\,a_x = -\tfrac{1}{2}\,(p\Delta)\,\Delta_x.$

Hence draw a plane through O and the two points given by the equation $(\Delta\Theta)\,\Delta_x\Theta_x = 0$, and already constructed, which will meet the curve in a third point representing the linear covariant $(p\Delta)\,\Delta_x$. In the same way the form $(\pi\nabla)\nabla_x$ is represented.

The two remaining linear covariants may be represented as follows. Suppose a plane drawn through O' and containing the chord through O, this plane will meet the curve in points given by the equation $\Delta_x^2 p_x$, which we shall call $\rho_x^3 = \Delta_x^2 p_x$; through the corresponding point of this plane, and the chord through O', draw a plane which will meet the curve in a third point given by the equation

$$(\rho\nabla)^2\rho_x = 0.$$

Now $$\rho_y^3 = \Delta_y^2\, p_y,$$

therefore $$3\rho_y^2\rho_x = 2\Delta_y\Delta_x p_y + \Delta_y^2 p_x,$$

or $$3\,(\rho\nabla)^2\rho_x = 2\,(\Delta\nabla)\,\Delta_x\,(p\nabla) + (\Delta\nabla)^2 p_x,$$

which expresses $(\rho\nabla)^2\rho_x$ in terms of Clebsch's forms.

In the same way we can represent the form

$$2\,(\pi\Delta)(\Delta\nabla)\,\nabla_x + (\nabla\Delta)^2\pi_x.$$

(13.) We now turn to the invariants of which there are seven, two being combinantive. We shall first discuss the combinants. Let us seek the condition that the point O may lie in the plane Φ, and we know, by (1), that when this condition is fulfilled O' lies in the plane F.

Expressing the condition that the point $(a_3, -a_2, a_1, -a_0)$ may lie in the plane $$a_0X + 3a_1Y + 3a_2Z + a_3W = 0,$$ we find $$(aa)^3 = 0 \text{ or } J = 0.$$

Hence J vanishes when O lies in the plane Φ; but in this case the line (F, Φ) becomes identical with its corresponding line OO', and this relation is evidently combinantive, since it depends only on the position of the line (F, Φ).

Clebsch denotes by $\Omega = 0$ the condition that must be satisfied in order that $f + k\phi$ may become a perfect cube; hence, when it vanishes, it will be possible to draw an osculating plane through the line (F, Φ). In this case, it is easy to see that the corresponding line meets the curve; and, for the same reason as before, this relation is combinantive.

Now Clebsch has shown analytically that $2\Omega = (p\pi)$, and we can show geometrically that when Ω vanishes, or the line OO' meets the curve in a point O'', that the points p and π coincide.

The point p is the point in which a plane containing O' and the chord through O meets the curve again; but this plane contains the line OO', which by hypothesis meets the curve in O''; hence, when $\Omega = 0$, p coincides with O''; and in like manner π is shown to coincide with O'', and therefore with p.

The two discriminants have been discussed before as the conditions that the points O and O' should lie on the developable, respectively.

We now come to the invariants

$$\Sigma = (aK)^3 = (\Theta\nabla)^2, \quad S = (aQ)^3 = (\Theta\Delta)^2.$$

Now it is easy to see that Σ is the condition that O should lie on the plane K, or that the line (F, K) should coincide with its corresponding line. S, in like manner, is the condition that O' should lie in the plane Q.

It remains to discuss the invariant $T = (\Delta\nabla)^2$. The vanishing of this invariant may be expressed in different ways; when $T = 0$, the planes through any chord and the four points in which the chords through O and O' meet the curve, form a harmonic system; again, when $T = 0$, the polar quadric of O passes through O', and *vice versâ*.

In this investigation I have adopted throughout the notation and procedure of Clebsch, as it lends itself more readily than other methods to the identification of binary forms with their geometrical significations.

NOTE.—Since this paper was written, I have found an article in the *Quarterly Mathematical Journal*, by Prof. Cayley, Vol. x, "On a Geometrical Interpretation of a Binary Cubic and its Covariants," in which Prof. Cayley gives to the Hessian of the cubic a geometrical interpretation which is practically the same as that given in this paper.

On Tangents to a Cubic forming a Pencil in Involution.

By R. A. ROBERTS, B.A.

[*Read Nov.* 10*th*, 1881.]

I propose to determine, in this paper, the locus of the vertex of a pencil in involution circumscribed about a non-singular cubic curve. The condition that a binary sextic should have its roots in involutions being of the fifteenth degree in the coefficients, it follows that the locus is of the forty-fifth degree (see Salmon's "Higher Plane Curves," Art. 94, and "Higher Algebra," Art. 251). The method which I adopt to obtain the equation of the locus will show that it breaks up into twelve cubics and nine lines.

I commence by showing that, if a binary sextic be written in the form $u^2 - kv^3$, where u is a cubic, and v a quadratic, and M the skew invariant of u and v vanish (Salmon's "Higher Algebra," Art. 198), then the roots of the sextic are in involution. If x, y be the factors of the Hessian of u, we may write $u \equiv x^3 + y^3$, $v \equiv ax^2 + by^2 + 2hxy$, when the condition $M = 0$ gives $a^3 - b^3 = 0$. Taking $a = b$, and putting $x^2 + y^2 = u_1$, $xy = u_2$, the sextic can be written

$$(u_1 + 2u_2)(u_1 - u_2)^2 - k(au_1 + 2hu_2)^3,$$

showing that the roots are in involution.

Referring the cubic to a canonical triangle, we have

$$U \equiv x^3 + y^3 + z^3 + 6mxyz,$$