

ON A CONNEXION BETWEEN THE FUNCTIONS OF HERMITE* AND THE FUNCTIONS OF LEGENDRE

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Introductory.

For positive integral values of n , the Hermite function is a polynomial and is defined by the relation

$$U_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Thus, if n is a positive integer,

$$U_n(x) = \frac{n!}{2\pi i} \int_D \frac{e^{x^2-t^2}}{(t-x)^{n+1}} dt,$$

where D is a simple contour containing x .

Hence
$$U_n(x) = \frac{n!}{2\pi i} \int_C e^{-2ux-u^2} u^{-n-1} du,$$

where C is a simple contour enclosing the origin.

It can easily be proved that $U_n(x)$ satisfies the differential equation

$$\frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + 2nu = 0. \quad (\text{A})$$

Whittaker (see footnote) then shows that for *all* values of n ,

$$\frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} e^{-2ux-u^2} u^{-n-1} du$$

* Hermite, *Comptes Rendus*, t. LVIII, pp. 266-273. Whittaker, "Functions associated with the Parabolic Cylinder," *Proc. London Math. Soc.*, 1903, pp. 420 *et seq.* Watson, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 393 *et seq.* Myller-Lebedoff, *Math. Ann.*, Vol. 64, pp. 388 *et seq.*

satisfies the differential equation (A), Γ_u being a contour in the plane of u , beginning at the positive end of the real axis, circulating in a counter-clockwise direction round the origin and returning again to positive infinity on the real axis. In the case when n is a positive integer this contour may clearly be deformed into a simple contour enclosing the origin without altering the value of the integral.

In general then $U_n(x)$ is defined by the relation

$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^{n+1}} du. \quad (\text{I})$$

In Whittaker's paper it is then shewn that a second independent solution of (A) is given by

$$V_n(x) = e^{x^2} U_{-n-1}(-ix). \quad (\text{II})$$

The object of this paper is to establish a connexion between these Hermite functions of the first and second kind and the Legendre functions of the first and second kind by means of relations involving integrals analogous to those connecting the Bessel and Legendre functions.

1. Suppose n is an even positive integer, then

$$U_n(x) = \frac{n!}{2\pi i} \int_D \frac{e^{-2ux-u^2}}{u^{n+1}} du,$$

where D is a simple contour enclosing the origin.

Hence, on expanding e^{-2ux} in ascending powers of ux ,

$$\begin{aligned} U_n(x) &= \sum_{r=0}^{\infty} \frac{(-2x)^r n!}{r! 2\pi i} \int_D e^{-u^2} u^{-n-1+r} du \\ &= \sum_{s=0}^{s=\frac{1}{2}n} \frac{(-2x)^{2s} n!}{(2s)! 2\pi i} \int_D e^{-u^2} u^{-n-1+2s} du \\ &= \sum_{s=0}^{s=\frac{1}{2}n} \frac{(2x)^{2s} n!}{(2s)!} \frac{(-1)^{\frac{1}{2}n-s}}{\left(\frac{n}{2} - s\right)!}. \end{aligned}$$

$$\text{Now } \int_{-\infty}^{\infty} e^{-t^2} t^{2s} dt = \Gamma(s+\tfrac{1}{2}) = \frac{\Gamma(s+\tfrac{1}{2}) \Gamma(s+1)}{\Gamma(s+1)} = \frac{\pi^{\frac{1}{2}} \Gamma(2s+1)}{2^{2s} \Gamma(s+1)};$$

therefore

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-t^2} U_n(xt) dt &= \sum_{s=0}^{s=\frac{1}{2}n} \frac{(2x)^{2s} n! (-1)^{\frac{1}{2}n-s} \pi^{\frac{1}{2}} (2s)!}{(2s)! \left(\frac{n}{2} - s\right)! 2^{2s} s!} \\ &= \frac{n! \pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} \sum_{s=0}^{\frac{1}{2}n} \frac{\left(\frac{n}{2}\right)! (-1)^{\frac{1}{2}n-s}}{s! \left(\frac{n}{2} - s\right)!} x^{2s} \\ &= \frac{n! \pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} (x^2 - 1)^{\frac{1}{2}n}.\end{aligned}$$

Now, from the definition of $U_n(x)$, viz.,

$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^{n+1}} du,$$

$$\text{clearly, } \frac{dU_n(x)}{dx} = -2 \frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^n} du = -2n U_{n-1}(x), \quad (\text{III})$$

and we have proved that

$$\int_{-\infty}^{\infty} e^{-t^2} U_n(xt) dt = \frac{n! \pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} (x^2 - 1)^{\frac{1}{2}n}.$$

On differentiating each side of this equation $\frac{1}{2}n$ times with respect to x , we therefore obtain

$$\begin{aligned}(-2)^{\frac{1}{2}n} \frac{n!}{\left(\frac{n}{2}\right)!} \int_{-\infty}^{\infty} e^{-t^2} t^{\frac{1}{2}n} U_{\frac{1}{2}n}(xt) dt &= \frac{n!}{\left(\frac{n}{2}\right)!} \pi^{\frac{1}{2}} \frac{d^{\frac{1}{2}n}}{dx^{\frac{1}{2}n}} (x^2 - 1)^{\frac{1}{2}n} \\ &= \frac{n!}{\left(\frac{n}{2}\right)!} \pi^{\frac{1}{2}} 2^{\frac{1}{2}n} \left(\frac{n}{2}\right)! P_{\frac{1}{2}n}(x).\end{aligned}$$

Hence for all positive integral values of m , replacing $\frac{1}{2}n$ by m ,

$$P_m(x) = \frac{(-1)^m}{\pi^{\frac{1}{2}} m!} \int_{-\infty}^{\infty} e^{-t^2} t^m U_m(xt) dt.$$

2. Expansion of $U_n(x)$ in ascending powers of x .

For all values of p it is well known that

$$\int_{\Gamma_u} e^{-u^2} u^{p-1} du = ie^{p\pi i} \sin p\pi \Gamma\left(\frac{p}{2}\right). \quad (\text{IV})$$

Now
$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_n} e^{-2ux-u^2} u^{-n-1} du \dots,$$

so that, if it is permissible to expand e^{-2ux} in ascending powers of x , and integrate term by term, the required expansion is at once obtained on applying the above formula.

Consider
$$\int_{\Gamma_n} \sum_{r=s}^{\infty} \frac{(-2x)^r u^r}{r!} u^{-n-1} e^{-u^2} du.$$

Suppose first of all that s is chosen large enough to secure that the real part of $s-n$ is positive.

Then this integral is equal to

$$(1 - e^{-2n\pi i}) \int_{\infty}^0 \sum_{r=s}^{\infty} \frac{(-2x)^r u^r}{r!} u^{-n-1} e^{-u^2} du.$$

Now let $|-2x|$ be denoted by ξ , and let the real part of n be represented by N , and then examine

$$\int_0^{\infty} \sum_{r=s}^{\infty} \frac{\xi^r u^r}{r!} u^{-N-1} e^{-u^2} du.$$

Choose a positive quantity λ , so that

$$\frac{s}{2\xi} < \lambda < \frac{s+1}{2\xi},$$

and rewrite the integral in the form

$$\int_0^{\lambda} \sum_{r=s}^{\infty} \frac{\xi^r u^r}{r!} u^{-N-1} e^{-u^2} du + \int_{\lambda}^{\infty} \sum_{r=s}^{\infty} \frac{\xi^r u^r}{r!} u^{-N-1} e^{-u^2} du,$$

which is less than

$$\int_0^{\lambda} \frac{\xi^s u^s}{s!} \frac{u^{-N-1} e^{-u^2}}{1 - \frac{\xi u}{s+1}} du + \int_{\lambda}^{\infty} e^{-u^2 + \xi u} u^{-N-1} du,$$

which is less than

$$\frac{2\xi^s}{s!} \int_0^{\lambda} u^{s-N-1} e^{-u^2} du + \int_{\lambda}^{\infty} e^{-u^2 + \xi u} u^{-N-1} du.$$

Now let λ tend to infinity as the result of s tending to infinity.

Then since $\int_0^{\infty} e^{-u^2 + \xi u} u^{-N-1} du$ is a convergent integral for all finite values of ξ and all finite values of N (α being a real positive quantity), it

follows that

$$\int_{\lambda}^{\infty} e^{-u^2 + \zeta u} u^{-N-1} du$$

tends to zero. While

$$\frac{2\zeta^s}{s!} \int_0^{\lambda} u^{s-N-1} e^{-u^2} du < \frac{2\zeta^s}{s!} \frac{1}{2} \Gamma\left(\frac{s-N}{2}\right),$$

and this tends to zero as s tends to infinity, for all finite values of ζ , and finite values of N .

Hence for all finite values of x , and all finite values of n , it is permissible to expand e^{-2ux} under the integral sign and integrate term by term. Thus

$$\begin{aligned} U_n(x) &= \frac{\Gamma(n+1)}{2\pi i} \sum_{r=0}^{\infty} \frac{(-2x)^r}{r!} \int_{\Gamma_u} e^{-u^2} u^{-n-1+r} du \\ &= \frac{\Gamma(n+1)}{2\pi i} \sum_{r=0}^{\infty} \frac{(-2x)^r}{r!} i e^{-(n-r)\pi i} \sin[-(n-r)] \pi \cdot \Gamma\left(-\frac{n-r}{2}\right) \\ &= \frac{e^{-n\pi i} \Gamma(n+1)}{2\pi} \sum_{r=0}^{\infty} \frac{(2x)^r}{r!} 2 \sin\left(-\frac{n-r}{2} \pi\right) \cos\left(-\frac{n-r}{2} \pi\right) \Gamma\left(-\frac{n-r}{2}\right) \\ &= e^{-n\pi i} \Gamma(n+1) \sum_{r=0}^{\infty} \frac{(2x)^r}{r!} \cos \frac{n-r}{2} \pi \frac{1}{\Gamma\left(\frac{n-r}{2} + 1\right)} \\ &= e^{-n\pi i} \Gamma(n+1) \sum_{s=0}^{\infty} \frac{(2x)^{2s}}{(2s)!} \cos \frac{n\pi}{2} \frac{(-1)^s}{\Gamma\left(\frac{n-2s}{2} + 1\right)} \\ &\quad + e^{-n\pi i} \Gamma(n+1) \sum_{s=0}^{\infty} \frac{(2x)^{2s+1}}{(2s+1)!} \sin \frac{n\pi}{2} \frac{(-1)^s}{\Gamma\left(\frac{n-2s-1}{2} + 1\right)} \\ &= \frac{e^{-n\pi i} \Gamma(n+1) \cos \frac{n\pi}{2}}{\Gamma\left(\frac{n}{2} + 1\right)} \sum_{s=0}^{\infty} \frac{(-1)^s \frac{n}{2} \left(\frac{n}{2} - 1\right) \dots \left(\frac{n}{2} - s + 1\right)}{(2s)!} x^{2s} \\ &\quad + \frac{e^{-n\pi i} \Gamma(n+1) \sin \frac{n\pi}{2}}{\Gamma\left(\frac{n+1}{2}\right)} \sum_{s=0}^{\infty} \frac{(-1)^s \frac{n-1}{2} \left(\frac{n-1}{2} - 1\right) \dots \left(\frac{n-1}{2} - s + 1\right)}{(2s+1)!} x^{2s+1}. \end{aligned} \tag{V}$$

If n is a positive integer one of the series vanishes and the other terminates.

If n is a negative integer, write $n = -m$, and then the factor multiplying each term of the first series is written

$$\begin{aligned} \frac{e^{m\pi i} \Gamma(-m+1) \cos \frac{m\pi}{2}}{\Gamma\left(-\frac{m}{2}+1\right)} &= e^{m\pi i} \frac{\pi}{\Gamma(m) \sin m\pi} \frac{\sin \frac{m\pi}{2} \Gamma\left(\frac{m}{2}\right)}{\pi} \cos \frac{m\pi}{2} \\ &= \frac{e^{m\pi i} \Gamma\left(\frac{m}{2}\right)}{2\Gamma(m)}, \end{aligned} \quad (\text{VI})$$

while the factor multiplying each term in the second series becomes

$$-\frac{e^{m\pi i} \Gamma(-m+1) \sin \frac{m\pi}{2}}{\Gamma\left(-\frac{m+1}{2}\right)} = -\frac{e^{m\pi i} \Gamma\left(\frac{m+1}{2}\right)}{2\Gamma(m)}.$$

Thus, if n is a negative integer, neither of the series vanishes, and neither of the series terminates.

3. Investigation of an upper limit for $|U_n(z)|$.

For brevity, let

$$A = \frac{\Gamma(n+1) \cos \frac{n\pi}{2}}{e^{n\pi i} \Gamma\left(\frac{n}{2}+1\right)}, \quad B = \frac{\Gamma(n+1) \sin \frac{n\pi}{2}}{e^{n\pi i} \Gamma\left(\frac{n+1}{2}\right)}.$$

Then

$$\begin{aligned} U_n(x) = & A \left[1 - \frac{n}{2!} 2x^2 + \frac{n(n-2)}{4!} (2x^2)^2 - \dots \right. \\ & \left. + (-1)^{k-1} \frac{n(n-2) \dots n+2k+4}{(2k-2)!} (2x^2)^{k-1} \right] \\ & + 2Bx \left[1 - \frac{n-1}{3!} 2x^2 + \frac{(n-1)(n-3)}{5!} (2x^2)^2 - \dots \right. \\ & \left. + (-1)^{k-1} \frac{(n-1)(n-3) \dots (n-2k+3)}{(2k-1)!} (2x^2)^{k-1} \right] \\ & + A\phi_1(x) + 2Bx\phi_2(x), \end{aligned}$$

where
$$\phi_1(x) = \sum_{p=\kappa}^{p=\infty} (-1)^p \frac{n(n-2)\dots(n-2p+2)}{(2p)!} (2x^2)^p,$$

and
$$\phi_2(x) = \sum_{p=\kappa}^{p=\infty} (-1)^p \frac{(n-1)(n-3)\dots(n-2p+1)}{(2p+1)!} (2x^2)^p.$$

Now, let $\nu = |n|$ and $\xi = |x|$. Then

$$\begin{aligned} |\phi_1(x)| &\leq \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2)}{(2p)!} 2^p \xi^{2p} \\ &\leq \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2)}{1.3\dots(2p-1)} \frac{\xi^{2p}}{p!}. \end{aligned}$$

Now, let κ be the smallest integer greater than (or equal to) $\frac{\nu}{2}$.

Then $|\phi_1(x)|$

$$\begin{aligned} &\leq \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2\kappa-2)}{(2\kappa+1)(2\kappa+3)\dots(2p-1)} \frac{(\nu+2p-2\kappa)\dots(\nu+2p-2)}{1.3\dots(2\kappa-1)} \frac{\xi^{2p}}{p!} \\ &\leq \sum_{p=\kappa}^{\infty} \frac{(\nu+2p-2\kappa)(\nu+2p-2\kappa+2)\dots(\nu+2p-2)}{1.3\dots(2\kappa-1)} \frac{\xi^{2p}}{p!} \\ &\leq \frac{1}{1.3\dots(2\kappa-1)} \sum_{p=\kappa}^{\infty} \frac{(\nu+2p-2\kappa)(\nu+2p-2\kappa+2)\dots(\nu+2p-2)}{(p-\kappa+1)(p-\kappa+2)\dots p} \frac{\xi^{2p}}{(p-\kappa)!} \\ &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{p=\kappa}^{\infty} \frac{\left(\frac{\nu}{2} + p - \kappa\right) \left(\frac{\nu}{2} + p - \kappa + 1\right) \dots \left(\frac{\nu}{2} + p - 1\right)}{(p-\kappa+1)(p-\kappa+2)\dots p} \frac{\xi^{2(p-\kappa)}}{(p-\kappa)!} \\ &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1 + \frac{\nu-2}{2(r+1)}\right) \left(1 + \frac{\nu-2}{2(r+2)}\right) \dots \left(1 + \frac{\nu-2}{2(r+\kappa)}\right) \frac{\xi^{2r}}{r!}. \end{aligned}$$

Hence, if ν is greater than 2,

$$\begin{aligned} |\phi_1(x)| &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1 + \frac{\nu-2}{2(r+1)}\right)^\kappa \frac{\xi^{2r}}{r!} \\ &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1 + \frac{\nu-2}{2}\right)^\kappa \frac{\xi^{2r}}{r!} \\ &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1.3\dots(2\kappa-1)} e^{\xi^2}, \end{aligned}$$

while, if ν is not greater than 2, it is essentially positive, and so

$$\begin{aligned} |\phi_1(x)| &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1 \cdot 3 \dots (2\kappa-1)} \sum_{r=0}^{\infty} \frac{\xi^{2r}}{r!} \\ &\leq \frac{2^\kappa \cdot \xi^{2\kappa}}{1 \cdot 3 \dots (2\kappa-1)} e^{\xi^2} \leq 2\xi^2 \cdot e^{\xi^2}, \end{aligned}$$

since $\kappa = 1$ in this case.

Similarly, we can prove that, if ν is greater than 1, then

$$|\phi_2(x)| \leq \frac{(\nu+1)^\kappa \cdot \xi^{2\kappa}}{1 \cdot 3 \dots (2\kappa+1)} e^{\xi^2},$$

while, if ν is not greater than 1,

$$|\phi_2(x)| \leq \frac{2\xi^2}{3} e^{\xi^2}.$$

4. If $y_n(v)$ is any solution of

$$\frac{d^2 u}{dv^2} - 2v \frac{du}{dv} + 2nu = 0,$$

then, under certain conditions governing x , it will be shewn that

$$I = \int_{\Gamma_n} e^{-u^2} u^n y_n(xu) du$$

satisfies Legendre's equation.

$$\begin{aligned} \text{For} \quad & (1-x^2) \frac{d^2 I}{dx^2} - 2x \frac{dI}{dx} + n(n+1) I \\ &= \int_{\Gamma_t} e^{-t^2} t^n [t^2(1-x^2) y_n''(xt) - 2xt y_n'(xt) + n(n+1) y_n(xt)] dt. \end{aligned}$$

$$\begin{aligned} \text{Now} \quad & \frac{d}{dt} [e^{-t^2} t^{n+1} \{n y_n(xt) - xt y_n'(xt)\}] \\ &= e^{-t^2} t^n [n(n+1) y_n(xt) - 2xt y_n'(xt) + t^2(1-x^2) y_n''(xt) \\ &\quad - t^2 \{y_n''(xt) - 2xt y_n'(xt) + 2n y_n(xt)\}] \\ &= e^{-t^2} t^n [n(n+1) y_n(xt) - 2xt y_n'(xt) + t^2(1-x^2) y_n''(xt)]. \end{aligned}$$

Hence

$$(1-x^2) \frac{d^2 I}{dx^2} - 2x \frac{dI}{dx} + n(n+1)I$$

$$= \int_{\Gamma_t} \frac{d}{dt} [e^{-t^2} t^{n+1} \{ny_n(xt) - xt y'_n(xt)\}] dt,$$

and this is equal to zero if

$$\lim_{t \rightarrow \infty} e^{-t^2} t^{n+1} \{ny_n(xt) - xt y'_n(xt)\} = 0.$$

This condition may be expressed in a simpler form.

Since

$$\int_{\Gamma_u} \frac{d}{du} (e^{-2ux-u^2} u^{-n}) du = 0;$$

therefore

$$-2x \int_{\Gamma_u} e^{-2ux-u^2} u^{-n} du - 2 \int_{\Gamma_u} e^{-2ux-u^2} u^{-n+1} du - n \int_{\Gamma_u} e^{-2ux-u^2} u^{-n-1} du = 0,$$

$$\text{i.e.,} \quad 2x U_{n-1}(x) + U_n(x) + 2(n-1) U_{n-2}(x) = 0. \quad (\text{IX})$$

Hence

$$n U_n(v) - v U'_n(v) = n U_n(v) + 2nv U_{n-1}(v) \quad [\text{by (III)}]$$

$$= -2n(n-1) U_{n-2}(v) \quad [\text{by (IX)}].$$

Further

$$V_n(v) = e^{v^2} U_{-n-1}(-iv) \quad [\text{by (II)}];$$

therefore

$$n V_n(v) - v V'_n(v)$$

$$= e^{v^2} [n U_{-n-1}(-iv) - 2v^2 U_{-n-1}(-iv) + 2(n+1)iv U_{-n-2}(-iv)] \quad [\text{by (III)}]$$

$$= e^{v^2} [n U_{-n-1}(-iv) + iv \{2iv U_{-n-1}(-iv) + 2(n+1) U_{-n-2}(-iv)\}]$$

$$= e^{v^2} [n U_{-n-1}(-iv) + iv U_{-n}(-iv)] \quad [\text{by (IX)}]$$

$$= \frac{1}{2} e^{v^2} U_{-n+1}(-iv) \quad [\text{by (IX)}].$$

Hence $ny_n(v) - v y'_n(v)$ must have the form

$$c_1 U_{-n-2}(v) + c_2 e^{v^2} U_{-n+1}(-iv),$$

since $y_n(v)$, being a solution of Hermite's differential equation, has the form

$$\gamma_1 U_n(v) + \gamma_2 V_n(v).$$

Hence we have to examine

$$\lim_{t \rightarrow \infty} e^{-t^2} t^{n+1} U_{n-2}(xt) \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-t^2+x^2 t^2} t^{n+1} U_{-n+1}(-ixt).$$

On applying the results of § 3, assuming n is finite, the first of these expressions tends to zero if $|x| < 1$, and the second tends to zero if the real part of $\{-1+x^2+|x|^2\}$ is negative.

In applying the results of paragraph 3, it is to be remembered that κ is merely the smallest integer greater than $\frac{|n|}{2}$.

The condition that the real part of $\{-1+x^2+|x|^2\}$ is negative may be more briefly expressed by saying that the real part of x is numerically less than $\frac{1}{\sqrt{2}}$. For, on putting $x = re^{\phi i}$,

$$\begin{aligned} -1+x^2+|x|^2 &= -1+r^2 \cos 2\phi + r^2 + ir^2 \sin 2\phi \\ &= -1+2r^2 \cos^2 \phi + ir^2 \sin 2\phi. \end{aligned}$$

Both conditions are satisfied if $|x| < \frac{1}{\sqrt{2}}$.

The following results, however, are obviously true.

If $|x| < 1$, then $\int_{\Gamma_t} e^{-t^2} t^n U_n(xt) dt$ satisfies Legendre's equation.

If the real part of $x| < \frac{1}{\sqrt{2}}$, then $\int_{\Gamma_t} e^{-t^2} t^n V_n(xt) dt$ is a solution.

$$5. \text{ Let } y = \frac{1}{i \cdot \pi^{\frac{1}{2}} \cdot \Gamma(n+1) \sin n\pi} \int_{\Gamma} e^{-t^2} t^n U_n(xt) dt.$$

This is a solution of Legendre's equation if $|x| < 1$. It will be identified with $P_n(x)$ by shewing that when x tends to zero y tends to $P_n(0)$, and dy/dx tends towards $P'_n(0)$.

$P_n(x) = F\left(-n, n+1, 1, \frac{1-x}{2}\right)$ if $|1-x| < 2$ (see, for instance, Whittaker, *Mod. Anal.*, § 118).

$$\text{Therefore} \quad P_n(0) = F(-n, n+1, 1, \tfrac{1}{2}).$$

But, by one of Kummer's formulæ,

$$F(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}, x) = F[\alpha, \beta, \alpha + \beta + \frac{1}{2}, 4x(1-x)]$$

if $|x| < 1$, and if $|4x(1-x)| \leq 1$.

Hence

$$\begin{aligned} P_n(0) &= F\left(-\frac{n}{2}, \frac{n+1}{2}, 1, 1\right) = \frac{\Gamma(1) \Gamma\left(1 + \frac{n}{2} - \frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{n}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right)} \\ &= \frac{\Gamma(\frac{1}{2}) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right)} = \frac{\Gamma\left(\frac{n+1}{2}\right) \cos \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n}{2} + 1\right)}. \quad (\text{XII}) \end{aligned}$$

Further $(x^2 - 1) \frac{dP_n(x)}{dx} = nx P_n(x) - n P_{n-1}(x);$

therefore $P'_n(0) = n P_{n-1}(0) = \frac{2 \cdot \Gamma\left(\frac{n+2}{2}\right) \sin \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n+1}{2}\right)}. \quad (\text{XIII})$

On turning back to the series for $U_n(x)$,

$$U_n(0) = \frac{\Gamma(n+1) \cos \frac{n\pi}{2}}{e^{n\pi i} \cdot \Gamma\left(\frac{n}{2} + 1\right)}, \quad U'_n(0) = \frac{2 \cdot \Gamma(n+1) \sin \frac{n\pi}{2}}{e^{n\pi i} \cdot \Gamma\left(\frac{n+1}{2}\right)}.$$

Hence

$$\begin{aligned} [y]_{x=0} &= \frac{1}{\pi^{\frac{1}{2}} \cdot i \cdot \Gamma(n+1) \sin n\pi} \frac{\Gamma(n+1) \cos \frac{n\pi}{2}}{e^{n\pi i} \cdot \Gamma\left(\frac{n}{2} + 1\right)} \\ &\quad \times i e^{(n+1)\pi i} \sin(n+1)\pi \cdot \Gamma\left(\frac{n+1}{2}\right) \quad [\text{using (IV)}] \\ &= \frac{\cos \frac{n\pi}{2} \Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n}{2} + 1\right)} = P_n(0), \end{aligned}$$

and

$$\begin{aligned}
 \left[\frac{dy}{dx} \right]_{x=0} &= \frac{1}{\pi^{\frac{1}{2}} \cdot i \cdot \Gamma(n+1) \sin n\pi} \int_{\Gamma_i} e^{-t^2} t^{n+1} [-2n U_{n-1}(xt)] dt \quad [\text{by (III)}] \\
 &= \frac{-2n}{\pi^{\frac{1}{2}} \cdot i \cdot \Gamma(n+1) \sin n\pi} \frac{\Gamma(n) \cos \frac{n-1}{2} \pi}{e^{(n-1)\pi i} \cdot \Gamma\left(\frac{n+1}{2}\right)} \\
 &\quad \times i e^{(n+2)\pi i} \sin(n+2)\pi \cdot \Gamma\left(\frac{n+2}{2}\right) \\
 &= \frac{2\Gamma\left(\frac{n}{2} + 1\right) \sin \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n+1}{2}\right)} = P'_n(0).
 \end{aligned}$$

Hence throughout the domain $|x| < 1$, which contains no singularities of $P_n(x)$,

$$P_n(x) = \frac{1}{i \cdot \pi^{\frac{1}{2}} \cdot \Gamma(n+1) \sin n\pi} \int_{\Gamma_i} e^{-t^2} t^n U_n(xt) dt.$$

If the real part of $(n+1)$ is positive, this result may be put into the form

$$\begin{aligned}
 P_n(x) &= \frac{e^{2n\pi i} - 1}{i \cdot \pi^{\frac{1}{2}} \cdot \Gamma(n+1) \sin n\pi} \int_0^\infty e^{-t^2} t^n U_n(xt) dt \\
 &= \frac{2e^{n\pi i}}{\pi^{\frac{1}{2}} \cdot \Gamma(n+1)} \int_0^\infty e^{-t^2} t^n U_n(xt) dt.
 \end{aligned}$$

If n is a positive integer, then $t^n U_n(xt)$ is an even function of t , and in that case

$$P_n(x) = \frac{(-1)^n}{\pi^{\frac{1}{2}} \cdot n!} \int_{-\infty}^\infty e^{-t^2} t^n U_n(xt) dt,$$

which is the result obtained in § 1.

6. Investigation of the corresponding relation between V_n and Q_n .

We shall follow the same procedure as in the preceding paragraph, but in determining $Q_n(0)$ a difficulty is encountered, for in defining $Q_n(x)$ a slit is made in the plane of x from $+1$ to -1 along the real axis.

Now we have shown that $\int_{\Gamma_i} e^{-t^2} t^n V_n(xt) dt$ exists, at any rate, inside the domain $|\text{real part of } x| < \frac{1}{\sqrt{2}}$.

The cut between -1 and $+1$ separates this strip into two half-strips, so we shall discuss the relation between $Q_n(x)$ and this integral, first, for a point x in the upper half-strip, and secondly for a point x in the lower half-strip.

For a point x actually on the real axis between $+\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ we shall use the usual definition

$$Q_n(x) = \lim_{\epsilon \rightarrow 0} \frac{Q_n(x+\epsilon i) + Q_n(x-\epsilon i)}{2}.$$

Let us first discuss the value of

$$\lim_{x \rightarrow 0} Q_n(x),$$

where x approaches 0 from above the real axis, which, for the sake of brevity, we will denote by

$$\lim_{x \rightarrow 0} Q_n(x).$$

Now it is well known that, if the real part of $(n+1)$ is positive, and if x is not a real number less than -1 , then

$$Q_n(x) = \int_0^\infty \{x + \sqrt{x^2 - 1} \cosh \theta\}^{-n-1} d\theta,$$

where that branch of $\{x + \sqrt{x^2 - 1} \cosh \theta\}^{-n-1}$ is to be taken which, on continuation along a straight line from x to 1, takes the value 1 at $x = 1$, ... (see, e.g., Whittaker, *Mod. Anal.*, § 125).

In order to pass from a real value of x greater than unity to a point in the upper half-plane near the origin we have to avoid $x = 1$ by means of a semicircle in the upper half of the plane. Thus $(x-1)$ has its argument *increased* by π on avoiding $x = 1$, so that for real values of x less than 1,

$$(x-1) = (1-x)e^{\pi i},$$

if x approaches the real axis from above. Thus

$$\begin{aligned} \lim_{x \rightarrow 0} Q_n(x) &= \lim_{x \rightarrow 0} \int_0^\infty \{x + \sqrt{(1-x^2)e^{\pi i}} \cosh \theta\}^{-n-1} d\theta \quad [\text{if } R(n+1) > 0] \\ &= e^{-\frac{1}{2}(n+1)\pi i} \int_0^\infty \cosh^{-n-1} \theta d\theta \quad [\text{if } R(n+1) > 0]. \end{aligned}$$

$$\text{Now} \quad \int_0^\infty \cosh^{-n-1} \theta d\theta = \int_0^1 \xi^{n+1} \xi^{-1} (1-\xi^2)^{-\frac{1}{2}} d\xi,$$

$$\text{on putting} \quad \cosh \theta = \frac{1}{\xi};$$

therefore

$$\lim_{x \rightarrow 0} Q_n(x) = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{-\frac{1}{2}(n+1)\pi i} \quad [\text{if } R(n+1) > 0]. \quad (\text{XIVa})$$

Now for all values of n ,

$$nQ_n(x) - (2n-1)xQ_{n-1}(x) + (n-1)Q_{n-2}(x) = 0$$

(see, e.g., Whittaker, *Mod. Anal.*, § 124); therefore

$$\lim_{x \rightarrow 0} Q_{n-2}(x) = -\frac{n}{n-1} \lim_{x \rightarrow 0} Q_n(x),$$

and by means of this relation it is clear that the condition $R(n+1) > 0$, of (XIVa), may be removed.

$$\text{Again, since} \quad (x^2-1) \frac{dQ_n(x)}{dx} = nxQ_n(x) - nQ_{n-1}(x);$$

$$\begin{aligned} \text{therefore} \quad \lim_{x \rightarrow 0} \frac{dQ_n(x)}{dx} &= n \lim_{x \rightarrow 0} Q_{n-1}(x) \\ &= \frac{n}{2} \frac{\Gamma\left(\frac{n}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{-(\frac{1}{2}n\pi i)} = \frac{\Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}} e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)}. \end{aligned} \quad (\text{XVa})$$

$$\text{Similarly,} \quad \lim_{x \rightarrow 0} Q_n(x) = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{+\frac{1}{2}(n+1)\pi i}, \quad (\text{XIVb})$$

$$\text{and} \quad \lim_{x \rightarrow 0} \frac{dQ_n(x)}{dx} = \frac{\Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{+(\frac{1}{2}n\pi i)}. \quad (\text{XVb})$$

$$\text{Now consider} \quad y = \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \int_{\Gamma} e^{-t^2} t^n V_n(xt) dt,$$

where x is restricted to the upper half of the strip $|R(x)| < \frac{1}{\sqrt{2}}$.

$$\text{Now} \quad V_n(v) = e^{v^2} U_{-n-1}(-iv);$$

therefore $V_n(0) = U_{-n-1}(0) = \frac{e^{(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)}$ [by (VI)], (XVI)

and $V'_n(0) = -iU'_{-n-1}(0) = -2i(n+1)U_{-n-2}(0)$ [by (III)]

$$= \frac{-2i(n+1)e^{(n+2)\pi i} \Gamma\left(\frac{n+2}{2}\right)}{2\Gamma(n+2)} = \frac{ie^{(n+1)\pi i} \Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)}; \quad (\text{XVII})$$

therefore

$$\begin{aligned} [y]_{x=0} &= \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \frac{e^{(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)} \frac{ie^{(n+1)\pi i} \sin(n+1)\pi \Gamma\left(\frac{n+1}{2}\right)}{[by (XVI) \text{ and (IV)}]} \\ &= \frac{-2^n \cdot i \cdot \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+2}{2}\right)}{2e^{\frac{1}{2}n\pi i} \Gamma(n+1) \Gamma\left(\frac{n+2}{2}\right)} \\ &= \frac{-2^n \cdot i \cdot \Gamma\left(\frac{n+1}{2}\right) \Gamma(n+1) \pi^{\frac{1}{2}} \cdot 2^{-n}}{2e^{\frac{1}{2}n\pi i} \Gamma(n+1) \Gamma\left(\frac{n+2}{2}\right)} \quad [\text{using formula (a)}] \\ &= \frac{e^{-\frac{1}{2}(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} = \lim_{x \rightarrow 0} Q_n(x) \quad [\text{by (XIVa)}]. \end{aligned}$$

While

$$\begin{aligned} \left[\frac{dy}{dx}\right]_{x=0} &= \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \int_{\Gamma} e^{-t^2} t^{n+1} V'_n(0) dt \\ &= \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \frac{ie^{(n+1)\pi i} \Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)} \frac{ie^{(n+2)\pi i} \sin(n+2)\pi \cdot \Gamma\left(\frac{n+2}{2}\right)}{[by (XVII) \text{ and (IV)}]} \\ &= \frac{2^n e^{-(\frac{1}{2}n\pi i)} \Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}} \cdot \Gamma(n+1) 2^{-n}}{\Gamma(n+1) \Gamma\left(\frac{n+1}{2}\right)} \\ &= \frac{e^{-(\frac{1}{2}n\pi i)} \Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} = \lim_{x \rightarrow 0} \frac{dQ_n(x)}{dx} \quad (\text{by XVa}). \end{aligned}$$

Hence we have proved that, throughout the upper half of the domain $|R(x)| < \frac{1}{\sqrt{2}}$, y satisfies Legendre's equation, and that when x is restricted to this domain and approaches the origin, then y and dy/dx respectively take the values that $Q_n(x)$ and $\frac{dQ_n(x)}{dx}$ take on approaching the origin in the same way.

Hence throughout this domain

$$Q_n(x) = y = \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \int_{\Gamma} e^{-t^2} t^n V_n(xt) dt.$$

On turning our attention to the lower half of the strip it is to be noticed that

$$\frac{\text{Lt}_{\substack{x \rightarrow 0 \\ x \rightarrow 0}} \frac{dQ_n(x)}{dx}}{\text{Lt}_{\substack{x \rightarrow 0 \\ x \rightarrow 0}} Q_n(x)} = - \frac{\text{Lt}_{\substack{x \rightarrow 0 \\ x \rightarrow 0}} \frac{dQ_n(x)}{dx}}{\text{Lt}_{\substack{x \rightarrow 0 \\ x \rightarrow 0}} Q_n(x)}.$$

Hence in seeking for a relation holding good in the lower half-strip it is not sufficient to alter the coefficient of the integral, and so we are led to consider

$$y_1 = - \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \int_{\Gamma_1} e^{-t^2} t^n V_n(-xt) dt.$$

Clearly $[y_1]_{x=0} = e^{(n+1)\pi i} [y]_{x=0}$

$$= \frac{e^{(n+1)\pi i} e^{-\frac{1}{2}(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} = \text{Lt}_{x \rightarrow 0} Q_n(x) \quad [\text{by (XIVb)}],$$

$$\text{while } \left[\frac{dy_1}{dx}\right]_{x=0} = e^{(n+1)\pi i} \frac{2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} (-1) \int_{\Gamma_1} e^{-t^2} t^{n+1} V'_n(0) dt$$

$$= e^{n\pi i} \left[\frac{dy}{dx}\right]_{x=0} = \frac{e^{\frac{1}{2}n\pi i} \Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} = \text{Lt}_{x \rightarrow 0} \left(\frac{dQ_n(x)}{dx}\right) \quad [\text{by (XVb)}]$$

Hence throughout the lower strip

$$Q_n(x) = y_1 = \frac{-2^n}{e^{\frac{1}{2}n\pi i} \sin n\pi} \int_{\Gamma_1} e^{-t^2} t^n V_n(-xt) dt.$$

Thus, throughout the domain $|R(x)| < \frac{1}{\sqrt{2}}$,

$$Q_n(x) = \frac{2^n \cdot i \cdot e^{\mp \frac{1}{2}(n+1)\pi i}}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_i} e^{-t^2} t^n V_n(\pm xt) dt,$$

where the upper or lower sign is to be taken according as x lies above or below the real axis.

If x lies on the real axis between $-\frac{1}{\sqrt{2}}$ and $+\frac{1}{\sqrt{2}}$,

$$Q_n(x) = \frac{2^n \cdot i}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_i} e^{-t^2} t^n \frac{e^{-\frac{1}{2}(n+1)\pi i} V_n(xt) + e^{\frac{1}{2}(n+1)\pi i} V_n(-xt)}{2} dt.$$

For a value of x lying on the real axis between $+1$ and -1 , it is known that

$$\lim_{\epsilon \rightarrow 0} [Q_n(x+\epsilon i) - Q_n(x-\epsilon i)] = -i\pi P_n(x) \dots$$

(Heine, *Kugelfunctionen*, Vol. 1, p. 130).

It is of interest to verify that the preceding results are consistent with this formula.

Using the results established in this paragraph,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} [Q_n(x+\epsilon i) - Q_n(x-\epsilon i)] \\ &= \frac{2^n \cdot i}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_i} e^{-t^2} t^n [e^{-\frac{1}{2}(n+1)\pi i} V_n(xt) - e^{\frac{1}{2}(n+1)\pi i} V_n(-xt)] dt, \end{aligned}$$

if x lies on the real axis between $+\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$.

Now, since $U_n(x)$, $V_n(x)$, and $V_n(-x)$ are three solutions of Hermite's equation, there must exist a relation of the form

$$U_n(x) = A V_n(x) + B V_n(-x),$$

where A and B are constants. We have

$$U_n(0) = (A+B) V_n(0) \quad \text{and} \quad U'_n(0) = (A-B) V'_n(0);$$

$$\text{therefore } A+B = \frac{U_n(0)}{V_n(0)} = \frac{e^{-n\pi i} \Gamma(n+1) \cos \frac{n\pi}{2}}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2\Gamma(n+1)}{e^{(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right)}$$

[by (V) and (XVI)];

$$\text{therefore } A+B = \frac{2e^{-(2n+1)\pi i} \Gamma(n+1) \cos \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot 2^{-n}},$$

$$\text{and } A-B = \frac{U'_n(0)}{V'_n(0)} = \frac{2 \cdot e^{-n\pi i} \Gamma(n+1) \sin \frac{n\pi}{2}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma(n+1)}{i \cdot e^{(n+1)\pi i} \Gamma\left(\frac{n+2}{2}\right)}$$

[by (V) and (XVII)];

$$\text{therefore } A-B = \frac{-2 \cdot i \cdot e^{-(2n+1)\pi i} \Gamma(n+1) \sin \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot 2^{-n}}.$$

$$\text{Hence } A = \frac{2^n \cdot e^{-(2n+1)\pi i} \Gamma(n+1)}{\pi^{\frac{1}{2}}} e^{-(\frac{1}{2}n\pi i)},$$

$$B = \frac{2^n \cdot e^{-(2n+1)\pi i} \Gamma(n+1)}{\pi^{\frac{1}{2}}} e^{+(\frac{1}{2}n\pi i)};$$

therefore

$$U_n(x) = \frac{2^n \cdot e^{-(2n+1)\pi i} \Gamma(n+1)}{\pi^{\frac{1}{2}}} [e^{-(\frac{1}{2}n\pi i)} V_n(x) + e^{(\frac{1}{2}n\pi i)} V_n(-x)];$$

therefore

$$U_n(xt) = \frac{2^n \cdot e^{-(2n+1)\pi i} \Gamma(n+1)i}{\pi^{\frac{1}{2}}} [e^{-\frac{1}{2}(n+1)\pi i} V_n(xt) - e^{\frac{1}{2}(n+1)\pi i} V_n(-xt)].$$

Hence

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} [Q_n(x+\epsilon i) - Q_n(x-\epsilon i)] \\ &= \frac{2^n}{e^{2n\pi i} \sin n\pi} \frac{\pi^{\frac{1}{2}}}{2^n \cdot e^{-(2n+1)\pi i} \Gamma(n+1)} \int_{\Gamma_i} e^{-t^2} t^n U_n(xt) dt \\ &= -\frac{\pi^{\frac{1}{2}}}{\sin n\pi \cdot \Gamma(n+1)} \frac{\pi^{\frac{1}{2}} \cdot i \cdot \Gamma(n+1) \sin n\pi}{1} P_n(x) \quad (\text{by § 5}) \\ &= -i\pi \cdot P_n(x). \end{aligned}$$

Q.E.D.

7a. Let $y_n(v)$ satisfy the equation

$$(1-v^2) \frac{d^2 y}{dv^2} - 2v \frac{dy}{dv} + n(n+1)y = 0.$$

Secondly, let L be a contour in the plane of t , beginning at a point where the real part of t is infinitely great and positive and greater in absolute

magnitude than the imaginary part of t , finishing at a point where the real part of t is infinitely great and negative and greater in absolute magnitude than the imaginary part of t , and avoiding the origin and the points $\pm ix$. Then it will be shown that

$$z = \int_L e^{-t^2} t^{-n} y_n \left(\frac{ix}{t} \right) dt$$

is a solution of Hermite's differential equation. For

$$\begin{aligned} \frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + 2nz \\ = \int_L e^{-t^2} t^{-n} \left[-\frac{1}{t^2} y_n'' \left(\frac{ix}{t} \right) - \frac{2ix}{t} y_n' \left(\frac{ix}{t} \right) + 2ny_n \left(\frac{ix}{t} \right) \right] dt. \end{aligned}$$

Now

$$\begin{aligned} \frac{d}{dt} \left[e^{-t^2} t^{-n-2} ix y_n' \left(\frac{ix}{t} \right) - e^{-t^2} t^{-n-1} ny_n \left(\frac{ix}{t} \right) \right] \\ = e^{-t^2} t^{-n} \left[-2 \frac{ix}{t} y_n' \left(\frac{ix}{t} \right) + 2ny_n \left(\frac{ix}{t} \right) - (n+2) \frac{ix}{t^3} y_n' \left(\frac{ix}{t} \right) \right. \\ \left. + n(n+1) \frac{1}{t^3} y_n \left(\frac{ix}{t} \right) + \frac{x^2}{t^4} y_n'' \left(\frac{ix}{t} \right) + n \frac{ix}{t^3} y_n' \left(\frac{ix}{t} \right) \right] \\ = e^{-t^2} t^{-n} \left[-\frac{1}{t^2} y_n'' \left(\frac{ix}{t} \right) - \frac{2ix}{t} y_n' \left(\frac{ix}{t} \right) + 2ny_n \left(\frac{ix}{t} \right) \right. \\ \left. + \frac{1}{t^2} \left\{ \left(1 + \frac{x^2}{t^2} \right) y_n'' \left(\frac{ix}{t} \right) - \frac{2ix}{t} y_n' \left(\frac{ix}{t} \right) + n(n+1) y_n \left(\frac{ix}{t} \right) \right\} \right] \\ = e^{-t^2} t^{-n} \left[-\frac{1}{t^2} y_n'' \left(\frac{ix}{t} \right) - \frac{2ix}{t} y_n' \left(\frac{ix}{t} \right) + 2ny_n \left(\frac{ix}{t} \right) \right]. \end{aligned}$$

Hence

$$\frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + 2nz = 0,$$

if, at the extremities of L , $e^{-t^2} t^{-n-1} \left[\frac{ix}{t} y_n' \left(\frac{ix}{t} \right) - ny_n \left(\frac{ix}{t} \right) \right]$ is zero.

Now, for infinite values of t , $y_n' \left(\frac{ix}{t} \right)$ and $y_n \left(\frac{ix}{t} \right)$ are finite quantities, since $y_n(v) = AP_n(v) + BQ_n(v)$, and the values of $P_n(0)$, $P_n'(0)$, $Q_n(0)$, $Q_n'(0)$ are finite [see (XII), (XIII), (XIV), and (XV)].

And, from the nature of the end-points of L , $e^{-t} t^{-n-1}$ becomes zero at each end-point.

Hence the proposition is proved.

7b. It will be proved that

$$U_n(x) = \int_{+\infty}^{-\infty} \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i} \pi^{\frac{1}{2}}} e^{-t} t^{-n} P_n\left(\frac{ix}{t}\right) dt,$$

where the origin is avoided by an anti-clockwise semicircle [and where the point $(-ix)$ is avoided by a similar semicircle if x is purely imaginary].

In proving this we shall require the value of

$$I_n = \int_{\infty}^{-\infty} e^{-t} t^{-n} dt,$$

where the origin is avoided in the manner mentioned.

It is easily seen that

$$\begin{aligned} I_n &= i e^{-\frac{1}{2}(n+1)\pi i} \sin\left(-\frac{n+1}{2}\pi\right) \Gamma\left(-\frac{n-1}{2}\right) \\ &= \frac{e^{-(\frac{1}{2}n\pi i)} \pi}{\Gamma\left(\frac{n+1}{2}\right)} \text{ for all values of } n. \end{aligned} \quad (\text{XVIII})$$

Now, let
$$z = \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i} \pi^{\frac{1}{2}}} \int_{+\infty}^{-\infty} e^{-t} t^{-n} P_n\left(\frac{ix}{t}\right) dt,$$

where singularities of the integrand are to be avoided in the prescribed manner. Then

$$\begin{aligned} [z]_{x=0} &= \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i} \pi^{\frac{1}{2}}} \frac{(-1) e^{-(\frac{1}{2}n\pi i)} \pi}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right) \cos \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n+2}{2}\right)} \quad [\text{by (XII)}] \\ &= \frac{\Gamma(n+1) \cos \frac{n\pi}{2}}{e^{n\pi i} \cdot \Gamma\left(\frac{n+2}{2}\right)} = U_n(0) \quad [\text{by (V)}], \end{aligned}$$

while

$$\begin{aligned}
 \left[\frac{dz}{dx} \right]_{x=0} &= \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i} \pi^{\frac{1}{2}}} iP'_n(0) I_{n+1} \\
 &= \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i} \pi^{\frac{1}{2}}} i \frac{2\Gamma\left(\frac{n+2}{2}\right) \sin \frac{n\pi}{2}}{\pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n+1}{2}\right)} \frac{(-1) e^{-\frac{1}{2}(n+1)\pi i} \pi}{\Gamma\left(\frac{n+2}{2}\right)} \\
 &\quad \text{[by (XIII) and (XVIII)]} \\
 &= \frac{2 \cdot \Gamma(n+1) \sin \frac{n\pi}{2}}{e^{n\pi i} \cdot \Gamma\left(\frac{n+1}{2}\right)} = U'_n(0) \quad \text{[by (V)],}
 \end{aligned}$$

and z is a solution of Hermite's equation.

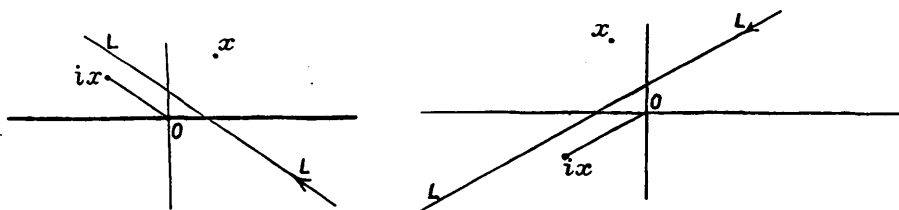
Therefore $z = U_n(x)$, and this is the formula to be established.

7c. In searching for a similar relation between V_n and Q_n the path of integration along which t travels must be such that ix/t does not cross over the cut between -1 and $+1$ used in defining $Q_n(z)$. To secure this we choose L to be a straight line parallel to the straight line joining 0 to ix , and proceeding to infinity in each direction.

In order that L shall satisfy the requirements of § 7a, it is clear that the argument of x must lie between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, or between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$.

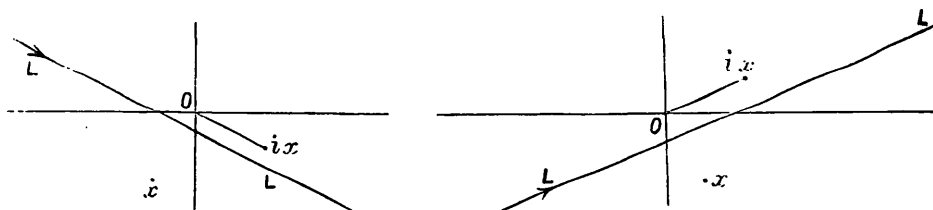
Let us further define L so that the argument of ix/t always lies between 0 and π , and let t travel along L so that the angular motion of t about the origin is anti-clockwise. Then the two cases occur illustrated by the following diagrams.

CASE I.



Here the argument of t at the beginning of the path lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, the argument of x lying between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

CASE II.



Here the argument of t at the beginning of the path lies between $-\frac{5}{4}\pi$ and $-\frac{3}{4}\pi$, the argument of x lying between $-\frac{3}{4}\pi$ and $-\frac{1}{4}\pi$.

Now in each of the two cases let us examine the integral

$$M = \int_L e^{-t^2} t^{-n} dt,$$

where that branch of t^{-n} is to be taken which, on the continuation of t along any path till its argument is zero and its modulus 1, takes the value 1.

In Case I, L may clearly be deformed into a contour beginning at positive infinity on the real axis, avoiding the origin by an anti-clockwise semi-circle, and ending at negative infinity on the real axis, that is to say, in Case I, $M = I_n$.

In Case II, the argument of t at the beginning of the path is between $-\frac{5}{4}\pi$ and $-\frac{3}{4}\pi$; so on applying the substitution $t = e^{-\pi i} u$, it is clear that in this case $M = e^{(n+1)\pi i} I_n = K_n$, say.

First suppose that the argument of x lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ and examine

$$y = \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi i} \int_L e^{-t^2} t^{-n} Q_n\left(\frac{ix}{t}\right) dt.$$

This satisfies Hermite's differential equation, and when L is defined and x is restricted as above, then $Q_n(ix/t)$ is a one-valued continuous function of (ix/t) for all values of t along L . Now suppose x travels along a straight line to the origin, then the one-valued character of $Q_n(ix/t)$ will be maintained for all values of t along L , and ix/t will approach the origin from above the real axis, and when x arrives at the origin,

$$Q_n\left(\frac{ix}{t}\right) \text{ takes the value } \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)} \quad [\text{by (XIVa)}],$$

$$Q'_n\left(\frac{ix}{t}\right) \text{ takes the value } \frac{\Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-\frac{1}{2}n\pi i}}{\Gamma\left(\frac{n+1}{2}\right)} \quad [\text{by (XVa)}];$$

therefore

$$\begin{aligned}
 [y]_{x=0} &= \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi i}^{\frac{1}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)} I_n \\
 &= \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi i}^{\frac{1}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)} \frac{(-1) e^{-(\frac{1}{2}n\pi i)} \pi}{\Gamma\left(\frac{n+1}{2}\right)} \quad [\text{by (XVIII)}] \\
 &= \frac{e^{(n-1)\pi i} \cdot \pi^{\frac{1}{2}}}{2^{n+1} \cdot \Gamma\left(\frac{n+2}{2}\right)} = \frac{e^{(n-1)\pi i} \cdot \pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{n+1}{2}\right)}{2^{n+1} \cdot \Gamma\left(\frac{n+2}{2}\right) \Gamma\left(\frac{n+1}{2}\right)} \\
 &= \frac{e^{(n-1)\pi i} \cdot \Gamma\left(\frac{n+1}{2}\right)}{2 \cdot \Gamma(n+1)} = [V_n(x)]_{x=0} \quad [\text{by (XVI)}],
 \end{aligned}$$

while

$$\begin{aligned}
 \left[\frac{dy}{dx}\right]_{x=0} &= \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi i} i \cdot I_{n+1} \frac{\Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)} \\
 &= \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi} \frac{(-1) e^{-\frac{1}{2}(n+1)\pi i} \pi}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\Gamma\left(\frac{n+2}{2}\right) \pi^{\frac{1}{2}} \cdot e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)} \\
 &= \frac{i \cdot e^{(n-1)\pi i} \pi^{\frac{1}{2}}}{2^n \cdot \Gamma\left(\frac{n+1}{2}\right)} = \frac{i \cdot e^{(n-1)\pi i} \Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)} = \left[\frac{d}{dx} V_n(x)\right]_{x=0} \\
 &\quad [\text{by (XVII)}];
 \end{aligned}$$

therefore

$$y = V_n(x).$$

Thus, if the argument of x lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, and if L is an infinite straight line parallel to the straight line joining ix to the origin, and drawn on that side of this straight line which makes the argument of ix/t lie always between 0 and π as t describes the straight line L in a direction anti-clockwise about the origin, then

$$V_n(x) = \frac{e^{(2n+1)\pi i}}{2^n \cdot \pi i} \int_L e^{-t} t^{-n} Q_n\left(\frac{ix}{t}\right) dt,$$

where that branch of t^{-n} is to be taken which, on the continuation of t along any path till its argument is zero and its modulus 1, gives the value 1.

Now suppose the argument of x lies between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$, and examine

$$y_1 = \frac{e^{n\pi i}}{2^n \cdot \pi i} \int_L e^{-t^2} t^{-n} Q_n \left(\frac{ix}{t} \right) dt.$$

Clearly $[y_1]_{x=0} = e^{-(n+1)\pi i} [y]_{x=0} \frac{K_n}{I_n} = [y]_{x=0} = [V_n(-x)]_{x=0},$

while $\left[\frac{dy_1}{dx} \right]_{x=0} = e^{-(n+1)\pi i} \left[\frac{dy}{dx} \right]_{x=0} \frac{K_{n+1}}{I_{n+1}} = - \left[\frac{dy}{dx} \right]_{x=0} = \left[\frac{d}{dx} V_n(-x) \right]_{x=0},$

since $\frac{K_n}{I_n} = e^{(n+1)\pi i}$ (see p. 257).

Hence, if the argument of x lies between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$, then the formula of the immediately preceding proposition must be replaced by

$$V_n(-x) = \frac{e^{n\pi i}}{2^n \cdot \pi i} \int_L e^{-t^2} t^{-n} Q_n \left(\frac{ix}{t} \right) dt.$$