SOME EXTENSIONS TO MULTIPLE SERIES OF ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

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1.

The object of this paper is to investigate certain extensions to multiple and repeated series of the following well-known theorem due to Abel:—

If the series

(1) $$a_0+a_1+a_2+\dots$$

is convergent, the series

(2) $$a_0+a_1x+a_2x^2+\dots$$

is absolutely convergent for all values of x whose modulus is less than unity, and if $f(x)$ denotes the function represented by the series (2), the limit of $f(x)$ when x approaches 1 along the straight line (0, 1) is equal to the sum of the series (1).†

Notation and Terminology.

It will be found essential in dealing with these questions to lay down as definite and concise a notation and as unambiguous a terminology as is possible, since those usually employed are in some ways misleading.

Suppose that

$$s_{m_1, m_2, \dots, m_n} = \sum_{i_1=0}^{i_1=m_1} \sum_{i_2=0}^{i_2=m_2} \dots \sum_{i_n=0}^{i_n=m_n} a_{i_1, i_2, \dots, i_n};$$

then we denote by

$$\sum_{(1, 2, \dots, p)(p+1, \dots, q)\dots(r+1, \dots, n)} a$$

* Mr. Hardy communicated his share of the paper on February 11th, 1904, and discovered shortly afterwards that Prof. Bromwich had at an earlier date arrived independently at the results of §§ 1–5. § 6 and §§ 12–17 are due more particularly to Mr. Hardy, and §§ 7–11 to Prof. Bromwich. Some of the earlier results (those relating to double series summed by rows or columns) were also obtained by Mr. A. Brown, to whom the subject had been suggested by Prof. Bromwich for a dissertation. As regards the latter part of the paper, each of the authors had arrived by conjecture at the other's results, but had not worked out formal proofs at the time when it was decided to unite them in one paper.

† The theorem is still true if x approaches 1 by any path (in the complex-plane) which does not touch the circle of convergence; but it is not with extensions of this kind that we shall be concerned now.

the result (if it be determinate) of making the suffixes $m_1, m_2, \ldots, m_n$ tend to infinity in groups, the group $m_{r+1}, \ldots, m_n$ being made first to tend *simultaneously* to infinity, and so on, the groups corresponding to the brackets written under the sign of summation. Thus, to take the simplest case—that of two integral parameters i_1, i_2—the expressions

$$\sum_{(1,2)} a, \quad \sum_{(1)(2)} a, \quad \sum_{(2)(1)} a$$

denote respectively the double series

$$\Sigma a_{i_1, i_2}$$

in Pringsheim's sense, and the two repeated series in which the sum is effected with respect to one parameter first. A similar notation will be used for limits. Thus,

$$\sum_{(1,2)} a = \lim_{(1,2)} s, \qquad \sum_{(1)(2)} a = \lim_{(1)(2)} s.$$

Where there is more than one bracket the operation of proceeding to the limit which corresponds to the bracket on the *right* is always to be performed first. The same notation applies to limits of functions of continuous variables. Thus, if $f(x_1, x_2)$ is a function of x_1 and x_2, both of these being positive and less than 1, $\lim\limits_{(1)(2)} f$ means $\lim\limits_{x_1=1} (\lim\limits_{x_2=1} f)$ and $\lim\limits_{(1,2)} f$ means the *double* limit $\lim\limits_{x_1=1,\, x_2=1} f$.

It is always to be understood that the limits of summation, unless the contrary is expressly stated, are zero and infinity, and the limiting value of every variable, which we shall always assume to be real and positive,* unless the contrary is expressly stated, is 1, and the term "double limit" will be used always as indicating that two variables (integral or continuous) are made to tend *simultaneously* to their limiting values. When there are several distinct passages to the limit the result is a *repeated* limit; thus,

$$\lim_{(1,2)(3,4)}$$

would denote a repeated limit—in this case the double limit of a double limit.

The expression $$\sum_{(1)} a$$

denotes the result of summing with respect to i_1 *only*, and so on. Also,

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* There is, of course, no such limitation on the value of a.

if b depends on $i_1, \ldots, i_n$,

$$\underset{(1)}{\Delta} b = b_{i_1+1, i_2, \ldots, i_n} - b_{i_1, i_2, \ldots, i_n},$$

$$\underset{(1,2)}{\Delta} b = \underset{(2,1)}{\Delta} b = \underset{(2)}{\Delta}\underset{(1)}{\Delta} b$$

$$= b_{i_1+1, i_2+1, i_3, \ldots, i_n} - b_{i_1, i_2+1, i_3, \ldots, i_n} - b_{i_1+1, i_2, i_3, \ldots, i_n} + b_{i_1, i_2, i_3, \ldots, i_n},$$

and so on.

Finally, all this notation may be generalised to denote, not limits, but maximum and minimum limits;* thus,

$$\underset{(\bar{1})(\underline{2})}{\Sigma} a$$

denotes the maximum limit for $i_1 = \infty$ of the minimum limit of s_{i_1, i_2} for $i_2 = \infty$, and

$$\underset{(\overline{1, 2})}{\Sigma} a$$

denotes the maximum limit of s_{i_1, i_2} when i_1 and i_2 tend together to infinity. And, again, exactly the same applies to such expressions as

$$\lim_{(\bar{1})(\underline{2})} f.$$

2. *Statement of the Analogue of Abel's Theorem for the General Series.*

If the simple series Σa_i is convergent, there is certainly a constant C, such that

$$|s_i| < C$$

for all values of i. We express this by saying that such a convergent series necessarily satisfies the *condition of finitude.* The same is not true for multiple series. This being so, we cannot affirm that, if, *e.g.*,

$$\underset{(1, 2, \ldots, n)}{\Sigma} a$$

is convergent, then

$$\underset{(1, 2, \ldots, n)}{\Sigma} a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

converges for values of $x_1, x_2, \ldots, x_n$ less than 1, and it is easy to see by examples that this is not necessarily the case.†

It is therefore essential to subject our series to some condition beyond that of mere convergence. We shall assume that it *does* satisfy the "condition of finitude," that is to say, that

(3) $$|s_{m_1, m_2, \ldots, m_n}| < C$$

* Sometimes called "upper and lower limits of indetermination."

† For instance, compare § 3, end.

for all values of $m_1, m_2, \ldots, m_n$. Doubtless this condition is unnecessarily narrow, but it is simple and fulfils all requirements.

The analogue of Abel's theorem is then as follows :—*If the condition of finitude is satisfied, and*

$$\sum_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} a \tag{4}$$

is convergent, then

$$\Sigma\, a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \tag{5}$$

is absolutely convergent for all values of $x_1, \ldots, x_n$ *whose moduli are less than* 1, *and if* $f(x_1, \ldots, x_n)$ *is the function represented by this series, then*

$$\lim_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} f \tag{6}$$

is determinate and equal to the sum of the series (4).

We shall prove this theorem first for double series and give some illustrations in which the series Σa has different sums when summed in different ways, so that f has different limits when we proceed to the limit in different ways.* We shall then consider some further extensions of a different kind connected with double series. Finally, we shall establish the general theorem by induction. In dealing with double series we shall use i, j, x, y for i_1, i_2, x_1, x_2 in order to avoid suffixes, and we shall write $\sum\limits_{(i)(j)}, \sum\limits_{(j)(i)}, \sum\limits_{(i,j)}, \lim\limits_{(x)(y)}, \lim\limits_{(y)(x)}, \lim\limits_{(x,y)}$ for $\sum\limits_{(1)(2)}, \ldots$.

3. *Double Series.*

Since $$a_{i,j} = \Delta_{(i,j)}\, s_{i-1,j-1}.$$

and $$|s_{m,n}| < C,$$

it follows that

$$|a_{i,j}| < 4C, \tag{7}$$

and hence that $$\Sigma a_{i,j} x^i y^j$$

is absolutely convergent. Let $f(x, y)$ denote its sum. Then

$$f(x, y) = \Sigma s_{i,j} (1-x)(1-y) x^i y^j, \tag{8}$$

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* This course seems best because this simple case affords the clearest illustration of the ideas on which our extensions of Abel's theorem are based, and its treatment does not involve the algebraical difficulties which occur in proving the more general theorems.

as is at once evident if we compare the coefficients and use condition (7).*

Now to say that $\sum_{(i,j)} a$ is convergent is the same as to say that there is a quantity s such that, however small be σ, we can determine M and N so that

$$| s_{m,n} - s | < \sigma,$$

if only $m \geqq M$ and $n \geqq N$. It is evident, moreover, that $| s | \leqq C$.

Now, since

$$\Sigma (1-x)(1-y) x^i y^j = 1,$$

it follows that $\quad f(x, y) - s = \Sigma (s_{i,j} - s)(1-x)(1-y) x^i y^j$

and $\quad |f(x, y) - s| \leqq \left| \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \right| + \left| \sum_{i=0}^{M-1} \sum_{j=N}^{\infty} \right| + \left| \sum_{j=0}^{N-1} \sum_{i=M}^{\infty} \right| + \left| \sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \right|.$

But $$\left| \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \right| < 2CMN(1-x)(1-y),$$

since $x < 1$, $y < 1$, and $| s_{i,j} - s | < 2C$; also

$$\left| \sum_{i=0}^{M-1} \sum_{j=N}^{\infty} \right| < 2CM(1-x) \sum_{j=0}^{\infty} y^j (1-y) < 2CM(1-x),$$

$$\left| \sum_{j=0}^{N-1} \sum_{i=M}^{\infty} \right| < 2CN(1-y),$$

and $$\left| \sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \right| < \sigma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^i y^j (1-x)(1-y) < \sigma.$$

Thus

$$|f(x, y) - s| < 2CMN(1-x)(1-y) + 2CM(1-x) + 2CN(1-y) + \sigma.$$

But when σ has been fixed M and N are fixed, and we can determine δ, ϵ, so that

$$|f(x, y) - s| < 2\sigma,$$

if $1-x < \delta$, $1-y < \epsilon$. Therefore

$$\lim_{(x,y)} f = s.$$

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* The transformation

$$a_0 + a_1 x + a_2 x^2 = \ldots = (1-x)(s_0 + s_1 x + s_2 x^2 + \ldots)$$

was given by Dirichlet and used as the basis of a proof of Abel's theorem identical in principle with the proof stated here of the corresponding theorem for double series, though (at any rate in the form in which he presents it) less simple than Abel's original proof. See Abel, *Œuvres*, Vol. I., p. 223; Dirichlet, *Werke*, Vol. II., p. 305; Pringsheim, *Münch. Ber.*, 1897, p. 344.

We may remark in passing that a similar proof applies to the general case when it is the convergence of the multiple series

$$\sum_{(1,2,\ldots,n)} a$$

which is given. The real difficulties begin when *repeated* limits are introduced.

We may further remark that the necessity of some such limitation as is implied by the condition of finitude becomes apparent when we consider that, for example, the double series defined by the scheme

$$\begin{array}{ccccc} a_0+b_0, & a_1-b_0, & a_2, & a_3, & \ldots, \\ -a_0+b_1, & -a_1-b_1, & -a_2, & -a_3, & \ldots, \\ b_2, & -b_2, & 0, & 0, & \ldots, \\ b_3, & -b_3, & 0, & 0, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

is convergent and has the sum 0 *whatever* be the quantities a, b; even if $a_\nu = b_\nu = \nu!$, in which case $\Sigma a_{i,j} x^i y^j$ is not convergent for any values of x and y except $x = 0$, $y = 0$ and $x = 1$, $y = 1$. If $a_\nu = b_\nu = 2^\nu$, the series is convergent and equal to $(1-y)/(1-2x)+(1-x)/(1-2y)$ if x and y are both $< \frac{1}{2}$, but divergent if $\frac{1}{2} \leqslant x < 1$ or $\frac{1}{2} \leqslant y < 1$.

4. *Repeated (Two-fold) Series.*

Now let us suppose that Σa is convergent when summed by *columns*, thus implying the convergence of every column, and that

$$\sum_{(i)(j)} a = s.$$

The series is of course absolutely convergent as before, in virtue of the condition of finitude. To illustrate the necessity of some such condition in this case we might suppose $a_{i,j}$ given by the scheme

$$\begin{array}{ccccc} 1, & 2, & 4, & 8, & \ldots, \\ -\frac{1}{2}, & -1, & -2, & -4, & \ldots, \\ -\frac{1}{4}, & -\frac{1}{2}, & -1, & -2, & \ldots, \\ -\frac{1}{8}, & -\frac{1}{4}, & -\frac{1}{2}, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \end{array}$$

Then $\sum_{(i)(j)} a = 0$, but the power series does not converge for any value of y if $\frac{1}{2} \leqslant x < 1$.

Let
$$b_i = \sum_{(j)} a_{i,j};$$
then, since
$$\left|\sum_{j=0}^{n} a_{i,j}\right| = \left|\Delta_{(i)} s_{i-1,n}\right| < 2C,$$
$$|b_i| \leqslant 2C$$
and

(9)
$$\sum_{(i)} b_i x^i$$
is absolutely convergent. Similarly,
$$\sum_{(i)} a_{i,j} x^i$$
is absolutely convergent. Further we can prove that

(10)
$$\sum_{(j)(i)} a_{i,j} x^i$$
is convergent, and its sum equal to that of (9).*

For, if we introduce the abbreviation
$$b_{i,j} = \sum_{l=0}^{j} a_{i,l} = \Delta_{(i)} s_{i-1,j},$$
then the series (10) is equal to the limit
$$\lim_{j=\infty} \left(\sum_{i=0}^{\infty} b_{i,j} x^i\right),$$
provided that this limit exists.

Now, by the condition of finitude,
$$|b_{i,j}| = |\Delta_{(i)} s_{i-1,j}| < 2C;$$
so that $|b_{i,j} - b_i| < 4C$, for all values of i, j.

Hence, for all values of j,
$$\left|\sum_{i=M}^{\infty} (b_{i,j} - b_i) x^i\right| < 4C \sum_{i=M}^{\infty} x^i = 4Cx^M/(1-x).$$

Let M be chosen so as to make $4Cx^M/(1-x)$ less than an assigned positive number σ; M being now fixed, N can be chosen so as to give
$$|b_{i,j} - b_i| < \sigma(1-x)$$

* This is a kind of converse of Weierstrass's theorem concerning series of power series.

for every value of $j \geqslant N$ and for $i = 0, 1, 2, \ldots, M-1$, since

$$\lim_{j=\infty} b_{i,j} = b_i.$$

Then
$$\left| \sum_{i=0}^{M-1} (b_{i,j} - b_i) x^i \right| < \sigma(1-x) \sum_{i=0}^{M-1} x^i < \sigma,$$

and hence
$$\left| \sum_{i=0}^{\infty} (b_{i,j} - b_i) x^i \right| \leqslant \left| \sum_{i=0}^{M-1} \right| + \left| \sum_{i=M}^{\infty} \right| < 2\sigma, \quad \text{if } j \geqslant N.$$

Thus*

$$\lim_{j=\infty} \sum_{i=0}^{\infty} b_{i,j} x^i = \sum_{i=0}^{\infty} b_i x^i;$$

that is to say, the series (10) converges and has the same sum as (9).

5.

Hence
$$\lim_{(x)(y)} f(x, y) = \lim_{(x)(y)} \sum_{(j)} y^j \sum_{(i)} a_{i,j} x^i = \lim_{(x)} \sum_{(j)} \sum_{(i)} a_{i,j} x^i$$

(by Abel's theorem)

$$= \lim \sum_{(i)} x^i \sum_{(j)} a_{i,j} \quad \text{(by § 4)}$$

$$= \sum_{(i)(j)} a_{i,j} \quad \text{(by Abel's theorem).}$$

An exactly similar proof applies to the case in which the convergence of $\sum_{(j)(i)} a$ is given. Hence, *if the condition of finitude is satisfied, and any one of the three series* $\sum_{(i,j)} a$, $\sum_{(i)(j)} a$, $\sum_{(j)(i)} a$ *is convergent, the corresponding one of the three limits* $\lim_{(x,y)} f$, $\lim_{(x)(y)} f$, $\lim_{(y)(x)} f$ *is determinate and equal to the sum of the series.*

By similar methods we can easily establish corresponding theorems, in case the series $\sum_{(i)(j)} a_{i,j}$, $\sum_{(j)(i)} a_{i,j}$, $\sum_{(i,j)} a_{i,j}$ do not converge, but oscillate.

* An alternative proof of this equation can be found by writing each side as a repeated limit, in the form

$$\lim_{(j,x)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right), \quad \lim_{(x,j)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right).$$

The equality can be then obtained by using conditions given by Bromwich (*Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 184).

Thus we find

$$\sum_{\underline{(i)}\,\underline{(j)}} a_{i,j} \leqslant \underline{\lim}_{(x)(y)} f(x, y) \leqslant \overline{\lim}_{(x)(y)} f(x, y) \leqslant \sum_{\overline{(i)}\,\overline{(j)}} a_{i,j},$$

$$\sum_{\underline{(j)}\,\underline{(i)}} a_{i,j} \leqslant \underline{\lim}_{(y)(x)} f(x, y) \leqslant \overline{\lim}_{(x)(y)} f(x, y) \leqslant \sum_{\overline{(i)}\,\overline{(j)}} a_{i,j},$$

$$\sum_{\underline{(i,j)}} a_{i,j} \leqslant \underline{\lim}_{(x,y)} f(x, y) \leqslant \overline{\lim}_{(x,y)} f(x, y) \leqslant \sum_{\overline{(i,j)}} a_{i,j}.$$

These results may be summed up in the statement that *the maximum and minimum limits of $f(x, y)$, when x, y approach unity in any one of the three standard ways, are included between the maximum and minimum limits of $\Sigma a_{i,j}$, when i, j approach infinity in the same way as x, y approach unity.*

6.

Before proceeding to the general case we shall illustrate this result by some examples:—

(i.) Suppose $$a_{i,j} = \frac{i-j}{2^{i+j}} \frac{(i+j-1)!}{i!\,j!} \qquad (i, j > 0)$$

and $a_{0,j} = -2^{-j}$ $(j > 0)$, $a_{i,0} = 2^{-i}$ $(i > 0)$, $a_{0,0} = 0$. Then, if $j > 0$,

$$\sum_{i=0}^{\infty} a_{i,j} = -2^{-j} + \frac{2^{-j}}{j!} \sum_{1}^{\infty} \frac{(i+j-1)!}{(i-1)!} 2^{-i} - \frac{2^{-j}}{(j-1)!} \sum_{1}^{\infty} \frac{(i+j-1)!}{i!} 2^{-i}$$

$$= -2^{-j} + 2^{-j-1}(1-\tfrac{1}{2})^{-j-1} - 2^{-j}\{(1-\tfrac{1}{2})^{-j} - 1\} = 0;$$

but $$\sum_{0}^{\infty} a_{i,0} = \sum_{1}^{\infty} 2^{-i} = 1.$$

Hence $$\sum_{(j)(i)} a = 1$$

and, as $$a_{j,i} = -a_{i,j},$$

$$\sum_{(i)(j)} a = -1.$$

It follows by a well known theorem of Pringsheim's that the double series $\sum_{(i,j)} a$ is not convergent. Hence we infer (assuming for a moment that the condition of finitude is satisfied) that

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1,$$

and therefore (by the same theorem) $\lim_{(x,y)} f$ is not determinate. It is interesting to note that in such a case as this we can make this last *negative* inference. In the case of Abel's theorem *no* negative inference is possible.

To verify that, as a matter of fact, the condition of finitude is satisfied, we have only to observe that, if $m = n$,

$$s_{n,n} = 0,$$

while, if $m > n$, $\quad s_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} = \sum_{i=n+1}^{m} \sum_{j=0}^{n} a_{i,j}.$

In this last expression every term is positive, and

$$s_{m,n} < \sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j};$$

but, since $s_{n,n} = 0$,

$$\sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} a_{i,j} = 1.$$

Thus we may take $C = 1$.

It is easy to verify our conclusions, for

$$\Sigma a_{i,j} x^i y^j = \frac{x-y}{2-x-y} = \frac{(1-y)-(1-x)}{(1-y)+(1-x)},$$

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1.$$

(ii.) Suppose that

$$\sin \frac{1}{1-x} = a_0 + a_1 x + a_2 x^2 + \dots \qquad (0 < x < 1),$$

and consider the double series defined by the scheme

$$\begin{array}{ccccc} a_0+a_0, & a_1-a_0, & a_2, & a_3, & \dots, \\ a_1-a_0, & -a_1-a_1, & -a_2, & -a_3, & \dots, \\ a_2, & -a_2, & 0, & 0, & \dots, \\ a_3, & -a_3, & 0, & 0, & \dots, \\ \dots & \dots & \dots & \dots & \dots. \end{array}$$

Then, if $m \geqslant 2$, $n \geqslant 2$, $s_{m,n} = 0$; so that

$$\sum_{(i,j)} a_{i,j} = 0.$$

But neither repeated series is convergent, since $a_0 + a_1 + a_2 + \dots$ is not convergent.* In this case

$$f(x, y) = (1-x) \sin \frac{1}{1-y} + (1-y) \sin \frac{1}{1-x};$$

so that
$$\lim_{(x,y)} f = 0$$

while neither repeated limit exists.

* For, if it were, $\sin \frac{1}{1-x}$ would by Abel's theorem have a limit for $x = 1$, which is not the case.

It is true that we have not in this case verified the condition of finitude, and it is difficult to see exactly how this can be done, as a_ν is a complicated function of ν. But it is only necessary to observe that to remove this objection we may replace $\sin\dfrac{1}{1-x}$ by any function of x which satisfies the following conditions :—

(i.) $$f(x) = a_0+a_1x+\dots \qquad (0 < x < 1);$$

(ii.) $$|a_0+a_1+\dots+a_\nu| < C;$$

(iii.) $f(x)$ oscillates between finite limits of indetermination for $x = 1$. Such functions certainly exist.*

7. *Statement of the Theorems of Frobenius and Hölder.*

Abel's theorem gives no information as to the behaviour near $x = 1$ of the function $f(x)$, in case the series (1) is not convergent; but if the series oscillates it is quite possible that the limit

$$\lim_{x=1} f(x)$$

may be finite and determinate, in spite of the divergence of the series.† Frobenius‡ was the first to obtain a result giving information about this case; his theorem may be stated as follows :—

Let $$s_n = \sum_{j=0}^{n} a_j;$$

in case s_n approaches no definite limit as n increases to infinity, it may

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* One may, in fact, be constructed as follows. Divide (0, 1) into the intervals

$$i_n = \left(1-\frac{1}{2^n},\ 1-\frac{1}{2^{n+1}}\right) \quad (n = 0, 1, 2, \dots).$$

Let σ be an assigned small positive quantity. Choose n_1 so that throughout i_{n_1}

$$|(1-2x)-(-1)| < \sigma.$$

Now choose p_2 so that throughout i_{n_1}

$$x^{p_2}(1+x+x^2+\dots) < \sigma,$$

n_2 so that throughout i_{n_2} $$|1-2x+2x^{p_2}-(+1)| < \sigma,$$

p_3 so that throughout i_{n_2} $$x^{p_3}(1+x+x^2+\dots) < \sigma,$$

and so on. Then it is easy to see that, if

$$f(x) = 1-2x^{p_1}+2x^{p_2}-2x^{p_3}+\dots \quad (p_1 = 1),$$

$f(x)$ differs from -1 by less than 3σ in $i_{n_1}, i_{n_3}, i_{n_5}, \dots$, and from $+1$ by less than 3σ in $i_{n_2}, i_{n_4}, i_{n_6}, \dots$. The numbers $p_1, p_2, p_3, \dots$ increase with very great rapidity.

† For example, let $f(x) = 1/(1+x) = 1-x+x^2-x^3+\dots$; then $\lim_{x=1} f(x)$ is equal to $\frac{1}{2}$, although $1-1+1-1+1-1+\dots$ is oscillatory. But it has been proved that if $\lim_{n=\infty} s_n = \infty$, then $\lim_{x=1} f(x) = \infty$.

‡ *Crelle's Journal*, Bd. LXXXIX., 1880, p. 262.

happen that the arithmetic mean

$$s_n^{(1)} = \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n)$$

approaches a limit l; then the limit

$$\lim_{x=1} f(x)$$

exists and is equal to l.

It may be noticed incidentally that, if s_j does approach a definite limit l, then the arithmetic mean $s_n^{(1)}$ will approach the same limit. For an integer n can be chosen so that

$$|s_j - l| < \sigma,$$

if $j \geqslant n$; n being fixed, choose N so that

$$|s_0+s_1+s_2+\ldots+s_{n-1}-nl| < N\sigma.$$

Then

$$|s_j^{(1)}-l| = \frac{1}{j+1}\left|(s_0+s_1+\ldots+s_{n-1}-nl)+(s_n-l)+(s_{n+1}-l)+\ldots+(s_j-l)\right|$$

$$< \frac{1}{j+1}[(N\sigma)+(j-n+1)\sigma] < 2\sigma,$$

if $j \geqslant n$ and N; that is, $\lim_{j=\infty} s_j^{(1)} = l$.

A similar method can be used to prove that if s_n tends to infinity with n, then the same is true of $s_n^{(1)}$.

The theorem of Frobenius was extended further by Hölder,* so as to cover cases in which the first arithmetic mean has no definite limit.

Hölder writes

$$s_n^{(1)} = \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n),$$

(11) $$s_n^{(2)} = \frac{1}{n+1}(s_0^{(1)}+s_1^{(1)}+s_2^{(1)}+\ldots+s_n^{(1)}),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

$$s_n^{(k)} = \frac{1}{n+1}(s_0^{(k-1)}+s_1^{(k-1)}+s_2^{(k-1)}+\ldots+s_n^{(k-1)}).$$

The extended theorem is then

$$\underline{\lim}_{(n)}\, s_n^{(k)} \leqslant \underline{\lim}_{(x)}\, f(x) \leqslant \overline{\lim}_{(x)}\, f(x) \leqslant \overline{\lim}_{(n)}\, s_n^{(k)},$$

provided that $|s_n^{(k)}| < C$ for all values of n.

* *Math. Annalen*, Bd. xx., 1882, p. 535.

8. *Extension of Frobenius's Theorem to Double Series.*

Let us write

(12) $$s_{m,n} = \sum_{i,j=0} a_{i,j}, \quad s_{m,n}^{(1)} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j};$$

so that $s_{m,n}^{(1)}$ is an arithmetic mean amongst the sums s_{mn}. Then the theorem is :—

If $$\lim_{(m,n)} s_{m,n}^{(1)} = l,$$

then also $$\lim_{(x,y)} f(x, y) = l,$$

provided that

(13) $$| s_{m,n}^{(1)} | < C$$

for all values of m, n (the present form of the condition of finitude).

In virtue of equations (12), we have

$$\Delta_{(i,j)} [ij\, s_{i-1,j-1}^{(1)}] = s_{i,j}, \quad \Delta_{(i,j)} [s_{i-1,j-1}] = a_{i,j}.$$

Hence, using (13), we deduce

(14) $$| s_{i\,j} | < C[(i+1)(j+1)+i(j+1)+(i+1)j+ij] < 4C(i+1)(j+1)$$

and

(15) $$| a_{i,j} | < 16C(i+1)(j+1).$$

It follows at once, from (13), (14), and (15), that the three series

$$\Sigma(i+1)(j+1)s_{i,j}^{(1)}x^i y^j, \quad \Sigma s_{i,j}x^i y^j, \quad \Sigma a_{i,j}x^i y^j$$

are all absolutely convergent, since their terms are less numerically than the corresponding terms in the series for

$$16C(1-x)^{-2}(1-y)^{-2}.$$

Further we find by direct multiplication that

$$(1-x)(1-y)\Sigma(i+1)(j+1)s_{i,j}^{(1)}x^i y^j = \Sigma s_{i,j}x^i y^j.$$

Thus, using (8), it is clear that

(16) $$f(x, y) = (1-x)^2(1-y)^2\Sigma(i+1)(j+1)s_{i,j}^{(1)}x^i y^j.$$

But, since the arithmetic means have the limiting value l, an integer N can be found such that

(17) $$| s_{i,j}^{(1)} - l | < \sigma, \text{ for } i, j \geqslant N,$$

however small the positive number σ may be; further, from (13), it follows that $$| l | \leqslant C, \quad | s_{i,j}^{(1)} - l | < 2C, \text{ for all values of } i, j.$$

Thus, since $(1-x)^2(1-y)^2\Sigma(i+1)(j+1)x^iy^j = 1,$

it follows, from (16), that

$$f(x, y)-l = (1-x)^2(1-y)^2\Sigma(i+1)(j+1)(s_{i,j}^{(1)}-l)x^iy^j$$
$$= (1-x)^2(1-y)^2\left[\sum_{i,j=0}^{N-1}+\sum_{i=0}^{N-1}\sum_{j=N}^{\infty}+\sum_{j=0}^{N-1}\sum_{i=N}^{\infty}+\sum_{i,j=N}^{\infty}\right].$$

But, from (17), it is evident that

$$\left|\sum_{i,j=N}^{\infty}\right| < \sigma\sum_{i,j=N}^{\infty}(i+1)(j+1)x^iy^j < \sigma(1-x)^{-2}(1-y)^{-2};$$

also $$\left|\sum_{i,j=0}^{N-1}\right| < 2C\sum_{i,j=0}^{N-1}(i+1)(j+1) = 2C[\tfrac{1}{2}N(N+1)]^2,$$

$$\left|\sum_{i=0}^{N-1}\sum_{j=N}^{\infty}\right| < 2C\sum_{i=0}^{N-1}(i+1)\sum_{j=N}^{\infty}(j+1)y^j < N(N+1)C(1-y)^{-2}.$$

$$\left|\sum_{j=0}^{N-1}\sum_{i=N}^{\infty}\right| < 2C\sum_{j=0}^{N-1}(j+1)\sum_{i=N}^{\infty}(i+1)x^i < N(N+1)C(1-x)^{-2}.$$

Combining these four inequalities, we obtain

(18) $$|f(x,y)-l| < \sigma+N(N+1)C[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2].$$

Now choose δ so that

$$N(N+1)C\delta^2[2+\tfrac{1}{2}N(N+1)\delta^2] < \sigma,$$

which is possible, since N is now fixed.* Then plainly

$$N(N+1)C[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2] < \sigma,$$

if $1-x<\delta$, $1-y<\delta$; and so (18) leads to the result

$$|f(x, y)-l| < 2\sigma,$$

if $1-x<\delta$, $1-y<\delta$; that is,

(19) $$\lim_{(x,y)} f(x, y) = l,$$

which is the analogue of Frobenius's theorem.

It is easy to prove, by a similar method, that, in case $s_{i,j}^{(1)}$ does not approach a definite limit, but oscillates between a maximum limit and a minimum limit, then

$$\varliminf_{(i,j)} s_{i,j}^{(1)} \leqslant \varliminf_{(x,y)} f(x, y) \leqslant \varlimsup_{(x,y)} f(x, y) \leqslant \varlimsup_{(i,j)} s_{i,j}^{(1)}.$$

Before considering the case of *repeated* limits of the double series, we shall give an example of the result contained in equation (19).

---

* One way of doing it is to take for δ the smaller of the two values $[\sigma/4N(N+1)C]^{\frac{1}{2}}$, $[\sigma/N^2(N+1)^2C]^{\frac{1}{4}}$: the smaller will usually be the first.

9. *Lord Kelvin's Series.*

In Lord Kelvin's discussion of the electrical force between two equal conducting spheres in contact,* he employs the double series given by

$$a_{i,j} = (-1)^{i+j} ij/(i+j)^2 \quad (i, j = 1, 2, 3, \ldots),$$

the scheme for which is

$$\begin{array}{cccccc} +\frac{1.1}{2^2}, & -\frac{2.1}{3^2}, & +\frac{3.1}{4^2}, & -\frac{4.1}{5^2}, & +\ldots, \\ -\frac{1.2}{3^2}, & +\frac{2.2}{4^2}, & -\frac{3.2}{5^2}, & +\frac{4.2}{6^2}, & -\ldots, \\ +\frac{1.3}{4^2}, & -\frac{2.3}{5^2}, & +\frac{3.3}{6^2}, & -\frac{4.3}{7^2}, & +\ldots, \\ -\frac{1.4}{5^2}, & +\frac{2.4}{6^2}, & -\frac{3.4}{7^2}, & +\frac{4.4}{8^2}, & -\ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots . \end{array}$$

He shows that† $\sum_{(i)(j)} a_{i,j} = \sum_{(j)(i)} a_{i,j} = \frac{1}{6}(\log 2 - \frac{1}{4}) = l,$

say, the method employed being, essentially, the same as that used below.

Before proceeding to the general discussion, we shall evaluate $f(x, x)$; now here $|a_{ij}| \leqslant \frac{1}{4}$, so that the series for $f(x, y)$ is absolutely convergent. Thus, we may write

$$f(x, x) = \sum_{n=2}^{\infty} x^n \left(\sum_{i=1}^{n-1} a_{i, n-i} \right).$$

But $$\sum_{i=1}^{n-1} a_{i, n-i} = (-1)^n \sum_{i=1}^{n-1} i(n-i)/n^2 = (-1)^n \tfrac{1}{6}(n - 1/n),$$

and thus

$$f(x, x) = \tfrac{1}{6} \sum_{n=2}^{\infty} (n - 1/n)(-x)^n = \tfrac{1}{6} \sum_{n=1}^{\infty} (n - 1/n)(-x)^n$$
$$= \tfrac{1}{6} [\log(1+x) - x/(1+x)].$$

From this equation it is plain that

$$\lim_{x=1} f(x, x) = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l,$$

---

* *Phil. Mag.*, April and August, 1853; *Reprint of Electrical Papers*, No. vi., Art. 140.

† It is of some interest to observe that it is the *repeated* summation which gives the correct expression for *the force between the spheres.* But this is NOT *the force between the two sets of images*; in fact, the latter force can only be regarded as $\lim s_{ij}$, where i, j approach infinity in such a way that i/j tends to the limit unity; but, as will be seen below, $\lim s_{ij}$ is then *not determinate.*

a result which has sometimes been used to evaluate the sum of Kelvin's series.*

Next, to find the general value of $f(x, y)$, we write

$$f(x, y) = \sum_{n=2}^{\infty} \left(\sum_{i=1}^{n-1} a_{i,\, n-i} x^i y^{n-i} \right);$$

but

$$\sum_{i=1}^{n-1} a_{i,\, n-i} x^i y^{n-i} = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \partial y} \left(\sum_{i=0}^{n} x^i y^{n-i} \right) = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \partial y} \left(\frac{x^{n+1} - y^{n+1}}{x-y} \right)$$

$$= (-1)^n \frac{1}{n^2} \left[(n+1) \frac{x^n + y^n}{(x-y)^2} - 2 \frac{x^{n+1} - y^{n+1}}{(x-y)^3} \right].$$

It will be observed that this expression is identically zero for $n = 1$, and so the summation may be extended to include $n = 1$; then we have

$$(x-y)^3 f(x, y) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} [(n+1)(x-y)(x^n + y^n) - 2(x^{n+1} - y^{n+1})]$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} [(x+y)(x^n - y^n) - n(x-y)(x^n + y^n)].$$

If we introduce the function $\phi(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n / n^2$, it is clear that

$$\phi'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}/n = \frac{1}{x} \log(1+x);$$

and then

$$(x-y)^3 f(x, y) = (x+y)[\phi(x) - \phi(y)] - (x-y)[x\phi'(x) + y\phi'(y)].$$

If we write, for the moment,

$$\xi = \tfrac{1}{2}(x+y), \qquad \eta = \tfrac{1}{2}(x-y),$$

it will be found (after some reductions which are tedious, but not difficult) that

$$f(x, y) = -\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R$$

where

$$|R| \leqslant (\tfrac{5}{2}|\xi| + 2|\eta|)\lambda < \tfrac{9}{2}\lambda,$$

λ being the greatest value of $|\phi^{\text{iv}}(\zeta)|$ when ζ takes all values from x to y, inclusive.

Thus

$$\lim_{(x,\, y)} f(x, y) = \lim_{\xi=1,\, \eta=0} [-\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R]$$

$$= -\tfrac{1}{6} \lim_{\xi=1} [\phi'''(\xi) + 3\phi''(\xi)]$$

$$= -\tfrac{1}{6}[(2\log 2 - \tfrac{5}{4}) + 3(\tfrac{1}{2} - \log 2)] = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l;$$

and it is clear that l is also the value of the two repeated limits

$$\lim_{(x)(y)} f(x, y) \qquad \text{and} \qquad \lim_{(y)(x)} f(x, y).$$

---

* For example, by Prof. Tarleton, in his book on *Attractions* (Ex. 9, p. 279), where the result is obtained by processes which can hardly be justified.

Next we consider the value of $s_{m,n}$; and, to find this, use the theorem

$$(i+j)^{-2} = \int_0^\infty e^{-(i+j)t}\, t\, dt\,;$$

so that

$$a_{i,j} = (-1)^{i+j} \int_0^\infty ij\, e^{-(i+j)t}\, t\, dt.$$

Hence

$$s_{m,n} = \int_0^\infty t\,dt \left[\sum_{i,j=0}^{m,n} (-1)^{i+j} ij\, e^{-(i+j)t}\right] = \int_0^\infty \phi(m,t)\,\phi(n,t)\,\frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4},$$

where

$$\phi(m,t) = 1+(-1)^{m-1}\{(m+1)e^{-mt}+me^{-(m+1)t}\}.$$

Now

$$\int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty \frac{me^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty me^{-mt}\,t\,dt = \frac{1}{m},$$

and accordingly

$$\lim_{m=\infty} \int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} = 0 = \lim_{m=\infty} \int_0^\infty \frac{(m+1)e^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4}.$$

Similarly

$$\lim_{m=\infty} \int_0^\infty \frac{m(n+1)e^{-(m+n+3)t}\,t\,dt}{(1+e^{-t})^4} = 0,$$

and so on; and hence

$$\lim_{(n)(m)} s_{m,n} = \int_0^\infty \frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4} = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l,$$

the value of the integral being obtained by direct integration.* In the same way,

$$\lim_{(m)(n)} s_{m,n} = l.$$

We have thus obtained an illustration of part of the theorem given in § 5; for we have proved directly that

$$\sum_{(i)(j)} a_{i,j} = \lim_{(x)(y)} f, \qquad \sum_{(j)(i)} a_{i,j} = \lim_{(y)(x)} f.$$

However, the double series $\sum_{(i,j)} a$ is not convergent, in spite of the fact

---

* The indefinite integral is

$$\tfrac{1}{6}\left[\frac{e^t}{(1+e^t)^2} - \frac{t(1+3e^t)}{(1+e^t)^3} - \log(1+e^{-t})\right].$$

This is the method employed by Kelvin, *loc. cit.*

that $\lim\limits_{(x,y)} f$ is perfectly determinate. For

$$\lim_{m=\infty} \int_0^\infty \frac{m^2 e^{-mt}\, t\, dt}{(1+e^{-t})^4} = \lim_{m=\infty} \int_0^\infty \frac{e^{-x}\, x\, dx}{(1+e^{-x/m})^4} = \tfrac{1}{16} \int_0^\infty e^{-x} x\, dx = \tfrac{1}{16},^*$$

from which it easily follows that

$$\lim_{m=\infty} s_{m,\, m+1} = l - \tfrac{1}{16}, \qquad \lim_{m=\infty} s_{m,\, m} = l + \tfrac{1}{16}.$$

It is not difficult to prove that these are the general values of

$$\underline{\lim}_{(m,\, n)} s_{m,\, n} \qquad \text{and} \qquad \overline{\lim}_{(m,\, n)} s_{m,\, n}.$$

If m, n tend to infinity in such a way that $\lim (m/n) = 1$, $s_{m,n}$ oscillates between these values; if in such a way that $\lim (m/n) = 0$ or ∞, $s_{m,n}$ tends to the determinate limit l.

It will be seen that, in agreement with § 5,

$$\underline{\Sigma}_{(i,j)}\, a_{i,j} < \lim_{(x,y)} f < \overline{\Sigma}_{(i,j)}\, a_{i,j}.$$

Next, if we form the arithmetic mean of $s_{m,n}$, it will be found that

$$s^{(1)}_{m,\, n} = \int_0^\infty \psi(m,\, t)\, \psi(n,\, t)\, \frac{e^{-2t}\, t\, dt}{(1+e^{-t})^4},$$

where $\psi(m,\, t) = 1 + \dfrac{(-)^{m-1}\, m e^{-(m+1)t}}{m+1} + \dfrac{2}{m+1}\, \dfrac{e^{-t} + (-)^{m-1} e^{-(m+1)t}}{1+e^{-t}}$.

This gives at once

$$\lim_{(m,\, n)} s^{(1)}_{m,\, n} = \int_0^\infty \frac{e^{-2t}\, t\, dt}{(1+e^{-t})^4} = l = \lim_{(x,\, y)} f(x,\, y);$$

and, to verify the condition of finitude, we observe that, since $|\psi(m,\, t)| < 4$,

$$|s^{(1)}_{m,\, n}| < 16 \int_0^\infty e^{-2t}\, t\, dt \quad \text{for all values of } m,\, n,$$

or

$$|s^{(1)}_{m,\, n}| < 4.$$

Thus the equation

$$\lim_{(m,\, n)} s^{(1)}_{m,\, n} = \lim_{(x,\, y)} f(x,\, y)$$

is in complete agreement with the theorem proved in § 8.

From the preceding work it is clear that there is no justification for *assuming* the equation

$$\sum_{(i)(j)} a_{i,j} = \sum_{(j)(i)} a_{i,j} = \lim_{x=1} f(x,\, x),$$

---

* It is easy to see that the conditions given by Bromwich (*l.c.*, p. 201) for this inversion of limits are satisfied.

until we have proved (i.) that the *repeated* sums $\sum_{(i)(j)} a_{i,j}$, $\sum_{(j)(i)} a_{i,j}$ are convergent; and (ii.) that the *double* limit $\lim_{(m,n)} s^{(1)}_{m,n}$ is determinate, in addition to verifying the condition of finitude.

It follows that this method of evaluating the repeated sums is really far more complicated than Kelvin's direct method of summation; although, superficially, the former method appears to be the easier.

10. *Extension to Repeated (Two-fold) Series of the Theorems of Frobenius and Hölder.*

Returning to the notation of § 4, suppose that the limit

$$\lim_{j=\infty} b_{i,j}$$

does not exist; it may then happen that the arithmetic means of $b_{i,j}$, namely,

$$b^{(1)}_{i,j} = \frac{1}{j+1} \sum_{n=0}^{j} b_{i,n},$$

approach a limit $b^{(1)}_i$; so that

$$\lim_{j=\infty} b^{(1)}_{i,j} = b^{(1)}.$$

Suppose further that the condition of finitude is satisfied in the form

$$| b^{(1)}_{i,j} | < C, \quad \text{for all values of } i, j;$$

it follows that the two series

$$\sum_{(i)} b^{(1)}_{i,j} x^i, \qquad \sum_{(i)} b^{(1)}_i x^i$$

are absolutely convergent. The same is true of the series

$$\sum a_{i,j} x^i,$$

since $$b_{i,j} = \Delta_{(j)} [j b^{(1)}_{i,j-1}], \qquad a_{i,j} = \Delta_{(j)} [b_{i,j-1}];$$

so that $$| b_{i,j} | < 2C(j+1), \qquad | a_{i,j} | < 4C(j+1).$$

Now write

$$X_j = \sum_{l=0}^{j} \sum_{(i)} a_{i,l} x^i = \sum_{(i)} b_{i,j} x^i$$

and $$X^{(1)}_j = \frac{1}{j+1} (X_0 + X_1 + X_2 + \ldots + X_j).$$

Then plainly

(20) $$X_j^{(1)} = \sum_{(i)} b_{i,j}^{(1)} x^i.$$

But, by the process adopted in proving the last equation of § 4, it follows that*

$$\lim_{j=\infty} \sum_{(i)} b_{i,j}^{(1)} x^i = \sum_{(i)} b_i^{(1)} x^i,$$

and so, from (20), we find

(21) $$\lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

Now it has been proved that

$$| a_{i,j} | < 4C(j+1),$$

and consequently $\Sigma a_{i,j} x^i y^j$ is absolutely convergent, its terms being less numerically than those in the expansion of $4C(1-x)^{-1}(1-y)^{-2}$. Thus

$$f(x, y) = \sum_{(j)} y^j \sum_{(i)} a_{i,j} x^i.$$

Frobenius's theorem can be applied to this series: and, in virtue of equation (21), it follows that

$$\lim_{(y)} f(x, y) = \lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

If now either the series $\sum_{(i)} b_i^{(1)}$ *converges to a sum* l, *or if the arithmetic mean process applied to* $b_i^{(1)}$ *gives a definite limit* l, *then*

$$\lim_{(x)(y)} f(x, y) = \lim_{(x)} \sum_{(i)} b_i^{(1)} x^i = l,$$

a result which follows at once from Abel's (or Frobenius's) theorem.

Obviously a similar method can be used to find the limit

$$\lim_{(y)(x)} f(x, y),$$

the necessary modifications being made in the hypotheses.

As an illustration, take the series given by

$$a_{i,j} = (-1)^{i+j},$$

---

* In § 4, the condition of finitude was stated in a slightly different form; but a glance at the proof will show that $| b_{i,j}^{(1)} | < C$ is sufficient for the truth of the conclusion.

which has the scheme

$$\begin{matrix} +1, & -1, & +1, & -1, & \ldots, \\ -1, & +1, & -1, & +1, & \ldots, \\ +1, & -1, & +1, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots . \end{matrix}$$

In this case $b_{i,j} = 0$, if j is odd; and $b_{i,j} = (-1)^i$, if j is even.

Hence $b_i^{(1)} = \lim_{j=\infty} b_{i,j}^{(1)} = \frac{1}{2}(-1)^i$, and $|b_{i,j}^{(1)}| < 1$ for all values of i, j. Thus

$$\lim_{(y)} f(x, y) = \sum_{(i)} \tfrac{1}{2}(-1)^i x^i.$$

The series $\sum_{(i)} \frac{1}{2}(-1)^i$ does not converge, but the arithmetic mean process leads to the limit $\frac{1}{4}$; so that

$$\lim_{(x)(y)} f(x, y) = \tfrac{1}{4},$$

which may be immediately verified, since $f(x, y) = (1+x)^{-1}(1+y)^{-1}$. In this case, as a matter of fact, the theorem of § 8 can be applied; for $s_{i,j} = 1$, if both i and j are even, while $s_{i,j} = 0$ in every other case. Thus

$$\lim_{(i,j)} s_{i,j}^{(1)} = \tfrac{1}{4},$$

and so

$$\lim_{(x,y)} f(x, y) = \tfrac{1}{4}.$$

It is clear that the method used in this paragraph is capable of immediate extension to any case in which a *finite* number* of arithmetic means must be taken in order to obtain a limit from each column of the scheme. A corresponding change must be made in the condition of finitude. Then, if the limits so found from the columns either form a convergent series with the sum l, or lead to a limit l after a finite number of arithmetic means, the equation

$$\lim_{(x)(y)} f(x, y) = l$$

is true.

A simple example which we do not pause to work out in detail is given by

$$a_{i,j} = (-)^{i+j}(i+1)^p(j+1)^q,$$

* This number may vary with i, so long as it has a finite maximum. This is clear, in consequence of a theorem proved in § 7, according to which, if a limit is obtained from an arithmetic mean of any order, the *same* limit will belong to all the subsequent arithmetic means.

or, more generally,

$$a_{i,j} = (i+1)^p (j+1)^q \exp\{(i\theta+j\phi)\sqrt{(-1)}\}.$$

11. *Extension of Hölder's Theorems to Double Series : Double Limit.*

Continuing the notation of equation (12), let us write

$$
\begin{aligned}
s_{m,n} &= \sum_{i,j=0}^{m,n} a_{i,j},\\
s^{(1)}_{m,n} &= \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j},\\
s^{(2)}_{m,n} &= \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(1)}_{i,j},\\
\ldots \quad & \ldots \quad \ldots \quad \ldots,\\
s^{(k)}_{m,n} &= \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(k-1)}_{i,j}.
\end{aligned} \tag{22}
$$

Suppose that the condition of finitude

$$|s^{(k)}_{i,j}| < C$$

is verified for all values of i, j; then, by a process analogous to that used in (14) and (15), we deduce

$$
\begin{aligned}
|a_{i,j}| &< 4^{k+1}(i+1)^k(j+1)^k C,\\
|s_{i,j}| &< 4^k(i+1)^k(j+1)^k C,\\
|s^{(k-r)}_{i,j}| &< 4^r(i+1)^r(j+1)^r C \quad (r = 0, 1, 2, \ldots, k-1).
\end{aligned} \tag{23}
$$

From (23) it is clear that each of the series

$$\Sigma a_{i,j} x^i y^j, \quad \Sigma s_{i,j} x^i y^j, \quad \Sigma s^{(r)}_{i,j} x^i y^j \quad (r = 1, 2, \ldots, k)$$

is absolutely convergent; since their terms are numerically less than the corresponding terms in $4^{k+1}(k!)^2 C(1-x)^{-(k+1)}(1-y)^{-(k+1)}$.

We prove next the following preliminary lemma:—

Assuming the truth of the equation

$$\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s^{(r)}_{i,j} x^i y^j = l, \tag{24}$$

where ϕ is a polynomial of the form

$$\phi(i,j) = \frac{i^p}{p!}\frac{j^q}{q!} + \text{ terms of lower degree,}$$

then also

$$\lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}^{(r-1)} x^i y^j = l, \tag{25}$$

provided that (24) *is valid for all integers* p, q.

To prove the lemma, we use the identity

$$s_{i,j}^{(r-1)} = \Delta_{(i,\,j)} [ij\, s_{i-1,\,j-1}^{(r)}],$$

which gives

$$\begin{aligned} \sum_{(i,\,j)} \phi(i,\,j) s_{i,j}^{(r-1)} x^i y^j = {} & (1-x)(1-y) \sum_{(i,\,j)} (i+1)(j+1)\, \phi(i,\,j) s_{i,j}^{(r)} x^i y^j \\ & -x(1-y) \sum_{(i,\,j)} (i+1)(j+1) [\Delta_{(i)} \phi(i,\,j)] s_{i,j}^{(r)} x^i y^j \\ & -y(1-x) \sum_{(i,\,j)} (i+1)(j+1) [\Delta_{(j)} \phi(i,\,j)] s_{i,j}^{(r)} x^i y^j \\ & +xy \sum_{(i,\,j)} (i+1)(j+1) [\Delta_{(i,\,j)} \phi(i,\,j)] s_{i,j}^{(r)} x^i y^j. \end{aligned} \tag{26}$$

But the polynomials appearing in these series are of the forms

$$(i+1)(j+1)\phi(i,\,j) = (p+1)(q+1) \frac{i^{p+1}}{(p+1)!} \frac{j^{q+1}}{(q+1)!} + \text{lower terms},$$

$$(i+1)(j+1)[\Delta_{(i)} \phi(i,\,j)] = p(q+1) \frac{i^p}{p!} \frac{j^{q+1}}{(q+1)!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(j)} \phi(i,\,j)] = (p+1)q \frac{i^{p+1}}{(p+1)!} \frac{j^q}{q!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(i,\,j)} \phi(i,\,j)] = pq \frac{i^p}{p!} \frac{j^q}{q!} + \cdots.$$

Thus, in virtue of (24), we find

$$\lim_{(x,\,y)} (1-x)^{p+2}(1-y)^{q+2} \sum_{(i,\,j)} (i+1)(j+1)\phi(i,\,j)\, s_{i,j}^{(r)} x^i y^j = (p+1)(q+1)\,l,$$

$$\lim_{(x,\,y)} x(1-x)^{p+1}(1-y)^{q+2} \sum_{(i,\,j)} (i+1)(j+1)[\Delta_{(i)} \phi(i,\,j)] s_{i,j}^{(r)} x^i y^j = p(q+1)\,l$$

$$\lim_{(x,\,y)} (1-x)^{p+2} y(1-y)^{q+1} \sum_{(i,\,j)} (i+1)(j+1)[\Delta_{(j)} \phi(i,\,j)] s_{i,j}^{(r)} x^i y^j = (p+1)ql,$$

$$\lim_{(x,\,y)} x(1-x)^{p+1} y(1-y)^{q+1} \sum_{(i,\,j)} (i+1)(j+1)[\Delta_{(i,\,j)} \phi(i,j)] s_{i,j}^{(r)} x^i y^j = pql.$$

Combining the last four equations with equation (26), we see that

$$\begin{aligned} \lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}^{(r-1)} x^i y^j & \\ = [(p+1)(q+1)-p(q+1)-(p+1)q+pq]\,l = l, & \end{aligned}$$

and this is equation (25). Thus the lemma is proved.

It is now clear that, *if the equation*

(27) $$\lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}^{(k)} x^i y^j = l$$

is true for all integers p, q and for any particular integer k, then also the equation

(28) $$\lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}\, x^i y^j = l$$

is true.

We shall now establish the truth of (27), on the hypothesis that

$$\lim_{(i,\,j)} s_{i,j}^{(k)} = l.$$

Let us write for brevity

$$\psi(i,\,j) = [(i+1)(i+2)\dots(i+p)(j+1)(j+2)\dots(j+q)]/p!\,q!,$$

so that $$\lim_{(i,\,j)} (\phi/\psi) = 1.$$

An integer N can now be found, corresponding to any assigned positive number σ, such that

$$|(\phi/\psi)\, s_{i,j}^{(k)} - l| < \sigma, \quad \text{if} \quad i,\,j \geqslant N.$$

Further, a number g can be found such that

$$|\phi/\psi| < g, \quad \text{for all values of } i,\,j;$$

and so, using the condition of finitude,

$$|\phi s_{i,j}^{(k)}| < gC\psi, \quad \text{for all values of } i,\,j,$$

and $$|l| \leqslant C;$$

so that $$|\phi s_{i,j}^{(k)} - l\psi| < (g+1)C\psi.$$

Now $$\sum_{(i,\,j)} (\phi s^k - l\psi) x^i y^j = \sum_{i,j=0}^{N-1} + \sum_{i=0}^{N-1} \sum_{j=N}^{\infty} + \sum_{j=0}^{N-1} \sum_{i=N}^{\infty} + \sum_{i,j=N}^{\infty}$$

and $$\left| \sum_{i,j=0}^{N-1} \right| < (g+1)C \sum_{i,j=0}^{N-1} \psi < (g+1)C \frac{(N+p)^{p+1}(N+q)^{q+1}}{(p+1)!\,(q+1)!},$$

$$\left| \sum_{i=0}^{N-1} \sum_{j=N}^{\infty} \right| < (g+1)C \sum_{i=0}^{N-1} \sum_{j=N}^{\infty} \psi y^i < (g+1)C \frac{(N+p)^{p+1}}{(p+1)!} (1-y)^{-(q+1)},$$

$$\left| \sum_{i=N}^{\infty} \sum_{j=0}^{N-1} \right| < (g+1)C \sum_{j=0}^{N-1} \sum_{i=N}^{\infty} \psi x^i < (g+1)C \frac{(N+q)^{q+1}}{(q+1)!} (1-x)^{-(p+1)},$$

$$\left| \sum_{i,j=N}^{\infty} \right| < \sigma \sum_{i,j=N}^{\infty} \psi x^i y^j < \sigma (1-x)^{-(p+1)} (1-y)^{-(q+1)}.$$

Hence we deduce

$$|(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s^{(k)}_{i,j}-l\psi)x^iy^j|$$

$$<\sigma+(q+1)C\left[\frac{(N+p)^{p+1}(N+q)^{q+1}}{(p+1)!\,(q+1)!}(1-x)^{p+1}(1-y)^{q+1}\right.$$

$$\left.+\frac{(N+p)^{p+1}}{(p+1)!}(1-x)^{p+1}+\frac{(N+q)^{q+1}}{(q+1)!}(1-y)^{q+1}\right],$$

and we can choose δ so that the right-hand side of this inequality is less than 2σ, provided that $1-x$, $1-y$ are each less than δ. Hence

$$\lim_{(x,y)}(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s^{(k)}_{i,j}-\psi l)x^iy^j=0.$$

But

$$(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}\psi l x^iy^j=l,$$

and equations (27), (28) follow at once.

If we now take in (28) the special values*

$$\phi(i,j)=1,\qquad p=0,\qquad q=0,$$

it will be seen that

$$\lim_{(x,y)}(1-x)(1-y)\sum_{(i,j)}s_{i,j}x^iy^j=l,$$

or, using equation (8),

$$\lim_{(x,y)}f(x,y)=l.$$

Thus the following theorem has been established:—

If, for all values of i, j, $|s^{(k)}_{i,j}|<C$, *and if*

$$\lim_{(i,j)}s^{(k)}_{i,j}=l,$$

then also

$$\lim_{(x,y)}f(x,y)=l.$$

This is the general extension of Hölder's theorem to double series; the method can be easily modified so as to include the possibility that $s^{(k)}_{i,j}$ may oscillate; the result is then

$$\underline{\lim}_{(i,j)}s^{(k)}_{i,j}\leqslant\underline{\lim}_{(x,y)}f(x,y)\leqslant\overline{\lim}_{(x,y)}f(x,y)\leqslant\overline{\lim}_{(i,j)}s^{(k)}_{i,j}.$$

12. *The General Theorem.*

We proceed now to the proof of the general theorem stated in § 2. It has been already pointed out that the argument of § 3 applies to the

* This appears to be the only case of practical importance, but the introduction of this specialization earlier does not materially simplify the work.

general case when it is the convergence of the multiple series proper

$$\sum_{(1, 2, \ldots, n)} a$$

which is given. To prove the theorem in its most general form it is convenient to proceed by induction. We shall adopt the following contracted notation. We denote the *groups* of suffixes $(i_1, i_2, \ldots, i_p)$, $(i_{p+1}, \ldots, i_q)$, $\ldots$, $(i_{r+1}, \ldots, i_n)$ by (α), (β), $\ldots$, (μ); so that the series summed in the manner explained at the top of p. 162 will be written as

(29) $$\sum_{(\alpha)(\beta)\ldots(\mu)} a.$$

Further, by $\sum_{\alpha=0}^{I} a$, we denote the sum in which i_1 ranges from 0 to I_1, i_2 from 0 to I_2, $\ldots$, i_p from 0 to I_p, and by $x^{(\alpha)}$ we denote $x_1^{i_1} x_2^{i_2} \ldots x_p^{i_p}$.

Let us then assume (i.) that the condition of finitude is satisfied, (ii.) that the series (29) is convergent, and (iii.) that the theorem holds in its most general form for any number of indices less than n. Let

(30) $$s_\alpha = \sum_{(\beta)\ldots(\mu)} a.$$

Then, since $$\sum_{\beta, \ldots, \mu=0}^{m} a = \Delta_{(\alpha)} s_{i_1-1, \ldots, i_p-1, m_{p+1}, \ldots, m_n},$$

it follows, from the condition of finitude, that

(31) $$| s_\alpha | \leqslant 2^p C$$

and that

(32) $$\sum_{(\alpha)} s_\alpha x^{(\alpha)}$$

is absolutely convergent. And, since

$$| a_{i_1, i_2, \ldots, i_n} | = | \Delta_{(1, 2, \ldots, n)} s_{i_1-1, \ldots, i_n-1} | < 2^n C,$$

the series

(33) $$\sum_{(\alpha)} a x^{(\alpha)}$$

is also absolutely convergent. We shall prove further that

(34) $$\sum_{(\beta)\ldots(\mu)} \sum_{(\alpha)} a x^{(\alpha)}$$

is convergent and equal to (32).

13.

Our first step will be to prove that

$$\sum_{(\mu)} \sum_{(a)} a x^{(a)} \tag{35}$$

is convergent and equal to

$$\sum_{(a)} x^{(a)} \sum_{(\mu)} a, \tag{36}$$

which is convergent for the same reasons as (32) and (33).

Let

$$\sum_{\mu=0}^{m} a = b_{a,m}$$

and

$$\sum_{\mu=0}^{\infty} a = \lim_{(m)} b_{a,m} = b_a \; *$$

(m of course being a *group* of suffixes). We have to prove that

$$\lim_{(m)} \sum_{(a)} (b_{a,m} - b_a) x^{(a)} = 0.$$

Now

$$\left| \left(\sum_{a=0}^{\infty} - \sum_{a=0}^{I-1} \right) (b_{a,m} - b_a) x^a \right|$$

$$< 2^{r+1} C \frac{1-(1-x_1^{I_1})(1-x_2^{I_2}) \ldots (1-x_p^{I_p})}{(1-x_1)(1-x_2) \ldots (1-x_p)} < 2^{r+1} C \frac{x_1^{I_1} + \ldots + x_p^{I_p}}{(1-x_1) \ldots (1-x_p)},$$

since $$| b_{a,m} - b_a | < 2^{r+1} C.$$

We can choose I so that this is $< \sigma$. Then, I being fixed, we can choose M so that $| b_{a,m} - b_a | < \sigma / I_1 I_2 \ldots I_p$ for all values of $(m) \geqslant M$, and all values of $(a) \leqslant I$; thus

$$\left| \sum_{(a)}^{I-1} (b_{a,m} - b_a) x^{(a)} \right| < \sigma \quad \text{and} \quad \left| \sum_{(a)} (b_{a,m} - b_a) x^{(a)} \right| < 2\sigma.$$

Hence (35) is convergent and equal to (36).

14.

This argument can now be repeated. Suppose that (λ) is the group of suffixes immediately preceding (μ). We have to show that

$$\sum_{(\lambda)(\mu)} \sum_{(a)} a x^{(a)} \tag{37}$$

---

* The existence of this limit is, of course, implied in our data.

is convergent and equal to

(38) $$\sum_{(a)} x^{(a)} \sum_{(\lambda)(\mu)} a,$$

which is convergent for the same reasons as the series (30), (33), and (36). To prove this we have only to observe that (37) may (after § 13) be written in the form
$$\sum_{(\lambda)} \sum_{(a)} x^{(a)} \sum_{(\mu)} a$$
and that a repetition of the preceding argument with $\sum_{(\mu)} a$ in place of a proves that this is convergent and equal to (38).

By repeating this line of argument as often as may be necessary we conclude finally that (34) is convergent and equal to (32).

15.

We are now in a position to prove the theorem. For
$$\lim_{(\beta)\dots(\mu)} f = \sum_{(\beta)\dots(\mu)} \sum_{(a)} a x^{(a)}$$
(since the theorem holds for any number of indices less than n) and therefore is equal to $\sum_{(a)} x^{(a)} \sum_{(\beta)\dots(\mu)} a$ (by §§ 13, 14). Hence, by a further application of the theorem for p indices,
$$\lim_{(a)(\beta)\dots(\mu)} f = \lim_{(a)} \sum_{(a)} x^{(a)} \sum_{(\beta)\dots(\mu)} a = \sum_{(a)(\beta)\dots(\mu)} a.$$
The theorem is therefore true for n indices if it is true for any number less than n; and therefore it is true generally.

16. *Multiplication of Series.*

It is well known that from Abel's theorem we can at once deduce that, *if the three series*
$$\sum a_i, \quad \sum b_i, \quad \sum c_i,$$
where
$$c_i = \sum_{(k+l=i)} a_k b_l,$$
are convergent, the third series is the product of the other two. We have in fact only to make the first two series absolutely convergent by introducing a factor x^i in each term, to multiply the resulting power series, and to proceed to the limit.

By an exactly similar process we deduce from the theorem proved in § 15 that, *if the three series*
$$\sum a_{i_1, i_2, \dots, i_n}, \quad \sum b_{i_1, i_2, \dots, i_n}, \quad \textit{and} \quad \sum c_{i_1, i_2, \dots, i_n},$$

where $$c_{i_1, i_2, \ldots, i_n} = \sum_{(k_1+l_1 = i_1, \ldots, k_n+l_n = i_n)} a_{k_1, \ldots, k_n} b_{l_1, \ldots, l_n},$$

satisfy the condition of finitude and are convergent when summed in the same way (e.g., in the way specified by $\sum\limits_{(1, 2, \ldots, p)(p+1, \ldots, q)\ldots(r+1, \ldots, n)}$*), then the third series is the product of the first two.*

Of course similar theorems can be proved for the product of any number of series.

17. *Mean Value Theorems for the General Series.*

It is easy to prove by the method of § 11 that, if $s^{(k)}_{i_1, \ldots, i_n}$ is the k-th arithmetic mean of $s_{i_1, \ldots, i_n}$, and $|s^{(k)}| < C$ for all suffixes, and

$$\lim_{(1, 2, \ldots, n)} s^{(k)} = s,$$

then

$$\lim_{(1, 2, \ldots, n)} f = s.$$

The form of the arithmetic mean theorem corresponding to the general theorem of §§ 11–15 is as follows:—

Let Σ' denote that a series is "summed" by taking any finite number of arithmetic means. *Suppose that*

$$\sum_{(\alpha)}{}' \sum_{(\beta)}{}' \ldots \sum_{(\mu)}{}' a$$

is determinate and equal to s, *and that a number C can be assigned such that the various quantities which we pass through before we arrive at* s *are all less than C; then*

$$\lim_{(\alpha)(\beta)\ldots(\mu)} f = s.$$