# SCIENCE

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## THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

## THE MESSAGE OF NON-EUCLIDEAN GEOMETRY.*

### 1. MATHEMATICS AND ITS HISTORY.

THE great Sylvester once told me that he and Kronecker, in attempting a definition of mathematics, got so far as to agree that it is poetry.

But the history of this poesy is itself poetry, and the creation of non-Euclidean geometry gives new vantage-ground from which to illuminate the whole subject, from before the time when Homer describes Proteus as finger-fitting-by-fives, or counting, his seals, past the epoch when Lagrange, confronted with the guillotine and asked how he can make himself useful in the new world, answers simply, 'I will teach arithmetic.'

Who has not wished to be a magician like the mighty Merlin, or Dr. Dee, who wrote a preface for the first English translation of Euclid, made by Henricus Billingsley, afterward, Aladdin-like, Sir Henry Billingsley, Lord Mayor of London?

Was not Harriot, whose devices in algebra our schoolboys now use, one of the three paid magi of the Earl of Northumberland? Do not our every-day numerals stand for Brahmin and Mohammedan, coming first into Europe from the land of the sacred Ganges, around by the way of the pyramids and the Moorish Alhambra?

* Address of the vice-president and chairman of Section A, American Association for the Advancement of Science, St. Louis meeting, December, 1903.

The appearance of courses on the history of mathematics in all our foremost universities is a fortunate and promising sign of the times. I had the honor of being the first to give such a course in America, at Princeton, in 1881.

2. GEOMETRY AND ITS FOUNDERS.

But something especially fascinating, pure, divine, seems to pertain to geometry.

When asked how God occupies himself, Plato answered, 'He geometrizes continually.'

It is a difficult, though highly interesting, undertaking to investigate the vestiges of primitive geometry. Geometric figures and designs appear in connection with the primitive arts; for example, the making of pottery. Arts long precede anything properly to be called science. The first creations by mankind are instruments for life, though it is surprising how immediately decoration appears; witness the sketches from life of mammoth and mastodon and horses by prehistoric man. But, in a sense, even the practical arts must be preceded by theoretical creative acts of the human mind. Man is from the first a creative thinker. Perhaps even some of our present theoretical presentation of the universe is due to creative mental acts of our pre-human ancestors. For example, that we inevitably view the world as consisting of distinct individuals, separate, distinct things, is a pre-human contribution to our working theory and representation of the universe. It is conscious science, as a potential presentation and explanation of everything, which comes so late.

Rude instruments were made for astronomy.

The creative imagination which put the bears and bulls and crabs and lions and scorpions into the random-lying stars made figures which occur in the Book of Job, more ancient than Genesis itself.

The daring astrologer, whose predictions foretold eclipses, saw no reason why his constructions should not equally fit human life. He chose to create a causal relation between the geometric configurations of the planets and the destinies of individuals. This was the way of science, where thought precedes and helps to make fact. No description or observation is possible without a precedent theory, which stays and sticks until some mind creates another to fight it, and perhaps to overshadow it.

That legend of the origin of geometry which attributes it to the necessity of refixing land boundaries in Egypt, where all were annually obliterated by the Nile overflow, is a too-ingenious hypothesis, made temporarily to serve for history. Some practical devices for measurement arose in Egypt, where periodic fertility fostered a consecutive occupancy, whose records, according to Flinders Petrie, we have for more than nine thousand years.

But in the Papyrus of the Rhind, measurements of volume come before those for surface.

Geometry as a self-conscious science waits for Thales and Pythagoras.

We find in Herodotus that Thales predicted an eclipse memorable as happening during a battle between the Lydians and Medes. The date was given by Baily as B. C. 610.

So we know about when geometry, we may say when science, began; for though primarily geometer, Thales taught the sphericity of the earth, was acquainted with the attracting power of magnetism, and noticed the excitation of electricty in amber by friction.

A greater than he, Pythagoras, was born B. C. 590 at Samos, traveled also into Egypt and the east, penetrating even into India. Returning in the time of the last Tarquin, and finding Samos under the dominion of the tyrant Polycrates, he went as a voluntary exile to Italy, settled at Croton (as Ovid mentions), and there created and taught new and sublimer hypotheses for our universe. The most diversely demonstrated and frequently applied theorem of geometry bears his name. The first solution of a problem in that most subtle and final of ways, by proving it impossible, is due to him; his solution of the problem to find a common submultiple of the hypothenuse and side of an isosceles right triangle, an achievement whereby he created incommensurability.

It is noteworthy that this making of incommensurables is confused by even the most respectable of the historians of mathematics with the creation of irrational numbers. But in the antique world there were no such numbers as the square root of two or the square root of three. Such numbers can not be discovered, and it was centuries before they were created. The Greeks had only rational numbers.

### 3. EUCLID.

Under the Horseshoe Falls at Niagara press on beyond the guide; risk life for the magnificent sensation of a waterspout, a cloudburst, an avalanche, a tumbling cathedral of waterblocks! It *must* end in an instant, this extravagant downpour of whole wealths of water. Then out; and look away down the glorious canyon, and read in that graven history how this momentary riotous chaos has been just so, precisely the same, for centuries, for ages, for thousands of years.

In the history of science a like antithesis of sensations is given by Euclid's geometry.

In the flood of new discovery and rich advance recorded in books whose mere names would fill volumes, we ask ourselves how any one thing can be permanent? Yet, looking back, we see this Euclid not only cutting his resistless way through the rock of the two thousand years that make the history of the intellectual world, but, what is still more astounding, we find that the profoundest advance of the last two centuries has only served to emphasize the consciousness of Euclid's perfection.

Says Lyman Abbott, if you want an infallible book go not to the Bible, but to Euclid.

In 'The Wonderful Century,' Alfred Russel Wallace says, speaking of all time before the seventeenth century: "Then going backward, we can find nothing of the first rank except Euclid's wonderful system of geometry, perhaps the most remarkable product of the earliest civilizations."

Says Professor Alfred Baker, of the University of Toronto: "Of the perfection of Euclid (B. C. 290) as a scientific treatise, of the marvel that such a work could have been produced two thousand years ago, I shall not here delay to speak. I content myself with making the claim that, as a historical study, Euclid is, perhaps, the most valuable of those that are taken up in our educational institutions."

At its very birth this typical product of the Greek genius assumed sway over the pure sciences. In its first efflorescence, through the splendid days of Theon and Hypatia, fanatics could not murder it as they did Hypatia, nor later could that dismal flood, the dark ages, drown it. Like the phœnix of its native Egypt it rises anew with the new birth of culture. An Anglo-Saxon, Adelhard of Bath, finds it clothed in Arabic vestments in the Moorish land of the Alhambra.

In 1120, Adelhard, disguised as a Mohammedan student, went to Cordova, obtained a Moorish copy of Euclid's 'Elements,' and made a translation from the Arabic into Latin.

The first translation into English (1570) was made by 'Henricus Billingsley,' afterward Sir Henry Billingsley, Lord Mayor of London, 1591. And up to this very year, throughout the vast system of examinations carried on by the British government, by Oxford and by Cambridge, to be accepted, no proof of a theorem in geometry should infringe Euclid's sequence of propositions. For two millenniums his axioms remained undoubted.

### 4. THE NEW IDEA.

The break from Euclid's charmed circle came not at any of the traditional centers of the world's thought, but on the circumference of civilization, at Maros-Vásárhely and Temesvár, and again at Kazan on the Volga, center of the old Tartar kingdom; and it came as the creation of a willful, wild Magyar boy of twenty-one and an insubordinate young Russian, who, a poor widow's son from Nijni-Novgorod, enters as a charity student the new university of Kazan.

The new idea is to deny one of Euclid's axioms and to replace it by its contradictory. There results, instead of chaos, a beautiful, a perfect, a marvelous new geometry.

### 5. HOW THE NEW DIFFERS FROM THE OLD.

Euclid had based his geometry on certain axioms or postulates which had in all lands and languages been systematically used in treatises on geometry, so that there was in all the world but one geometry. The most celebrated of these axioms was the so-called parallel-postulate, which, in a form due to Ludlam, is simply this: 'Two straight lines which cut one another can not *both* be parallel to the same straight line.'

Now this same Magyar, John Bolyai, and this Russian, Lobachevski, made a geometry based not on this axiom or postulate, but on its direct contradiction. Wonderful to say, this new geometry, founded on the flat contradiction of what had been forever accepted as axiomatic, turned out to be perfectly logical, true, self-consistent and of marvelous beauty. In it many of the good old theorems of Euclid and our own college days are superseded in a surprising way. Through any point outside any given straight line can be drawn an infinity of straight lines in the same plane with the given line, but which nowhere would meet it, however far both were produced.

### 6. A CLUSTER OF PARADOXES.

In Euclid, Book I., Proposition 32 is that the sum of the angles in every rectilineal triangle is *just exactly* two right angles. In this new or non-Euclidean geometry, on the contrary, the sum of the angles in every rectilineal triangle is *less* than two right angles.

In the Euclidean geometry parallels *never* approach. In this non-Euclidean geometry parallels *continually* approach.

In the Euclidean geometry all points equidistant from a straight line are on a *straight* line. In this non-Euclidean geometry all points equidistant from a straight line are on a *curve* called the equidistantial.

In the Euclidean geometry the limit approached by a circumference as the radius increases is a *straight* line. In the non-Euclidean geometry this is a *curve* called the oricycle. Thus the method of Kempe's book 'How to draw a straight line,' would here draw not a straight line, but a curve.

In the Euclidean geometry, if three angles of a quadrilateral are right, then the fourth is *right,* and we have a rectangle. In this non-Euclidean geometry, if three angles of a quadrilateral are right, then the fourth is *acute,* and we never can have any rectangle.

In the Euclidean geometry two perpendiculars to a line remain *equidistant.* In this non-Euclidean geometry two perpendiculars to a line *spread away from each other* as they go out; their points two inches from the line are farther apart than their points one inch from the line.

In the Euclidean geometry every three points are either on a *straight* line or a *circle*. In this non-Euclidean geometry there are triplets of points which are neither *costraight* nor *concyclic*. Thus three points each one inch above a straight line are neither on a straight line nor on a circle.

7. MISTAKE OF THE INEXPERT.

These seeming paradoxes could be multiplied indefinitely, and they form striking examples of this new geometry. They seem so bizarre, that the first impression produced on the inexpert is that the traditional geometry could easily be proved, as against this rival, by careful experiments. Into this error have fallen Professors Andrew W. Phillips and Irving Fisher, of Yale University. In their 'Elements of Geometry,' 1898, page 23, they say: "Lobachevski proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from what we know it to be through experience. Lobachevski tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, whereas the space in which we really live is called *Euclidean,* because Euclid (about 300 B. C.) first wrote a systematic geometry of our space."

Now, strangely enough, no one, not even the Yale professors, can ever prove this naïve assertion. If any one of the possible geometries of uniform space could ever be proved to be the system actual in our external physical world, it certainly could not be Euclid's.

Experience can never give, for instance, such absolutely exact metric results as precisely, perfectly two right angles for the angle sum of a triangle. As Dr. E. W. Hobson says: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

8. THE ARTIFICIALLY CREATED COMPONENT IN SCIENCE.

In connecting a geometry with experience there is involved a process which we find in the theoretical handling of any empirical data, and which, therefore, should be familiarly intelligible to any scientist.

The results of any observations are always valid only within definite limits of exactitude and under particular conditions. When we set up the axioms, we put in place of these results statements of absolute precision and generality. In this idealization of the empirical data our addition is at first only restricted in its arbitrariness in so much as it must seem to approximate, must apparently fit, the supposed facts of experience, and, on the other hand, must introduce no logical contradiction. Thus our actual space to-day may very well be the space of Lobachevski or Bolyai.

If anything could be proved or disproved about the nature of space or geometry by experiments, by laboratory methods, then our space could be proved to be the space of Bolyai by inexact measurements, the only kind which will ever be at our disposal. In this way it might be known to be *non*-Euclidean. It never can be known to be Euclidean.

### 9. DARWINISM AND GEOMETRY.

The doctrine of evolution as commonly expounded postulates a world independent of man, and teaches the production of man from lower forms of life by wholly natural and unconscious causes. In this statement of the world of evolution there is need of some rudimentary approximative practical geometry.

The mighty examiner is death. The puppy, though born blind, must still be able to superimpose his mouth upon the source of his nourishment. The little chick must be able, responding to the stimulus of a small bright object, to bring his beak into contact with the object so as to grasp and then swallow it. The springing goat, that too greatly misjudges an abyss, does not survive and thus is not the fittest.

So, too, with man. We are taught that his ideas must in some way and to some degree of approximation correspond to this independent world, or death passes upon him an adverse judgment.

But it is of the very essence of the doctrine of evolution that man's knowledge of this independent world, having come by gradual betterment, trial, experiment, adaptation, and through imperfect instruments, for example the eye, can not be metrically exact.

If two natural hard objects, susceptible of high polish, be ground together, their surfaces in contact may be so smoothed as to fit closely together and slide one on the other without separating. If now a third surface be ground alternately against each of these two smooth surfaces until it accurately fits both, then we say that each of the three surfaces is approximately plane. If one such plane surface cut through another, we say the common boundary or line where they cross is approximately a straight line. If three approximately plane surfaces on objects cut through a fourth, in general they make a figure we may call an approximate triangle. Such triangles vary greatly in shape. But no matter what the shape, if we cut off the six ends of any two such and place them side by side on a plane with their vertices at the same point, the six are found, with a high degree of approximation, just to fill up the plane about the point. Thus the six angles of any two approximate triangles are found to be together approximately four right angles.

Now, does the exactness of this approximation depend only on the straightness of the sides of the original two triangles, or also upon the size of these triangles?

If we know with absolute certitude, as the Yale professors imagine, that the size of the triangles has nothing to do with it, then we know something that we have no right to know, according to the doctrine of evolution; something impossible for us ever to have learned evolutionally.

### 10. THE NEW EPOCH.

Yet before the epoch-making ideas of Lobachevski and John Bolyai every one made this mistake, every one supposed we were perfectly sure that the angle-sum of an actual approximate triangle approached two right angles with an exactness dependent only on the straightness of the sides, and not at all on the size of the triangle.

### 11. THE SLIPS OF PHILOSOPHY.

The Scotch philosophy accounted for this absolute metrically exact knowledge by teaching that there are in the human mind certain synthetic theorems, called intuitions, supernaturally inserted there. Dr. McCosh elaborated this doctrine in a big book entitled 'The Intuitions of the Mind Inductively Investigated.' One of these supernatural intuitions was Euclid's parallel-postulate! *Voilà!*

'Yet,' to quote a sentence from Wenley's criticism in SCIENCE, of McCosh's disciple Ormond, 'we may well doubt whether a thinker, standing with one foot firmly planted on the Rock of Ages and the other pointing heavenward, has struck the attitude most conducive to progress.'

Kant, supposing that we knew Euclid's geometry and Aristotle's logic to be true absolutely and necessarily, accounted for the paradox by teaching that this seemingly universal synthetic knowledge was in reality particular, being part of the apparatus of the human mind itself.

But now the very foundations are cut away from under the Kantian system of philosophy by this new geometry which is in simple and perfect harmony with experience, with experiment, with the properties of the solid bodies and the motions about us. Thus this new geometry has given explanation of what in the old geometry was accepted without explanation.

### 12. WHAT GEOMETRY IS.

At last we really know what geometry is. Geometry is the science created to give understanding and mastery of the external relations of things; to make easy the explanation and description of such relations, and the transmission of this mastery. Geometry is the most perfect of the sciences. It precedes experiment and is safe above all experimentation.

The pure idea of a plane is something we have made, and by aid of which we see surfaces as perfectly plane, over-riding imperfections and variations, which themselves we can see only by help of our self-created precedent idea. Just so the straight line is wholly a creation of our own.

### 13. ARE THERE ANY LINES?

I was once consulted by an eminent theologian and a powerful chemist as to whether there are really any such things as lines. I drew a chalk-mark on the blackboard, and used the boundary idea. Along the sides of the chalk-mark is there a common boundary where the white ends and the black begins, neither white nor black, but common to both?

Said the theologian, yes. Said the chemist, no.

Though lines are my trade, I sympathized with the chemist.

There is nothing there until I create a line and then see it there, if I may say I see what is an invisible creation of my mind.

Geometry is in structure a system of theorems deduced in pure logical way from certain unprovable assumptions precreated by auto-active animal and human minds.

### 14. THE REQUIREMENT OF RIGOR IN REASONING.

Some unscientific minds have a personal antipathy to 'a perfect logical system,' 'deduced logically from simple fundamental truths.' But as Hilbert says: "The requirement of rigor, which has become proverbial in mathematics, corresponds to a universal philosophic necessity of our understanding; and, on the other hand, only by satisfying this requirement do the thought content and the suggestiveness of the problem attain their full effect. Besides, it is an error to believe that rigor in the proof is the enemy of simplicity. On the contrary, we find it confirmed by numerous examples that the rigorous method is at the same time the simpler and the more easily comprehended. The very effort for rigor forces us to find out simpler methods of proof.

"Let us look at the principles of analysis and geometry. The most suggestive and notable achievements of the last century in this field are, as it seems to me, the arithmetical formulation of the concept of the continuum, and the discovery of non-Euclidean geometry.''

The importance of the advance they had made was fully realized by John Bolyai and Lobachevski, who claimed at once, unflinchingly, that their discovery or creation marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or science, demonstrating, as had never been proved before, the supremacy of pure reason, at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

### 15. THE YOUTH LOBACHEVSKI.

Young Lobachevski at the University of Kazan, though a charity student, and, as seeking a learned career, utterly dependent on the authorities, yet plunged into all sorts of insubordination and wildness. Among other outbursts, one night at eleven o'clock he scandalized the despotic Russian authorities of the Tartar town by shooting off a great skyrocket, which prank put him promptly in prison. However, he continued to take part in all practical jokes and horse-play of the more daring students, and the reports of the commandant and inspector are never free from bitter complaints against the outrageous Lobachevski. His place as 'Kammerstudent' he lost for too great indulgence toward the misbehavior of the younger students at a Christmas festivity. In spite of all, he ventured to attend a strictly forbidden masked ball, and what was worse, in discussing the supposed interference of God to make rain, etc., he used expressions which subjected him to the suspicion of atheism. From the continual accusing reports of the commandant to the Rektor, the latter took a grudge against the troublesome Lobachevski, and reported his badness to the curator, who, in turn, with expressions of intense regret that Lobachevski should so tarnish his brilliant qualities, said he would be forced to inform the minister of education. Lobachevski seemed about to pay dear for his youthful wantonness. He was to come up as a candidate for the master's degree, but was refused by the senate, explicitly because of his bad behavior. But his friend, the foreign professor of mathematics, now rallied the three other foreign professors to save him, if he would appear before the senate, declare that he rued his evil behavior, and solemnly promise complete betterment.

This was the mettle of the youth, the dare-devil, the irrepressible, who startled the scientific sleep of two thousand years, who contemptuously overthrew the great Legendre, and stood up beside Euclid, the god of geometers; this the Lobachevski who knew he was right against a scornful world, who has given us a new freedom to explain and understand our universe and ourselves.

### 16. THE BOY BOLYAI.

Of the boy Bolyai, joint claimant of the new world, we have a brief picture by his father. ''My $(13 + \frac{1}{4})$ year old son, when he reached his ninth year, could do nothing more than speak and write German and Magyar, and tolerably play the violin by note. He knew not even to add. I began at first with Euclid; then he became familiar with Euler; now he not only knows of Vega (which is my manual in the college) the first two volumes completely, but has also become conversant with the third and fourth volumes. He loves differential and integral calculus, and works in them with extraordinary readiness and ease. Just so he lightly carries the bow through the hardest runs in violin concerts. Now he will soon finish my lectures on physics and chemistry. On these once he also passed with my grown pupils a public examination given in the Latin language, an examination worthy of all praise, where in part others questioned him *ad aperturam,* and in part, as opportunity served, I let him carry out some proofs in mechanics by the integral calculus, such as variable motion, the *tautochronism* of the cycloid, and the like. Nothing more could be wished. The simplicity, clearness, quickness and ease were enrapturing even for strangers. He has a quick and comprehensive head, and often flashes of *genius,* which many paths at once with a glance find and penetrate. He loves pure deep theories and astronomy. He is handsome and rather strongly built, and appears restful, except that he plays with other children very willingly and with fire. His character is, as far as one can judge, firm and noble. I have destined him as a sacrifice to mathematics. He also has consecrated himself thereto."

His mother, née Zsuzsanna Benkö Arkosi, wonderfully beautiful, fascinating, of extraordinary mental capacity, but always nervous, so idolized this only child that when in his fifteenth year he was to be sent to Vienna to the K. K. Ingenieur-Akademie, she said it seemed he should go, but his going would drive her distracted. And so it did.

Appointed 'sous-lieutenant,' and sent to Temesvár, he wrote thence to his father a letter in Magyar, which I had the good fortune to see at Maros-Vásárhely:

*My Dear and Good Father:*

I have so much to write about my new inventions that it is impossible for the moment to enter into great details, so I write you only on one fourth of a sheet. I await your answer to my letter of two sheets; and perhaps I would not have written you before receiving it if I had not wished to address to you the letter I am writing to the Baroness, which letter I pray you to send her.

First of all I reply to you in regard to the binomial.

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Now to something else, so far as space permits. I intend to write, as soon as I have put it into order, and when possible to publish, a work on parallels.

At this moment it is not yet finished, but the way which I have followed promises me with certainty the attainment of the goal, if it in general is attainable.

It is not yet attained, but I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it.

Now I can not say more, only so much: *that from nothing I have created another wholly new world.*

All that I have hitherto sent you compares to this only as a house of cards to a castle.

P. S.—I dare to judge absolutely and with conviction of these works of my spirit before you, my father; I do not fear from you any false interpretation (that certainly I would not merit), which signifies that, in certain regards, I consider you as a second self.

Nor was the young Magyar deceived. The early flashings of his genius culminated here in a piercing search-light penetrating and dissolving the enchanted walls in which Euclid had for two thousand years held captive the human mind.

The potential new universe, whose creation this letter announces, afterward set forth with master strokes in his 'Science Absolute of Space,' contains the old as nothing more than a special case of the new.

Already all the experts of the mathematical world are his disciples.

### 17. SOLVING THE UNIVERSE.

Henceforth the non-Euclidean geometry must be reckoned with in all culture, in all scientific thinking. It shows that the riddle of the universe is an indeterminate equation capable of entirely different sets of solutions. It shows that our universe is largely man-made, and must be often remade to be solved.

In Science for November 20, 1903, page 643, W. S. Franklin, under a heading for which he shows scant warrant, expresses himself after the following naïve fashion:

A clear understanding of the essential limitations of systematic physics is important to the engineer; it is I think equally important to the biologist, and it is of vital importance to the physicist, for, in the case of the physicist, to raise the question as to limitations is to raise the question as to whether his science does after all deal with realities, and the conclusion which must force itself on his mind is, I think, that his science, the systematic part of it, comes very near indeed to being a science of unrealities.

Of course, we deeply sympathize with this seemingly sad perception, with its accompanying 'simple weeps,' 'trailing weeps' and 'steady weeps,' but are tempted to prescribe a tonic or bracer in the form of a correspondence course in non-Euclidean geometry.

At least in part, space is a creation of the human mind entering as a subjective contribution into every physical experiment. Experience is, at least in part, created by the subject said to receive it, but really in part making it.

In rigorously founding a science, the ideal is to create a system of assumptions containing an exact and complete description of the relations between the elementary concepts of this science, its statements following from these assumptions by pure deductive logic.

18. GEOMETRY NOT EXPERIMENTAL.

Now, geometry, though a natural science, is not an experimental science. If it ever had an inductive stage, the experiments and inductions must have been made by our pre-human ancestors.

Says one of the two greatest living mathematicians, Poincaré, reviewing the work of the other, Hilbert's transcendently beautiful 'Grundlagen der Geometrie':

What are the fundamental principles of geometry? What is its origin; its nature; its scope? These are questions which have at all times engaged the attention of mathematicians and thinkers, but which took on an entirely new aspect, thanks to the ideas of Lobachevski and of Bolyai.

For a long time we attempted to demonstrate the proposition known as the *postulate of Euclid;* we constantly failed; we know now the reason for these failures.

Lobachevski succeeded in building a logical edifice as coherent as the geometry of Euclid, but in which the famous postulate is assumed false, and in which the sum of the angles of a triangle is always less than two right angles. Riemann devised another logical system, equally free from contradiction, in which this sum is on the other hand always greater than two right angles. These two geometries, that of Lobachevski and that of Riemann, are what are called the *non-Euclidean geometries*. The postulate of Euclid then can not be demonstrated; and this impossibility is as absolutely certain as any mathematical truth whatsoever. * * *

The first thing to do was to enumerate all the axioms of geometry. This was not so easy as one might suppose; there are the axioms which one sees and those which one does not see, which are introduced unconsciously and without being noticed.

Euclid himself, whom we suppose an impeccable logician, frequently applies axioms which he does not expressly state.

Is the list of Professor Hilbert final? We may take it to be so, for it seems to have been drawn up with care.

But just here this gives us a startling incident: the two greatest living mathematicians both in error. In my own class a young man under twenty, R. L. Moore, proved that of Hilbert's 'betweenness' assumptions, axioms of order, one of the five is redundant, and by a proof so simple and elegant as to be astonishing. Hilbert has since acknowledged this redundancy.

The same review touches another fundamental point as follows:

Hilbert's Fourth Book treats of the measurement of plane surfaces. If this measurement can be easily established without the aid of the principle of Archimedes, it is because two equivalent polygons can either be decomposed into triangles in such a way that the component triangles of the one and those of the other are equal each to each (so that, in other words, one polygon can be converted into the other after the manner of the Chinese puzzle [by cutting it up and rearranging the pieces]), or else can be regarded as the difference of polygons capable of this mode of decomposition (this is really the same process, admitting not only positive triangles but also negative triangles).

But we must observe that an analogous state of affairs does not seem to exist in the case of two equivalent polyhedra, so that it becomes a question whether we can determine the volume of the pyramid, for example, without an appeal more or less disguised to the infinitesimal calculus. It is, then, not certain whether we could dispense with the axiom of Archimedes as easily in the measurement of volumes as in that of plane areas. Moreover, Professor Hilbert has not attempted it.

Max Dehn, a young man of twenty-one, in *Mathematische Annalen,* Band 55, proved that the treatment of equivalence by cutting into a finite number of parts congruent in pairs, can never be extended from two to three dimensions.

Poincaré's review first appeared in September, 1902. But on July 1, 1902, I had already presented, before this very section, a complete solution of the question or problem he proposes, the determination of volume without any appeal to the infinitesimal calculus, without any use of the axiom of Archimedes.

### 19. THE TEACHING OF GEOMETRY.

As Study has said: "Among conditions to a more profound understanding of even very elementary parts of the Euclidean geometry, the knowledge of the non-Euclidean geometry can not be dispensed with."

How shall we make this new creation, so fruitful already for the theory of knowledge, for kenlore, bear fruit for the teaching of geometry? What new ways are opened by this masterful explosion of pure genius, shattering the mirrors which had so dazzlingly protected from perception both the flaws and triumphs of the old Greek's marvelous, if artificial, construction?

One advance has been safely won and may be rested on. There should be a preliminary course of intuitive geometry which does not strive to be rigorously demonstrative, which emphasizes the sensuous rather than the rational, giving full scope for those new fads, the using of pads of squared paper, and the so-called laboratory methods so well adapted for the feeble-minded. Hailmann, in his preface, sums up 'the purpose throughout' in these significant words: 'And thus, *incidentally,* to stimulate genuine vital interest in the study of geometry.'

I remember Sylvester's smile when he told me he had never owned a mathematical or drawing instrument in his life.

His great twin brother, Cayley, speaks of space as 'the representation [creation] lying at the foundation of all external experience.' 'And these objects, points, lines, circles, etc., in the mathematical sense of the terms, have a likeness to, and are represented more or less imperfectly, and from a geometer's point of view, *no matter how imperfectly,* by corresponding physical points, lines, circles, etc.'

But geometry, always relied upon for training in the logic of science, for teaching what demonstration really is, must be made more worthy the world's faith. There is need of a text-book of rational geometry really rigorous, a book to give every clear-headed youth the benefit of his living after Bolyai and Hilbert.

### 20. THE NEW RATIONAL GEOMETRY.

The new system will begin with still simpler ideas than did the great Alexandrian, for example, the 'betweenness' assumptions; but can confound objectors by avoiding the old matters and methods which have been the chief points of objection and contest. For example, says Mr. Perry, 'I wasted much precious time of my life on the fifth book of Euclid.' Says the great Cayley: 'There is hardly anything in mathematics more beautiful than his wondrous fifth book.'

For my own part, nothing ever better repaid study. But the contest is over, for now, at last, without sacrificing a whit of rigor, we are able to give the whole matter by an algebra as simple as if only approximate, like Euclid, including incommensurables without even mentioning them.

Again, we shall regain the pristine purity of Euclid in the matter of what Jules Andrade calls 'cette malheureuse et illogique definition' (Phillips and Fisher, §7): 'A straight line is a line which is the shortest path between any two of its points.'

As to this hopeless muddle, which has been condemned *ad nauseam,* notice that it is senseless without a definition for the length of a curve. Yet, Professor A. Lodge, in a discussion on reform, says: "I believe we could not do better than adopt some French text-book as our model. Also I., 24, 25, being obviously related to I., 4, are made to immediately follow it in such of the French books as define a straight line to be the shortest distance between two points." Professor Lodge, then, does not know that the French themselves have repudiated this nauseous pseudo-definition. Of it Laisant says (p. 223):

This definition, almost unanimously abandoned, represents one of the most remarkable examples of the persistence with which an absurdity can propagate itself throughout the centuries.

In the first place, the idea expressed is incomprehensible to beginners, since it presupposes the notion of the length of a curve; and further, it is a vicious circle, since the length of a curve can only be understood as the limit of a sum of rectilinear lengths; moreover, it is not a definition at all, since, on the contrary, it is a demonstrable proposition.

As to what a tremendous affair this proposition really is, consult Georg Hamel in *Mathematische Annalen* for this very year (p. 242), who employs to adequately express its content the refinements of the integral calculus and the modern theory of functions.

Moreover, underneath all this even is the assumption of the theorem, Euclid, I., 20: 'Any two sides of a triangle are together greater than the third side'; upon which proposition, which the Sophists said even donkeys knew, Hilbert has thrown brilliant new light in the *Proceedings of the London Mathematical Society,* 1902, pp. 50–68, where he creates a geometry in which the donkeys are mistaken, a geometry in which two sides of a triangle may be together *less* than the third side, exhibiting as a specific and definite example a right triangle in which the sum of the two sides is less than the hypothenuse.

Any respectably educated person knows that in general the length of a curve is defined by the aggregate formed by the lengths of a proper sequence of inscribed polygons.

The curve of itself has no length. This definition in ordinary cases creates for the curve a length; but in case the aggregate is not convergent, the curve is regarded as not rectifiable. It had no length, and even our creative definition has failed to endow it with length; so it has no length, and lengthless it must remain.

If, however, it can be shown that the lengths of these inscribed polygons form a convergent aggregate which is independent of the particular choice of the polygons of the sequence, the curve is rectifiable, its length being defined by the number given by the aggregate.

### 21. GEOMETRY WITHOUT ANY CONTINUITY ASSUMPTION.

Euclid in his very first proposition and again in I., 22, 'to make a triangle from given sides,' uses unannounced a continuity assumption. But nearly the whole of Euclid can be obtained without any continuity assumption whatever, and this great part it is which forms the real domain of elementary geometry.

Continuity belongs, with limits and infinitesimals, in the Calculus.

Professor W. G. Alexejeff, of Dorpat, in 'Die Mathematik als Grundlage der Kritik wissenschaftlich-philosophischer Weltanschauung' (1903), shows how men of science have stultified themselves by ignorantly presupposing continuity. He calls that a higher standpoint which takes account of the individuality of the elements, and gives as examples of this discrete or discontinuous mathematics the beautiful enumerative geometry, the invariants of Sylvester and Cayley, and in chemistry the atomic-structure theory of Kekulé and the periodic system of the chemical elements by Mendelejev, to which two theories, both exclusively discrete in character, we may safely attribute almost entirely the present standpoint of the science.

Still more must discontinuity play the chief rôle in biology and sociology, dealing as they do with differing individuals, cells and persons. How desirable, then, that the new freedom should appear even as early as in elementary geometry.

After mathematicians all knew that number is in origin and basis entirely independent of measurement or measurable magnitude; after in fact the dominant trend of all pure mathematics was its arithmetization, weeding out as irrelevant any fundamental use of measurement or measurable quantity, there originated in Chicago from the urbane Professor Dewey (whom, in parenthesis, I must thank for his amiable courtesy throughout the article in the *Educational Review* which he devoted to my paper on the 'Teaching of Geometry'), the shocking tumble or reversal that the origin, basis and essence of number is measurement.

Many unfortunate teachers and professors of pedagogy ran after the new darkness, and even books were issued trying to teach how to use these dark lines in the spectrum for illuminating purposes.

There is a ludicrous element in the parody of all this just now in the domain of geometry.

After mathematicians all know of the wondrous fruit and outcome of the non-Euclidean geometry in removing all the difficulties of pure elementary geometry, there comes another philosopher, a Mr. Perry, who never having by any chance heard of all this, advises the cure of these troubles by the abolition of rational geometry.

Just as there was a Dewey movement so is there a Perry movement, and books on geometry written by persons who never read 'Alice in Wonderland' or its companion volume, 'Euclid and his Modern Rivals.'

But the spirits of Bolyai and Lobachevski smile on this well-meaning strenuosity, and whisper, 'It is something to know what proof is and what it is not; and where can this be better learned than in a science which has never had to take one footstep backward?'

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## *THE SOCIETY FOR PLANT MORPHOLOGY AND PHYSIOLOGY.*

THE seventh regular annual meeting of this society was held, in conjunction with the meetings of several other scientific societies, at the University of Pennsylvania, Philadelphia, Pa., December 28–30, 1903. In the absence of the president and vice-president, the most recent past president, Dr. Erwin F. Smith, presided. Though not large in point of numbers the meeting