In accordance with the general reasoning of Hilbert, we infer that all combinants of three binary forms are integral algebraic functions of invariants of J, and therefore, *a fortiori*, of the coefficients of J.

The results for any number of binary forms are exactly the same.

Addition Theorems for Hyperelliptic Integrals. *By* A. L. DIXON.

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The present communication is a continuation of my paper on "An Addition Theorem for Hyperelliptic Theta-Functions," presented to the Society in December, 1900 (*Proc. Lond. Math. Soc.*, Vol. XXXIII., No. 755).

The method there given of deducing theorems in the theory of hyperelliptic integrals from the geometrical properties of confocals is applied to the investigation of addition theorems for the integrals of the second and third kinds.*

I must record my obligation to a paper by Herr O. Staude, on the "Geometrische Deutung der Additionstheoreme der hyperelliptischen Integrale" (*Math. Ann.*, Bd. XXII., 1883). In particular the fundamental idea of § 4 has been taken from that paper.

References to my first paper are prefixed by the number I.

Integrals of the Second Kind.

1. Taking the equations (11), I., § 2, of the straight lines through the point h_ι, which lie in the surfaces S and T, one of them is given by

$$\frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \frac{\xi_q}{\sqrt{q-s\,.\,q-t\,.\,r-p}} = \frac{\xi_r}{\sqrt{r-s\,.\,r-t\,.\,p-q}},$$

$$\xi_s = 0, \quad \xi_t = 0.$$

Let S be the distance measured along this line from h_ι. Then

$$S = \sqrt{\Sigma\,(x_\iota - h_\iota)^2} = \sqrt{\xi_p^2 + \xi_q^2 + \xi_r^2}; \tag{1}$$

and therefore

$$\frac{\iota S}{\sqrt{q-r\,.\,r-p\,.\,p-q}} = \frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \dots \tag{2}$$

* A paper on the application of the method to confocal conicoids in ordinary space and the deduction of theorems for elliptic integrals has appeared in the *Quarterly Journal*, No. 131, 1902.

Therefore also

$$\frac{2\iota\, dS}{\sqrt{q-r\,.\,r-p\,.\,p-q}}=\frac{2ds_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}}=\dots \tag{3}$$

$$=\frac{\sqrt{p-q\,.\,p-r}}{\sqrt{q-r}}\,\frac{dp}{\sqrt{P}}=\dots, \tag{4}$$

$$2dS=(p-q)(p-r)\frac{dp}{\sqrt{P}}=(q-r)(q-p)\frac{dq}{\sqrt{Q}}=(r-p)(r-q)\frac{dr}{\sqrt{R}};$$

and therefore $\qquad 2dS=p^2\dfrac{dp}{\sqrt{P}}+q^2\dfrac{dq}{\sqrt{Q}}+r^2\dfrac{dr}{\sqrt{R}}. \qquad (5)$

Integrating, we get

$$2S=\int p^2\frac{dp}{\sqrt{P}}-\int p_0^2\frac{dp_0}{\sqrt{P_0}}+\int q^2\frac{dq}{\sqrt{Q}}-\int q_0^2\frac{dq_0}{\sqrt{Q_0}}+\int r^2\frac{dr}{\sqrt{R}}-\int r_0^2\frac{dr_0}{\sqrt{R_0}},$$

where $\qquad S=\sqrt{\Sigma\,(x_\iota-h_\iota)^2}=\sqrt{\dfrac{p_0-q_0\,.\,p_0-r_0}{p_0-s\,.\,p_0-t}}\,\xi_p=\dots.$

2\. Since
$$\Sigma x_\iota^2=\Sigma a_\iota+p+q+r+s+t,$$
$$\Sigma h_\iota^2=\Sigma a_\iota+p_0+q_0+r_0+s+t,$$

we get

$$S^2=2\Sigma a_\iota+2s+2t+p+p_0+q+q_0+r+r_0$$
$$-2\Sigma_\iota\frac{(a_\iota+s)(a_\iota+t)}{f'(-a_\iota)}\sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}, \tag{7}$$

where s and t may be given any value we please, and, in fact, the coefficients of $s+t$ and st vanish by I. (15).

Putting $s=-a_4$, $t=-a_5$, I get

$$S^2=2\,(a_1+a_2+a_3)+p+p_0+q+q_0+r+r_0$$
$$-2\,\frac{\sqrt{a_1+p\,.\,a_1+p_0\,.\,a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0}}{(a_1-a_2)(a_1-a_3)}$$
$$-2\,\frac{\sqrt{a_2+p\,.\,a_2+p_0\,.\,a_2+q\,.\,a_2+q_0\,.\,a_2+r\,.\,a_2+r_0}}{(a_2-a_3)(a_2-a_1)}$$
$$-2\,\frac{\sqrt{a_3+p\,.\,a_3+p_0\,.\,a_3+q\,.\,a_3+q_0\,.\,a_3+r\,.\,a_3+r_0}}{(a_3-a_1)(a_3-a_2)} \tag{8}$$

Also $$\xi_p = \frac{\Sigma_\iota \dfrac{h_\iota x_\iota}{a_\iota + p_0} - 1}{\left(\Sigma \dfrac{h_\iota^2}{(a_\iota+p_0)^2}\right)^{\frac{1}{2}}} = \frac{(s-p_0)(t-p_0)\,\Sigma \dfrac{h_\iota x_\iota}{(a_\iota+p_0)(a_\iota+s)(a_\iota+t)}}{\left(\Sigma \dfrac{h_\iota^2}{(a_\iota+p_0)^2}\right)^{\frac{1}{2}}},$$

since $$\xi_r = 0, \quad \xi_s = 0;$$

and therefore

$$S = \sqrt{P_0}\left\{\Sigma_\iota \frac{\sqrt{a_\iota+p \,.\, a_\iota+q \,.\, a_\iota+q_0 \,.\, a_\iota+r \,.\, a_\iota+r_0}}{f'(-a_\iota)\sqrt{a_\iota+p_0}}\right\}. \tag{9}$$

3. Another expression for S, which will be used hereafter, is obtained as follows. We have

$$S = \frac{\sqrt{p_0-q_0 \,.\, p_0-r_0}}{\sqrt{p_0-s \,.\, p_0-t}}\,\xi_p = \frac{\sqrt{P_0}}{(p_0-s)(p_0-t)}\left(\Sigma \frac{h_\iota x_\iota}{a_\iota+p_0} - 1\right), \tag{10}$$

where s and t are arbitrary constants.

Putting $s = -a_1$, $t = -a_2$, I get

$$\frac{\sqrt{a_1+p_0 \,.\, a_2+p_0}}{\sqrt{a_3+p_0 \,.\, a_4+p_0 \,.\, a_5+p_0}}\,S$$
$$= \Sigma \frac{\sqrt{a_3+p \,.\, a_3+p_0 \,.\, a_3+q \,.\, a_3+q_0 \,.\, a_3+r \,.\, a_3+r_0}}{(a_3-a_4)(a_3-a_5)(a_3+p_0)} - 1, \tag{11}$$

the other two terms in the Σ corresponding to a_4 and a_5. Also, interchanging p and p_0, q and q_0, r and r_0, I get

$$\frac{\sqrt{a_1+p \,.\, a_2+p}}{\sqrt{a_3+p \,.\, a_4+p \,.\, a_5+p}}\,(-S)$$
$$= \Sigma \frac{\sqrt{a_3+p \,.\, a_3+p_0 \,.\, a_3+q \,.\, a_3+q_0 \,.\, a_3+r \,.\, a_3+r_0}}{(a_3-a_4)(a_3-a_5)(a_3+p)} - 1.$$

Then, by subtraction,

$$\frac{S}{p-p_0}\left\{\frac{\sqrt{a_1+p_0 \,.\, a_2+p_0}}{\sqrt{a_3+p_0 \,.\, a_4+p_0 \,.\, a_5+p_0}} + \frac{\sqrt{a_1+p \,.\, a_2+p}}{\sqrt{a_3+p \,.\, a_4+p \,.\, a_5+p}}\right\}$$
$$= \frac{\sqrt{a_3+q \,.\, a_3+q_0 \,.\, a_3+r \,.\, a_3+r_0}}{(a_3-a_4)(a_3-a_5)\sqrt{a_3+p \,.\, a_3+p_0}} + \frac{\sqrt{a_4+q \,.\, a_4+q_0 \,.\, a_4+r \,.\, a_4+r_0}}{(a_4-a_5)(a_4-a_3)\sqrt{a_4+p \,.\, a_4+p_0}}$$
$$+ \frac{\sqrt{a_5+q \,.\, a_5+q_0 \,.\, a_5+r \,.\, a_5+r_0}}{(a_5-a_3)(a_5-a_4)\sqrt{a_5+p \,.\, a_5+p_0}}. \tag{12}$$

Integrals of the Third Kind.

4. To find corresponding expressions applicable to integrals of the third kind, let us take the generalized conception of distance as given by Cayley in his sixth memoir upon quantics (*Coll. Works*, Vol. II., pp. 583–592).

Taking for the absolute the continuum

$$\Sigma_\iota \frac{x_\iota^2}{a_\iota+n} = 1^* \quad (\iota = 1, 2, 3, 4, 5),$$

the distance S' between any two points x_ι and h_ι is given by

$$\cos S' = \frac{\Sigma \frac{x_\iota h_\iota}{a_\iota+n} - 1}{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}} \left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}}}; \tag{13}$$

and therefore

$$\sin^2 S' = \frac{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)\left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right) - \left(\Sigma \frac{h_\iota x_\iota}{a_\iota+n} - 1\right)^2}{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)\left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right)}$$

$$= \frac{-\Pi\,(a_\iota+n)}{(n-p)(n-q)(n-r)(n-s)(n-t)}$$

$$\times \left\{ \frac{\xi_p^2}{n-p_0} + \frac{\xi_q^2}{n-q_0} + \frac{\xi_r^2}{n-r_0} + \frac{\xi_s^2}{n-s} + \frac{\xi_t^2}{n-t} \right\}.\dagger \tag{14}$$

5. To find an expression for dS' at any point, suppose the point h_ι to move up to and ultimately coincide with the point x_ι, and we get, writing

$$N \equiv \Pi\,(a_\iota+n), \quad N' \equiv \Pi\,(n-\lambda) \quad (\lambda = p, q, r, s, t),$$

$$dS'^2 = -\frac{N}{N'}\left\{ \frac{ds_p^2}{n-p} + \frac{ds_q^2}{n-q} + \frac{ds_r^2}{n-r} + \frac{ds_s^2}{n-s} + \frac{ds_t^2}{n-t} \right\}\dagger \tag{15}$$

$$= -\tfrac{1}{4}\frac{N}{N'}\Sigma \frac{(p-q)(p-r)(p-s)(p-t)}{(n-p)\,P}\,dp^2. \tag{16}$$

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* Staude, *loc. cit.*, *Math. Ann.*, Bd. XXII., p. 23, § 7.

† ξ_p, ξ_q, d_{sp}, d_{sq}, ... have exactly the same meaning here as in the last section, that is, they represent the same expressions in x_ι, or in p, q, r, s, t.

Now along the straight lines considered, namely those which lie in both the surfaces S and T,

$$ds = 0, \quad dt = 0,$$

and, as in the preceding section, we get

$$\frac{(n-p)(n-q)(n-r)\, 2dS'}{\sqrt{N}\,.\,q-r\,.\,r-p\,.\,p-q} = \frac{2ds_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \ldots \tag{17}$$

$$= \frac{\sqrt{p-q\,.\,p-r}}{\sqrt{q-r}}\,\frac{dp}{\sqrt{P}} = \ldots, \tag{18}$$

$$\frac{(n-p)(n-q)(n-r)}{\sqrt{N}}\, 2\iota dS' = (p-q)(p-r)\,\frac{dp}{\sqrt{P}} = (q-r)(q-p)\,\frac{dq}{\sqrt{Q}}$$

$$= (r-p)(r-q)\,\frac{dr}{\sqrt{R}},$$

$$\frac{2\iota dS'}{\sqrt{N}} = \frac{dp}{(n-p)\sqrt{P}} + \frac{dq}{(n-q)\sqrt{Q}} + \frac{dr}{(n-r)\sqrt{R}}. \tag{19}$$

Integrating, we get

$$2\iota S' = \int \frac{\sqrt{N}\,dp}{(n-p)\sqrt{P}} - \int \frac{\sqrt{N}\,dp_0}{(n-p_0)\sqrt{P_0}} + \int \frac{\sqrt{N}\,dq}{(n-q)\sqrt{Q}} - \int \frac{\sqrt{N}\,dq_0}{(n-q_0)\sqrt{Q_0}}$$

$$+ \int \frac{\sqrt{N}\,dr}{(n-r)\sqrt{R}} - \int \frac{\sqrt{N}\,dr_0}{(n-r_0)\sqrt{R_0}}. \tag{20}$$

6. One expression for S' is given by

$$\cos S' = \frac{\Sigma \dfrac{x_\iota h_\iota}{a_\iota+n} - 1}{\left(\Sigma \dfrac{h_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}} \left(\Sigma \dfrac{x_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}}}$$

$$= \frac{N\left\{\Sigma_\iota \dfrac{(a_\iota+s)(a_\iota+t)}{(a_\iota+n)f'(-a_\iota)} \sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0} - 1\right\}}{(n-s)(n-t)\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}} \tag{21}$$

where s and t may have any value. Putting $s = \infty$, $t = -a_5$, I get

$$\cos S' = \frac{(a_1+n)(a_2+n)(a_3+n)(a_4+n)}{\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}}$$

$$\times \left\{\Sigma \frac{\sqrt{a_1+p\,.\,a_1+p_0\,.\,a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0}}{(a_1+n)(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right\}, \tag{22}$$

the other three terms of the Σ corresponding to a_2, a_3, and a_4. Also, putting $s = t = \infty$, I get another form, viz.,

$$\cos S' = \frac{N\Sigma_\iota \dfrac{\sqrt{a_\iota+p \,.\, a_\iota+p_0 \,.\, a_\iota+q \,.\, a_\iota+q_0 \,.\, a_\iota+r \,.\, a_\iota+r_0}}{(a_\iota+n)\, f'(-a_\iota)}}{\sqrt{n-p \,.\, n-p_0 \,.\, n-q \,.\, n-q_0 \,.\, n-r \,.\, n-r_0}}. \quad (23)$$

Another expression for S' follows from (14) and (2). For, putting

$$\frac{\xi_p}{\sqrt{p_0-s \,.\, p_0-t \,.\, q_0-r_0}} = \frac{\xi_q}{\sqrt{q_0-s \,.\, q_0-t \,.\, r_0-p_0}}$$

$$= \frac{\xi_r}{\sqrt{r_0-s \,.\, r_0-t \,.\, p_0-q_0}} = \frac{\iota S}{\sqrt{q_0-r_0 \,.\, r_0-p_0 \,.\, p_0-q_0}}$$

in (14), we get

$$\sin^2 S' = -\frac{NS^2}{(n-p)(n-q)(n-r)(n-p_0)(n-q_0)(n-r_0)},$$

or
$$\sin S' = \frac{\iota\sqrt{NS}}{\sqrt{n-p \,.\, n-p_0 \,.\, n-q \,.\, n-q_0 \,.\, n-r \,.\, n-r_0}}. \quad (24)$$

Confocals of Revolution.

7. It is also interesting from the geometrical point of view to consider the results obtained when two of the parameters a are equal to one another, and one of the families degenerates into the system of planes through an axis.

It will be found that in this way a real geometrical construction is obtained for the sum of integrals of the third kind.

Take
$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} = 1 \quad (\iota = 1, 2, 3, 4, 5, 6),$$

where
$$x_1^2 = y^2+z^2,$$

so that y, z, x_2, x_3, x_4, x_5, x_6 are Cartesian coordinates, as the equation of a set of confocal ${}_2R_6'$'s of revolution in a space S_7, and let q, r, s, t, u, v be the values of λ for the six members of the set through any point. The degenerate seventh member of the set corresponding to the parameter p is given by

$$y = z \tan\theta.$$

Then, as before,

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda}-1 \equiv -\frac{(\lambda-q)(\lambda-r)(\lambda-s)(\lambda-t)(\lambda-u)(\lambda-v)}{\Pi\,(a_\iota+\lambda)};$$

$$x_\iota^2 = -\frac{(a_\iota+q)(a_\iota+r)(a_\iota+s)(a_\iota+t)(a_\iota+u)(a_\iota+v)}{f'(-a_\iota)},$$

writing $$f(\lambda) \equiv \Pi\,(a_\iota+\lambda);$$

and $$4ds_q^2 = \Sigma_\iota \frac{x_\iota^2}{(a_\iota+q)^2}\,dq^2 = \frac{(q-r)(q-s)(q-t)(q-u)(q-v)}{f(q)}\,dq^2, \ldots,$$

but ds_p^2 is replaced by $x_1^2 d\theta^2$, that is, by

$$-\frac{(a_1+q)(a_1+r)(a_1+s)(a_1+t)(a_1+u)(a_1+v)}{f'(-a_1)}\,d\theta^2.$$

8. In considering the "tangent cone," we may without loss of generality take the coordinates of the point h_ι to be $0, h_1, h_2, h_3, h_4, h_5, h_6$; so that its equation is

$$\left(\frac{y^2+z^2}{a_1+\lambda}+\Sigma_\iota \frac{x_\iota^2}{a_\iota+\lambda}-1\right)\left(\frac{h_1^2}{a_1+\lambda}+\Sigma_\iota \frac{h_\iota^2}{a_\iota+\lambda}-1\right)$$
$$= \left(\frac{zh_1}{a_1+\lambda}+\Sigma_\iota \frac{x_\iota h_\iota}{a_\iota+\lambda}-1\right)^2 \quad (\iota = 2, 3, 4, 5, 6),$$

which when referred to its principal axes takes the form

$$\frac{y^2}{a_1+\lambda}+\frac{\xi_q^2}{\lambda-q}+\frac{\xi_r^2}{\lambda-r}+\frac{\xi_s^2}{\lambda-s}+\frac{\xi_t^2}{\lambda-t}+\frac{\xi_u^2}{\lambda-u}+\frac{\xi_v^2}{\lambda-v}=0.$$

Then, exactly as before (I., § 12), the common points of the three surfaces T, U, V and the three tangent planes T', U', V' are given by

$$\xi_t = 0, \quad \xi_u = 0, \quad \xi_v = 0,$$

$$\frac{y^2}{(a_1+t)(a_1+u)(a_1+v)(q-r)(r-s)(s-q)}$$
$$= \frac{-\xi_q^2}{(q-t)(q-u)(q-v)(r-s)(a_1+s)(a_1+r)}$$
$$= \frac{\xi_r^2}{(r-t)(r-u)(r-v)(a_1+s)(a_1+q)(q-s)}$$
$$= \frac{-\xi_s^2}{(s-t)(s-u)(s-v)(a_1+q)(q-r)(a_1+r)}, \qquad (A)$$

putting $-a_1$ for ρ in the equations of I., § 12.

Now, writing, for y, $x_1 d\theta$; for ξ_q, ds_q; &c., we get, as the differential equations of the eight lines in the surfaces T, U, V,

$$\frac{1}{(q-r)(r-s)(s-q)}\frac{2\,d\theta}{\sqrt{f'(-a_1)}} = \frac{\epsilon}{(r-s)(a_1+r)(a_1+s)}\frac{dq}{\sqrt{a_1+q}\sqrt{f(q)}}$$
$$= \frac{\epsilon'}{(s-q)(a_1+s)(a_1+q)}\frac{dr}{\sqrt{a_1+r}\sqrt{f(r)}}$$
$$= \frac{\epsilon''}{(q-r)(a_1+q)(a_1+r)}\frac{ds}{\sqrt{a_1+s}\sqrt{f(s)}}.$$

Writing $\quad Q \equiv (a_2+q)(a_3+q)(a_4+q)(a_5+q)(a_6+q), \ \ldots,$

these are equivalent to

$$\epsilon\frac{dq}{\sqrt{Q}}+\epsilon'\frac{dr}{\sqrt{R}}+\epsilon''\frac{ds}{\sqrt{S}} = 0,$$

$$\epsilon\frac{q\,dq}{\sqrt{Q}}+\epsilon'\frac{r\,dr}{\sqrt{R}}+\epsilon''\frac{s\,ds}{\sqrt{S}} = 0,$$

$$\frac{2\,d\theta}{\sqrt{f'(-a_1)}}+\frac{\epsilon\,dq}{(a_1+q)\sqrt{Q}}+\frac{\epsilon'\,dr}{(a_1+r)\sqrt{R}}+\frac{\epsilon''\,ds}{(a_1+s)\sqrt{S}} = 0,$$

and the integral of these is

$$\cos\theta\,\frac{(a_1+\lambda)(a_1+\mu)\sqrt{a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0\,.\,a_1+s\,.\,a_1+s_0}}{f'(-a_1)}$$
$$+\Sigma_\iota\frac{(a_\iota+\lambda)(a_\iota+\mu)\sqrt{a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0\,.\,a_\iota+s\,.\,a_\iota+s_0}}{f'(-a_\iota)}+1 = 0$$
$$(\iota = 2, 3, 4, 5, 6),$$

which is the same as (21).

9. Also putting $\qquad S^2 = y^2+\xi_q^2+\xi_r^2+\xi_s^2,$

we get, from (A),

$$\frac{S^2}{(a_1+q_0)(a_1+r_0)(a_1+s_0)} = \frac{y^2}{(a_1+t)(a_1+u)(a_1+v)};$$

and therefore, substituting for y,

$$S = \frac{\sqrt{a_1+q\,.\,a_1+r\,.\,a_1+s\,.\,a_1+q_0\,.\,a_1+r_0\,.\,a_1+s_0}}{\sqrt{a_2-a_1\,.\,a_3-a_1\,.\,a_4-a_1\,.\,a_5-a_1\,.\,a_6-a_1}}\sin\theta,$$

which is equation (24).

In fact θ represents the "distance" between two points, when the absolute is taken to be

$$y^2+z^2 = 0.$$

Paraboloids.

10. The degenerate case of paraboloids may also be briefly noticed. The particular case here worked out gives an integral of the two equations

$$\sum_{r=1}^{r=6} u_r = 0, \quad \sum_{r=1}^{r=6} \Pi(u_r a) = 0,$$

where u is an ordinary elliptic integral of the first kind, and $\Pi(u, a)$ one of the third kind. Starting with

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} = 4a(x+a\lambda) \quad (\iota = 1, 2, 3, 4),$$

I shall get

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} - 4a(x+a\lambda) = -\frac{4a^2(\lambda-p)(\lambda-q)(\lambda-r)(\lambda-s)(\lambda-t)}{\prod_\iota (a_\iota+\lambda)}.$$

Then $$x_\iota^2 = \frac{4a^2(a_\iota+p)(a_\iota+q)(a_\iota+r)(a_\iota+s)(a_\iota+t)}{f'(-a_\iota)},$$

where $$f(\lambda) = \Pi(a_\iota+\lambda).$$

Also $$-\frac{x}{a} = p+q+r+s+t,$$

and so

$$4ds_p^2 = dp^2\left(\Sigma \frac{x_\iota^2}{(a_\iota+p)^2} + 4a^2\right) = \frac{4a^2(p-q)(p-r)(p-s)(p-t)}{f(p)}\,dp^2.$$

The results of I., §§ 2, 3 will not be altered, and I shall get an algebraical integral of the equations

$$\int\frac{dp}{\sqrt{P}} - \int\frac{dp_0}{\sqrt{P_0}} + \epsilon\left(\int\frac{dq}{\sqrt{Q}} - \int\frac{dq_0}{\sqrt{Q_0}}\right) + \epsilon'\left(\int\frac{dr}{\sqrt{R}} - \int\frac{dr_0}{\sqrt{R_0}}\right) = 0,$$

$$\int\frac{p\,dp}{\sqrt{P}} - \int\frac{p_0\,dp_0}{\sqrt{P_0}} + \epsilon\left(\int\frac{q\,dq}{\sqrt{Q}} - \int\frac{q_0\,dq_0}{\sqrt{Q_0}}\right) + \epsilon'\left(\int\frac{r\,dr}{\sqrt{R}} - \int\frac{r_0\,dr_0}{\sqrt{R_0}}\right) = 0,$$

where $$\Theta \equiv f(\theta) \equiv (a_1+\theta)(a_2+\theta)(a_3+\theta)(a_4+\theta),$$

in the form

$$\Sigma\frac{(a_\iota+\lambda)\sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}}{f'(-a_\iota)}$$
$$+p+p_0+q+q_0+r+r_0+2\lambda = 0 \quad (\iota = 1, 2, 3, 4),$$

where λ is an arbitrary constant.

11. I proceed to express the results obtained in the notation adopted in the former communication (I., §§ 5, 6). Making the substitutions

$$p = \frac{1}{a-x}, \quad a_1 = \frac{1}{b-a}, \quad a_2 = \frac{1}{c-a}, \quad \ldots,$$

and writing $\Theta \equiv a-\theta . b-\theta . c-\theta . d-\theta . e-\theta . f-\theta$,

I take

$$\left. \begin{aligned} u &= \int \frac{(e-x)\,dx}{(e-f)\sqrt{X}} - \int \frac{(e-y)\,dy}{(e-f)\sqrt{Y}} \\ v &= \int \frac{(f-x)\,dx}{(f-e)\sqrt{X}} - \int \frac{(f-y)\,dy}{(f-e)\sqrt{Y}} \end{aligned} \right\}, \tag{25}$$

from which

$$\left. \begin{aligned} (x-y)\frac{dx}{\sqrt{X}} &= (f-y)\,du + (e-y)\,dv \\ (x-y)\frac{dy}{\sqrt{Y}} &= (f-x)\,du + (e-x)\,dv \end{aligned} \right\}. \tag{26}$$

Then $\int \frac{p^2 dp}{\sqrt{P}} - \int \frac{p_0^2 dp_0}{\sqrt{P_0}}$ becomes

$$\sqrt{b-a.c-a.d-a.e-a.f-a}\left\{\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{(a-y)\sqrt{Y}}\right\},$$

and

$$\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{(a-y)\sqrt{Y}} = \int \frac{a+f-x-y}{(a-x)(a-y)}\,du + \int \frac{a+e-x-y}{(a-x)(a-y)}\,dv.$$

But

$$F^2 = \zeta^2 (f-x)(f-y),$$

$$A^2 = \alpha^2 (a-x)(a-y);$$

therefore

$$F^2/\zeta^2 - A^2/\alpha^2 = (f-a)(a+f-x-y),$$

and so

$$\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{a-y\sqrt{Y}} = \frac{u}{a-f} + \frac{v}{a-e} + \int \frac{\alpha^2 F^2 du}{(f-a)\,\zeta^2 A^2} + \int \frac{\alpha^2 E^2 dv}{(e-a)\,\epsilon^2 A^2},$$

and therefore $\int \frac{p^2 dp}{\sqrt{P}} - \int \frac{p_0^2 dp_0}{\sqrt{P_0}}$ becomes

$$\sqrt{b-a.c-a.d-a.e-a.f-a}$$
$$\times \left\{\frac{u}{a-f} + \frac{v}{a-e} + \int \frac{\alpha^2 F^2 du}{(f-a)\,\zeta^2 A^2} + \int \frac{\alpha^2 E^2 dv}{(e-a)\,\epsilon^2 A^2}\right\}$$

Now $b-a$, $a-f$, ... can be expressed in terms of $\alpha, \beta, \gamma, \ldots$,* and I finally get equation (6) in the form

$$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}} = \Sigma_r \int \frac{\alpha F_r^2 du_r}{(\alpha\zeta)\,\zeta A_r^2} + \frac{\alpha E_r^2 dv_r}{(\alpha\epsilon)\,\epsilon A_r^2}, \tag{27}$$

where, as in I., § 9, $(\alpha\zeta)$ is written as an abbreviation for

$$(\alpha\beta\zeta)(\alpha\gamma\zeta)(\alpha\delta\zeta)(\alpha\epsilon\zeta).$$

Then, using $Z_A(u, v)$ to denote the function

$$\int \frac{1}{A^2}\left(\frac{\alpha F^2 du}{(\alpha\zeta)\,\zeta} + \frac{\alpha E^2 dv}{(\alpha\epsilon)\,\epsilon}\right),$$

I have
$$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}} = Z_A(u_1, v_1) + Z_A(u_2, v_2) + Z_A(u_3, v_3), \tag{28}$$

where
$$u_1+u_2+u_3 = 0 \quad\text{and}\quad v_1+v_2+v_3 = 0.$$

12. Let us now consider the transformation of S^2 as given in § 2 (8). We may obviously put

$$S^2 = \lambda_0 + \lambda\Sigma\frac{B_r^2}{A_r^2} + \mu\Sigma\frac{C_r^2}{A_r^2} + \nu\Sigma\frac{D_r^2}{A_r^2} + \lambda'\frac{B_1B_2B_3}{A_1A_2A_3} + \mu'\frac{C_1C_2C_3}{A_1A_2A_3} + \nu'\frac{D_1D_2D_3}{A_1A_2A_3},$$

where λ_0, λ, λ', ... are coefficients to be determined. Then, firstly, since there is a linear relation between the squares of A, B, C, D, λ_0 may be merged in λ, μ, ν; and, secondly, since S^2 must vanish when

$$u_3, v_3 = 0, 0 \quad\text{and}\quad u_1, v_1 = -u_2, -v_2,$$

we have
$$\beta\lambda' + 2\lambda\alpha = 0, \ldots,$$

and we may therefore put

$$\begin{aligned} S^2 = \;& \lambda\left(\frac{B_1^2}{A_1^2} + \frac{B_2^2}{A_2^2} + \frac{B_3^2}{A_3^2} - 2\frac{\alpha}{\beta}\,\frac{B_1B_2B_3}{A_1A_2A_3}\right) \\ &+ \mu\left(\frac{C_1^2}{A_1^2} + \frac{C_2^2}{A_2^2} + \frac{C_3^2}{A_3^2} - 2\frac{\alpha}{\gamma}\,\frac{C_1C_2C_3}{A_1A_2A_3}\right) \\ &+ \nu\left(\frac{D_1^2}{A_1^2} + \frac{D_2^2}{A_2^2} + \frac{D_3^2}{A_3^2} - 2\frac{\alpha}{\delta}\,\frac{D_1D_2D_3}{A_1A_2A_3}\right). \end{aligned}$$

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* For the values of α, $(\alpha\beta\zeta)$, ... in terms of a, b, c, d, e, f, see Cayley, *Coll. Works*, Vol. x., pp. 502, 503. Neglecting a fourth root of unity which occurs as a coefficient, it is easily found that

$$a-b = (\alpha\beta)/\alpha\beta, \ldots,$$

and that
$$\sqrt{b-a\,.\,c-a\,.\,d-a\,.\,e-a\,.\,f-a} = (\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}/\alpha^2.$$

But the coefficient of $-2\frac{B_1B_2B_3}{A_1A_2A_3}$ in S^2 as determined from (8) is $\frac{(c-a)(d-a)}{(b-a)(b-c)(b-d)}\frac{\alpha^3}{\beta^3}$, which, on substituting $c-a=(\overline{\alpha\gamma})/\alpha\gamma$, becomes on reduction $\frac{\alpha^2}{(\alpha\epsilon\zeta)^2}\frac{(\beta\epsilon\zeta)^2}{(\overline{\alpha\beta})}$, and so we get, finally,

$$S^2=\frac{\alpha\beta\,(\beta\epsilon\zeta)^2}{(\overline{\alpha\beta})(\alpha\epsilon\zeta)^2}\left(\frac{B_1^2}{A_1^2}+\frac{B_2^2}{A_2^2}+\frac{B_3^2}{A_3^2}-2\frac{\alpha}{\beta}\frac{B_1B_2B_3}{A_1A_2A_3}\right)$$
$$+\frac{\alpha\gamma\,(\gamma\epsilon\zeta)^2}{(\overline{\alpha\gamma})(\alpha\epsilon\zeta)^2}\left(\frac{C_1^2}{A_1^2}+\frac{C_2^2}{A_2^2}+\frac{C_3^2}{A_3^2}-2\frac{\alpha}{\gamma}\frac{C_1C_2C_3}{A_1A_2A_3}\right)$$
$$+\frac{\alpha\delta\,(\delta\epsilon\zeta)^2}{(\overline{\alpha\delta})(\alpha\epsilon\zeta)^2}\left(\frac{D_1^2}{A_1^2}+\frac{D_2^2}{A_2^2}+\frac{D_3^2}{A_3^2}-2\frac{\alpha}{\delta}\frac{D_1D_2D_3}{A_1A_2A_3}\right), \quad (29)$$

where $$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}}=Z_A(u_1,v_1)+Z_A(u_2,v_2)+Z_A(u_3,v_3),$$

which is one form of the addition theorem for integrals of the second kind.

13. The transformation of formula (12), § 3, leads similarly to

$$S\frac{(AEF)_1}{(\alpha\epsilon\zeta)}\frac{\alpha^2}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}}\frac{A_1^2}{\alpha^2}\frac{\beta\gamma\delta}{B_1C_1D_1}=\Sigma\frac{\beta^2\gamma\delta}{(\overline{\beta\gamma})(\overline{\beta\delta})}\frac{\alpha B_2B_3A_1}{\beta A_2A_3B_1};$$

and therefore

$$S=\frac{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}}{(\overline{\beta\gamma})(\overline{\beta\delta})(\overline{\gamma\delta})}\frac{\alpha\,(\alpha\epsilon\zeta)\,B_1C_1D_1}{(AEF)_1A_1A_2A_3}$$
$$\times\left\{(\overline{\gamma\delta})\frac{B_2B_3}{B_1}+(\overline{\beta\delta})\frac{C_2C_3}{C_1}-(\overline{\beta\gamma})\frac{D_2D_3}{D_1}\right\}. \quad (30)$$

The signs of the terms in the bracket are determined by putting

$$(u_1,v_1)=(ef),\quad (u_2,v_2)=-(u_3,v_3)-(ef),$$

when S is seen to vanish by the help of the identical relation

$$(\gamma\delta\epsilon)(\gamma\delta\zeta)(BEF)\,B+(\beta\delta\epsilon)(\beta\delta\zeta)(CEF)\,C-(\beta\gamma\epsilon)(\beta\gamma\zeta)(DEF)\,D=0.$$

This gives the result that, with

$$\Sigma_r u_r=0,\quad \Sigma_r v_r=0\quad (r=1,2,3),$$

$$\Sigma_r Z_A(u_r,v_r)=\frac{2\alpha\,(\alpha\epsilon\zeta)\,B_1C_1D_1}{(AEF)_1A_1A_2A_3}$$
$$\times\left\{\frac{B_2B_3}{(\overline{\beta\gamma})(\overline{\beta\delta})\,B_1}+\frac{C_2C_3}{(\overline{\beta\gamma})(\overline{\gamma\delta})\,C_1}-\frac{D_2D_3}{(\overline{\beta\delta})(\overline{\gamma\delta})\,D_1}\right\}. \quad (31)$$

14. In exactly the same way $\int \frac{\sqrt{N}\,dp}{(n-p)\sqrt{P}}$ (§ 5) is replaced by $\int \frac{\sqrt{N}\,dx}{(n-x)\sqrt{X}}$, and $\int \frac{\sqrt{N}\,dx}{(n-x)\sqrt{X}} - \frac{\sqrt{N}\,dy}{(n-y)\sqrt{Y}}$ is equal to

$$\int \frac{\sqrt{N}F'^2du}{(f-n)\,\zeta^2\,(n-x)(n-y)} + \int \frac{\sqrt{N}E'^2dv}{(e-n)\,\epsilon^2\,(n-x)(n-y)}.$$

Now take

$$\left.\begin{array}{l} B(a,\beta) = B' = \beta\sqrt{b-n\,.\,b-a} \\ C(a,\beta) = C' = \gamma\sqrt{c-n\,.\,c-a} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}, \tag{32}$$

so that a, β are parameters of double Θ-functions for which $A(a,\beta)$ vanishes identically. Then, since

$$\frac{(n-x)(n-y)}{(b-n)(c-n)(d-n)}$$

$$= \frac{(b-x)(b-y)}{(b-n)(b-c)(b-d)} + \ldots$$

$$= \frac{\beta\gamma\delta}{a\,(\overline{a\epsilon})(\overline{a\zeta})}\left\{(a\beta\epsilon)^2(a\beta\zeta)^2\frac{B^2}{B'^2} - (a\gamma\epsilon)^2(a\gamma\zeta)^2\frac{C^2}{C'^2} + (a\delta\epsilon)^2(a\delta\zeta)^2\frac{D^2}{D'^2}\right\},$$

where the signs are determined from the identity

$$(a\beta\epsilon)^2(a\beta\zeta)^2 - (a\gamma\epsilon)^2(a\gamma\zeta)^2 + (a\delta\epsilon)^2(a\delta\zeta)^2 = 0,$$

I get

$$\int \frac{\sqrt{N}}{(n-x)(n-y)}\left(\frac{F'^2du}{(f-n)\,\zeta^2} + \frac{E'^2dv}{(e-n)\,\epsilon^2}\right)$$

$$= \int \frac{a\beta\gamma\delta\,(a\epsilon\zeta)}{(\overline{a\epsilon})(\overline{a\zeta})}\,\frac{B'C'D'}{(AEF)}$$

$$\times \frac{(\overline{a\zeta})\,E'^2F^2du + (\overline{a\epsilon})\,\zeta F'^2E^2dv}{(a\beta\epsilon)^2(a\beta\zeta)^2B^2C'^2D'^2 - (a\gamma\epsilon)^2(a\gamma\zeta)^2C^2D'^2B'^2 + (a\delta\epsilon)^2(a\delta\zeta)^2D^2B'^2C'^2}. \tag{33}$$

This I shall denote by $2\Pi(u, v;\, n)$, or by $2\Pi(u, v;\, a, \beta)$, and for shortness I write

$$M^2 \equiv (a\beta\epsilon)^2(a\beta\zeta)^2B^2C'^2D'^2 - (a\gamma\epsilon)^2(a\gamma\zeta)^2C^2D'^2B'^2 + (a\delta\epsilon)^2(a\delta\zeta)^2D^2B'^2C'^2$$

$$\equiv \frac{(a\beta\gamma\delta\epsilon\zeta)^2}{a^2}(n-x)(n-y). \tag{34}$$

Then formula (20) becomes

$$S' = \Pi(u_1, v_1;\, n) + \Pi(u_2, v_2;\, n) + \Pi(u_3, v_3;\, n). \tag{35}$$

The transformation of (22) gives

$$\cos S' = \frac{(\overline{a\zeta})}{a\beta\gamma\delta\epsilon\zeta}\,\frac{B'^2C'^2D'^2E'^2}{M_1M_2M_3}$$

$$\times\left\{\frac{(\overline{a\beta})^2(\overline{\beta\zeta})\,B_1B_2B_3}{B'^2}+\frac{(\overline{a\gamma})^2(\overline{\gamma\zeta})\,C_1C_2C_3}{C'^2}+\frac{(\overline{a\delta})^2(\overline{\delta\zeta})\,D_1D_2D_3}{D'^2}\right.$$

$$\left.-\frac{(\overline{a\epsilon})^2(\overline{\epsilon\zeta})\,E_1E_2E_3}{E'^2}\right\}, \quad (36)$$

and, of (24),

$$\sin S' = \frac{\iota S a^2\beta\gamma\delta\epsilon\zeta\,(a\epsilon\zeta)\,A_1A_2A_3B'C'D'E'^2F'^2}{(\overline{a\epsilon})(\overline{a\zeta})\,M_1M_2M_3\,(AEF)'}, \quad (37)$$

which may be written

$$\Sigma\Pi\,(u_r,\,v_r;\;a,\,\beta)$$

$$= \sinh^{-1}\frac{a\,(a\beta\gamma\delta\epsilon\zeta)^{\frac{3}{2}}\,(a\epsilon\zeta)\,A_1A_2A_3B'C'D'E'^2F'^2\,\Sigma Z_A(u_r,\,v_r)}{2\,(\overline{a\epsilon})(\overline{a\zeta})\,M_1M_2M_3\,(AEF)'},$$

when $\Sigma u_r = 0,\quad \Sigma v_r = 0\quad (r = 1,\,2,\,3).$ (38)

Linear Groups in an Infinite Field. By L. E. Dickson, Ph.D.

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1. *Introduction.*

Various branches of analytic group theory may be coordinated and generalized by the study of groups of transformations in an arbitrary field or domain of rationality. A *field* (Körper) is a set of elements within which the rational operations of algebra may be performed. Thus the totality of rational numbers forms a field R; the totality of all complex numbers $a+b\sqrt{-1}$ forms a field C. A finite field is completely defined by its order, which is necessarily a power of a prime number p, the latter being the *modulus* of the field. Although certain infinite fields may have a modulus p, so that $\mu+p\equiv\mu$, $\tau p\equiv 0$, for arbitrary elements μ, τ in the field, such fields do not seem to have been investigated. An example is given by the aggregate of the Galois fields of orders p^n, for $n = 1, 2, 3, \ldots$.