"Über eine Aufgabe der Ausgleichungsrechnung," von H. Bruns; 4to pamphlet; Leipzig, 1886.

"Zur Geschichte des menschlichen Rückenmarkes und der Nervenwurzeln," von Wilhelm His; 4to pamphlet; Leipzig, 1886.

Nine pamphlets (4to) by L. Kronecker: from the author.

On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants. *By* E. B. ELLIOTT.

[*Read Dec. 9th*, 1886.]

CONTENTS.

1. Mr. Leudesdorf (*Proceedings*, Vol. XVII., pp. 216—219) has noticed that a pure ternary reciprocant must have two quadro-linear differential annihilators, and has calculated a few terms of one of them. In the present paper a systematic search is made for a complete system of annihilators; and the results may, it will be found, be summarised as follows. Every pure ternary reciprocant, regarded as a function of the derivatives of a dependent variable z with regard to two independent variables x and y, must satisfy six linear partial differential equations, forming three pairs; and conversely, a homogeneous function of the second and higher derivatives, which does satisfy the six equations, must necessarily be a reciprocant. Two of the six equations are of forms akin to Euler's equation of homogeneous functions, involve a single constant which is different for

different reciprocants, and express facts with regard to degree and weight, homogeneity and isobarism. Two more are lineo-linear, and express that any pure ternary reciprocant is a full invariant of the system of emanants, as was proved in my former paper on ternary reciprocants (Vol. XVII., p. 181). The remaining pair are quadro-linear, that is to say, linear in the differential coefficients, but with coefficients quadratic in the elements, and are directly analogous to the well-known equation $VR = 0$ of ordinary or binary reciprocants, which equation either of them includes.

Other results which it is hoped may prove to have interest will be arrived at incidentally.

2. In this article the method to be followed will be exemplified by its application to the known theory of ordinary or binary reciprocants. This course seems justifiable, as the process has some novelty of form, and as the annihilator V will be arrived at in a compact symbolical shape, not, I believe, hitherto noticed.

Let, for every positive integral value of r, y_r denote $\frac{1}{r!}\frac{d^r y}{dx^r}$, so that $y_1, y_2, y_3, y_4, \ldots$ are the $t, a, b, c, \ldots$ of Professor Sylvester, and the $y_1, \frac{y_2}{2!}, \frac{y_3}{3!}, \frac{y_4}{4!}, \ldots$ of Mr. Leudesdorf. Then, if ξ, η be corresponding finite increments of x and y, Taylor's theorem tells us that

$$y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots = \eta-y_1\xi = -y_1(\xi-x_1\eta)$$

$$= -\frac{1}{x_1}(x_2\eta^2+x_3\eta^3+x_4\eta^4+\ldots) \quad \ldots\ldots(1),$$

where x_r denotes $\frac{1}{r!}\frac{d^r x}{dy^r}$.

Now, suppose the form of the connecting relation between x and y so to change that x_1 alters infinitesimally, but $x_2, x_3, x_4, \ldots$ remain constant. Also let the increment η remain unaltered. The corresponding ξ for which (1) remains true will, in general, receive a change. By differentiation of (1), therefore,

$$\frac{dy_2}{dx_1}\xi^2+\frac{dy_3}{dx_1}\xi^3+\frac{dy_4}{dx_1}\xi^4+\ldots+(2y_2\xi+3y_3\xi^2+4y_4\xi^3+\ldots)\frac{d\xi}{dx_1}$$

$$= \frac{1}{x_1^2}(x_2\eta^2+x_3\eta^3+x_4\eta^4+\ldots)$$

$$= -\frac{1}{x_1}(y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots), \text{ by (1)}.$$

Moreover, since $\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots,$

$$\frac{d\xi}{dx_1} = \eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots.$$

Hence we obtain, upon substitution,

$$\frac{dy_2}{dx_1}\xi^2 + \frac{dy_3}{dx_1}\xi^3 + \frac{dy_4}{dx_1}\xi^4 + \ldots$$
$$= -(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)(2y_2\xi + 3y_3\xi^2 + 4y_4\xi^3 + \ldots)$$
$$-y_1(y_2\xi^2 + y_3\xi^3 + y_4\xi^4 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2);$$

so that $\frac{dy_2}{dx_1}, \frac{dy_3}{dx_1}, \frac{dy_4}{dx_1}, \ldots$ are the coefficients of $\xi^2, \xi^3, \xi^4, \ldots$ in the expanded right-hand side of this identity. In other words,

$$\frac{dy_2}{dx_1} = -3y_1y_2,$$

$$\frac{dy_3}{dx_1} = -4(y_1y_3 + \tfrac{1}{2}y_2^2),$$

$$\frac{dy_4}{dx_1} = -5(y_1y_4 + y_2y_3),$$

$$\frac{dy_5}{dx_1} = -6(y_1y_5 + y_2y_4 + \tfrac{1}{2}y_3^2),$$

$$\ldots\ldots\ldots,$$

so that, since also $\frac{dy_1}{dx_1} = -y_1^2$, if ϕ is any function whatever,

$$-\frac{d}{dx_1}\phi(y_1, y_2, y_3, y_4, \ldots)$$
$$= y_1^2\frac{d\phi}{dy_1} + 3y_1y_2\frac{d\phi}{dy_2} + 4(y_1y_3 + \tfrac{1}{2}y_2^2)\frac{d\phi}{dy_3}$$
$$+ 5(y_1y_4 + y_2y_3)\frac{d\phi}{dy_4} + 6(y_1y_5 + y_2y_4 + \tfrac{1}{2}y_3^2)\frac{d\phi}{dy_5}$$
$$+ 7(y_1y_6 + y_2y_5 + y_3y_4)\frac{d\phi}{dy_6} + \ldots$$
$$= y_1\left(y_1\frac{d\phi}{dy_1} + 3y_2\frac{d\phi}{dy_2} + 4y_3\frac{d\phi}{dy_3} + \ldots\right) + V\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where V denotes, as usual,

$$4\frac{y_2^2}{2}\frac{d}{dy_3}+5y_2y_3\frac{d}{dy_4}+6\,(y_2y_4+\tfrac{1}{2}y_3^2)\,\frac{d}{dy_5}+\dots.$$

This is, if we remember the difference of notation, Mr. Leudesdorf's formula [*Proceedings*, Vol. XVII., p. 199, (1)]; so that the rest of the argument may proceed as in his paper.

Now, the terms free from y_1 in (2) are those which give the coefficients that occur in V. But these terms are

$$-(y_2\xi^2+y_3\xi^3+y_4\xi^4+\dots)(2y_2\xi+3y_3\xi^2+4y_4\xi^3+\dots),$$

which may be written

$$-\frac{1}{2}\frac{d}{d\xi}\left\{(y_2\xi^2+y_3\xi^3+y_4\xi^4+\dots)^2\right\}.$$

We have, then, that the coefficients of $\frac{d}{dy_3}$, $\frac{d}{dy_4}$, $\frac{d}{dy_5}$, ... in V are exactly the coefficients with sign changed of ξ^3, ξ^4, ξ^5, ... in this expansion. The expression with sign changed may consequently be itself taken as a symbolical representation of V, becoming V as it does when, after the expansion and differentiation are performed, for every power ξ^r in it the corresponding $\frac{d}{dy_r}$ is substituted. Or we may say, still more briefly, that, symbolically,

$$2V=\frac{d}{d\xi}\left\{(\eta-y_1\xi)^2\right\}\dots\dots\dots\dots(4);$$

the meaning being that, in this right-hand side, η, regarded as an increment of y, is to be expanded by Taylor's theorem in terms of powers of ξ, that the squaring and partial differentiation with regard to ξ are to be performed, and that then, for all values of the number r, ξ^r is to be replaced by the symbol of partial differentiation $\frac{d}{dy_r}$.

Other results can readily be obtained by differentiation of (1) partially with regard to each of the derivatives x_r. The method is exactly that of the case already discussed of differentiation with regard to x_1. The process is, however, even more simple for values of r exceeding unity, and need not be here given, as the analogous reasoning when there are three variables x, y, z will be gone through at length in § 6 below. The result is that, r being any number but unity, and the expression operated on upon the right being any function whatever of y_1, y_2, y_3, ..., whose unknown equivalent in terms of

$x_1, x_2, x_3, \ldots$ is referred to on the left,

$$(r+1)\frac{d}{dx_r} = -\frac{d}{d\xi}\left\{(y_1\xi+y_2\xi^2+y_3\xi^3+\ldots)^{r+1}\right\}$$

$$= -\frac{d}{d\xi}(\eta^{r+1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the symbolization on the right being exactly as before.

It may be noticed, in passing, that the terms in the dexter of (5) which do not involve y_1, *i.e.*, the terms symbolically represented by

$$\frac{d}{d\xi}\left\{(\eta-y_1\xi)^{r+1}\right\},$$

constitute Captain MacMahon's operator

$$(2r+2,\ 1;\ r+1,\ 2r-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5a).$$

Hence it follows, without difficulty, that this operator is an annihilator of *pure* functions where transforms do not involve x_r.*

3. Passing now to our main subject, the corresponding theory when there are three variables x, y, z connected by a single relation, let ξ, η, ζ be corresponding finite increments of those variables. Two of these may be chosen at will, and the third is then determinate in virtue of the relation connecting x, y, z. The notation used in my former paper must be discarded as unsuitable for general investigations, and as unfortunate in its peculiar choice of suffixes. The most suitable for adoption would seem to be the following. Let, for all possible combinations of two numbers r, s, $\frac{1}{r!\,s!}\frac{d^{r+s}z}{dx^r dy^s}$ be represented

* Note that the most general MacMahon operator of four elements $(\mu, \nu;\ m, n)$ may be written symbolically,

$$\nu\frac{d}{d\xi}\left\{\xi^{\mu/\nu-2m}(\eta-y_1\xi)^m\right\},$$

in which, ξ and η being regarded in the first place as corresponding increments of x and y, the substitution for η in terms of ξ by Taylor's theorem, the expansion by the multinomial theorem, and the partial differentiation with regard to ξ, are to be performed, and then, for each value of s occurring, $\xi^{s+\mu/\nu}$ is to be replaced by the operative symbol $\frac{d}{dy_{n+s+3}}$, which in the Sylvester-MacMahon notation is called $\frac{d}{da_{n+s+1}}$. This remark may probably lead to the interpretation of other large classes of these operators, as well as of those of (5*a*) above.

The close connection may be noticed between this symbolization and that of Mr. Hammond on p. 63 of the present volume. The two were simultaneously and independently arrived at, and were communicated to the Society on the same evening.

by z_{rs}, $\frac{1}{r!\,s!}\frac{d^{r+s}x}{dy^r dz^s}$ by x_{rs}, and $\frac{1}{r!\,s!}\frac{d^{r+s}y}{dz^r dx^s}$ by y_{rs}. The connection between ξ, η, ζ may then be written, by Taylor's theorem, in three forms:

$$\left.\begin{aligned}\zeta-z_{10}\xi-z_{01}\eta &= (z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\\ \xi-x_{10}\eta-x_{01}\zeta &= (x_{20}\eta^2+x_{11}\eta\zeta+x_{02}\zeta^2)+(x_{30}\eta^3+x_{21}\eta^2\zeta+x_{12}\eta\zeta^2+x_{03}\zeta^3)+\dots\\ \eta-y_{10}\zeta-y_{01}\xi &= (y_{20}\zeta^2+y_{11}\zeta\xi+y_{02}\xi^2)+(y_{30}\zeta^3+y_{21}\zeta^2\xi+y_{12}\zeta\xi^2+y_{03}\xi^3)+\dots\end{aligned}\right\} \quad\dots\dots\dots\dots(6);$$

of which the second [*Proceedings*, Vol. XVII., p. 182 (32)] is only the first divided by either of the equal quantities

$$-z_{10},\ -\frac{1}{x_{01}},\ \left(\frac{z_{10}z_{01}}{x_{10}x_{01}}\right)^{\frac{1}{2}},$$

and the third is the first divided by either

$$-z_{01},\ -\frac{1}{y_{10}},\ \text{or}\ \left(\frac{z_{10}z_{01}}{y_{10}y_{01}}\right)^{\frac{1}{2}}.$$

We have, for instance, from the first two, the equivalence

$$(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots$$

$$=-\frac{1}{x_{01}}\left\{(x_{20}\eta^2+x_{11}\eta\zeta+x_{02}\zeta^2)+(x_{30}\eta^3+x_{21}\eta^2\zeta+x_{12}\eta\zeta^2+x_{03}\zeta^3)+\dots\right\} \quad\dots\dots\dots\dots(7).$$

This will now help us to express the partial differential coefficients of functions of the derivatives z_{rs} of z with regard to the derivatives x_{mn} of x, of which they can of course be regarded as functions. Primarily we seek the effect of the operations $\frac{d}{dx_{10}}$, $\frac{d}{dx_{01}}$, where x_{10}, x_{01} are the first derivatives $\frac{dx}{dy}$, $\frac{dx}{dz}$ of x, upon z-functions.

4. First let the connecting relation between x, y, z be so varied infinitesimally, that x_{01} receives an infinitesimal increment, while x_{10}, x_{20}, x_{11}, x_{02}, x_{30}, ... remain unaltered. Keep also the η and ζ in (6) and (7) unchanged. The corresponding ξ for which they remain true will vary infinitesimally in consequence of the change in x_{01}.

Thus, from (7),

$$\left(\frac{dz_{20}}{dx_{01}}\xi^2+\frac{dz_{11}}{dx_{01}}\xi\eta+\frac{dz_{02}}{dx_{01}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{01}}\xi^3+\frac{dz_{21}}{dx_{01}}\xi^2\eta+\dots\right)+\dots$$

$$+\frac{d\xi}{dx_{01}}\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\}$$

$$=\quad\frac{1}{x_{01}^2}\left\{(x_{20}\eta^2+x_{11}\eta\zeta+x_{02}\zeta^2)+(x_{30}\eta^3+x_{21}\eta^2\zeta+x_{12}\eta\zeta^2+x_{03}\zeta^3)+\dots\right\}$$

$$=-\frac{1}{x_{01}}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\},$$

by (7). But, by the second of equations (6),

$$\frac{d\xi}{dx_{01}}=\zeta$$

$$=(z_{10}\xi+z_{01}\eta)+(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots.$$

Consequently, upon inserting this value above,

$$\left(\frac{dz_{20}}{dx_{01}}\xi^2+\frac{dz_{11}}{dx_{01}}\xi\eta+\frac{dz_{02}}{dx_{01}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{01}}\xi^3+\dots\right)+\left(\frac{dz_{40}}{dx_{01}}\xi^4+\dots\right)+\dots$$

$$=-\left\{(z_{10}\xi+z_{01}\eta)+(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+\dots\right\}$$

$$\times\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$-z_{10}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$=-z_{10}\left(\xi\frac{d}{d\xi}+1\right)\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$-z_{01}\eta\frac{d}{d\xi}\qquad\left\{\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\right\}$$

$$-\frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}^2\right]\quad\dots\dots\dots(8).$$

Thus $\frac{dz_{20}}{dx_{01}}$, $\frac{dz_{11}}{dx_{01}}$, $\frac{dz_{02}}{dx_{01}}$, $\frac{dz_{30}}{dx_{01}}$, ... are the coefficients of ξ^2, $\xi\eta$, η^2, ξ^3, ... in the expansion after differentiation for ξ of this right-hand side. Also, since [*Proceedings*, Vol. XVII., p. 174, (4)],

$$z_{10}=\frac{1}{x_{01}},\ \text{and}\ z_{01}=-\frac{x_{10}}{x_{01}},$$

we have, further,

$$\frac{dz_{10}}{dx_{01}} = -\frac{1}{x_{01}^2} = -z_{10}^2, \text{ and } \frac{dz_{01}}{dx_{01}} = \frac{x_{10}}{x_{01}^2} = -z_{10}z_{01} \quad \ldots\ldots\ldots(8a).$$

The means have been obtained, therefore, of writing down the partial differential coefficient with regard to x_{01} of the function of the suffixed x's which is equal to any given function of the suffixed z's. Thus, if

$$\mathfrak{F}(z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, z_{30}, \ldots) = F(x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, x_{30}, \ldots) \ldots(9),$$

then $$\frac{dF}{dx_{01}} = \frac{d\mathfrak{F}}{dz_{10}}\cdot\frac{dz_{10}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{01}}\cdot\frac{dz_{01}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{20}}\cdot\frac{dz_{20}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{11}}\cdot\frac{dz_{11}}{dx_{01}} + \ldots$$

$$= -z_{10}\left\{z_{10}\frac{d\mathfrak{F}}{dz_{10}} + z_{01}\frac{d\mathfrak{F}}{dz_{01}}\right\}$$

+ the result of writing in the right-hand member of (8), after differentiation and expansion, $\frac{d\mathfrak{F}}{dz_{rs}}$ instead of $\xi^r\eta^s$ for each pair of values of r and s.

If, then, we adopt $\xi^r\eta^s$, in an expanded form, as a symbolical representation of $\frac{d}{dz_{rs}}$,

$$\begin{aligned}\frac{d}{dx_{01}} = &-z_{10}\{z_{10}\xi + z_{01}\eta + (3z_{20}\xi^2 + 2z_{11}\xi\eta + z_{02}\eta^2) \\ &\qquad + (4z_{30}\xi^3 + 3z_{21}\xi^2\eta + 2z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots\} \\ &-z_{01}\{\qquad (2z_{20}\xi\eta + z_{11}\eta^2) + \qquad (3z_{30}\xi^2\eta + 2z_{21}\xi\eta^2 + z_{12}\eta^3) + \ldots\} \\ &-\frac{1}{2}\frac{d}{d\xi}\left[\{(z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) \right. \\ &\qquad \left. + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots\}^2\right] \ldots\ldots(10).\end{aligned}$$

Now, suppose the function $\mathfrak{F}(z_{10}, \ldots)$, in the identity (9), to be such that its transform F does not involve x_{01}. The right-hand side of (10) will then be an annihilator of $\mathfrak{F}(z_{10}, \ldots)$, and, conversely, for this to be an annihilator will be a sufficient condition for the absence of x_{01}, no less than a necessary consequence of such absence.

To particularise, take for $\mathfrak{F}(z_{10}, \ldots)$ an expression of the form $z_{10}^{-\mu}P$, where P is a pure function of the suffixed z's, *i.e.*, one which does not involve the first derivatives z_{10}, z_{01}. It results that

$$\frac{d}{dx_{01}}\left(\frac{P}{z_{10}^{\mu}}\right)=\frac{1}{z_{10}^{\mu-1}}\left\{\mu-\left(3z_{20}\frac{d}{dz_{30}}+2z_{11}\frac{d}{dz_{11}}+z_{02}\frac{d}{dz_{02}}\right)\right.$$

$$\left.-\left(4z_{30}\frac{d}{dz_{30}}+3z_{21}\frac{d}{dz_{21}}+2z_{12}\frac{d}{dz_{12}}+z_{03}\frac{d}{dz_{03}}\right)-\dots\right\}P$$

$$-\frac{z_{01}}{z_{10}^{\mu}}\left\{\left(2z_{20}\frac{d}{dz_{11}}+z_{11}\frac{d}{dz_{02}}\right)+\left(3z_{30}\frac{d}{dz_{21}}+2z_{21}\frac{d}{dz_{12}}+z_{12}\frac{d}{dz_{03}}\right)+\dots\right\}P$$

$$-\frac{1}{z_{10}^{\mu}}\cdot\frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\}^2\right]P,$$
$$\dots\dots\dots\dots(11),$$

in which the last line is symbolical; and consequently that, if the transform of $z_{10}^{-\mu}P$ is independent of x_{01}, P must satisfy the linear partial differential equation obtained by equating the right-hand member of this identity to zero; and conversely.

But we can say more than this. The right-hand member of (11) consists of three distinct parts, having factors $\frac{1}{z_{10}^{\mu-1}}$, $\frac{z_{01}}{z_{10}^{\mu}}$, $\frac{1}{z_{10}^{\mu}}$, respectively, and not containing either z_{10} or z_{01} in their other factors. Now no term whose z_{10} factor is $z_{10}^{-\mu+1}$ can disappear when added to another term whose z_{10} factor is $z_{10}^{-\mu}$. Nor can a term having z_{01} as a factor be cancelled by addition to a term not involving z_{01}. If, then, the right-hand side of (11) vanish, its three parts must vanish separately. We conclude that, for the transform of $z_{10}^{-\mu}P$ not to involve x_{01}, the necessary and sufficient conditions are three in number; viz., that P satisfy the three linear partial differential equations $E_1P=0$, $\Omega_1P=0$, $V_1P=0$, where the meanings of the three annihilators are

$$E_1\equiv\mu-\left(3z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+z_{02}\frac{d}{dz_{02}}\right)$$
$$-\left(4z_{30}\frac{d}{dz_{30}}+3z_{21}\frac{d}{dz_{21}}+2z_{12}\frac{d}{dz_{12}}+z_{03}\frac{d}{dz_{03}}\right)-\dots\quad\dots\dots(A),$$

$$\Omega_1\equiv\left(2z_{20}\frac{d}{dz_{11}}+z_{11}\frac{d}{dz_{02}}\right)+\left(3z_{30}\frac{d}{dz_{21}}+2z_{21}\frac{d}{dz_{12}}+z_{12}\frac{d}{dz_{03}}\right)+\dots\quad\dots(B),$$

$$V_1\equiv\frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\}^2\right]$$

$$\equiv\frac{1}{2}\frac{d}{d\xi}\left\{(\zeta-z_{10}\xi-z_{01}\eta)^2\right\}\dots\dots\dots\dots(C),$$

the last of which is symbolically written, and is to be taken in the sense that ζ, an increment of z, is first to be expanded in terms of ξ, η, corresponding increments of x and y, by Taylor's theorem, that the expansion and the partial differentiation with regard to ξ are to be performed, and that then, in the result, $\xi^r\eta^s$ is to be replaced by $\dfrac{d}{dz_{rs}}$ for every pair of values of r and s.

The first two of the three conditions are readily interpreted. Thus, $E_1P = 0$ tells us that, if P be homogeneous, as we shall see later must be the case when it is a reciprocant, it is necessarily isobaric in first suffixes, *i.e.*, such that in every term of which it consists the sum of the first suffixes of the factors is constant; and also that, if w_1 be this constant sum, the first partial weight say, and i the degree of P,

then $$i+w_1 = \mu.$$

Again, $\Omega_1P = 0$ expresses that P is a ξ-seminvariant, or at any rate a sum of such seminvariants, of the system of emanants

$$(z_{20}, z_{11}, z_{02} \between \xi, \eta)^2, \quad (z_{30}, z_{21}, z_{12}, z_{03} \between \xi, \eta)^3, \text{ \&c.};$$

in connection with which see my former paper on "Ternary Reciprocants" (Vol. XVII., pp. 181, 182).

The third condition $V_1P = 0$, a quadro-linear equation, is the first analogue of the equation $VR = 0$ [see (3) or (4) above] of ordinary pure reciprocants.

5. The determination of an expression for $\dfrac{d}{dx_{10}}$ in terms of the operators $\dfrac{d}{dz_{rs}}$ is a much shorter matter than that of $\dfrac{d}{dx_{01}}$. Returning to (7), let x_{10} vary, while η, ζ and all the doubly suffixed x's except x_{10} remain constant. We thus obtain

$$\left(\frac{dz_{20}}{dx_{10}}\xi^2+\frac{dz_{11}}{dx_{10}}\xi\eta+\frac{dz_{02}}{dx_{10}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{10}}\xi^3+\frac{dz_{21}}{dx_{10}}\xi^2\eta+\frac{dz_{12}}{dx_{10}}\xi\eta^2+\frac{dz_{03}}{dx_{10}}\eta^3\right)+\dots$$

$$= -\frac{d\xi}{dx_{10}}\cdot\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\}:$$

and in this, since by the second of equations (6) $\dfrac{d\xi}{dx_{10}} = \eta$, the right-hand member may be written

$$-\left\{(2z_{20}\xi\eta+z_{11}\eta^2)+(3z_{30}\xi^2\eta+2z_{21}\xi\eta^2+z_{12}\eta^3)+\dots\right\};$$

whence, equating coefficients,

$$\frac{dz_{20}}{dx_{10}}=0,\quad \frac{dz_{11}}{dx_{10}}=-2z_{20},\quad \frac{dz_{02}}{dx_{10}}=-z_{11},$$

$$\frac{dz_{30}}{dx_{10}}=0,\quad \frac{dz_{21}}{dx_{10}}=-3z_{30},\quad \frac{dz_{12}}{dx_{10}}=-2z_{21},\quad \frac{dz_{03}}{dx_{10}}=-z_{12},$$

&c., &c.,

which, with the facts obtained as in (8*a*),

$$\frac{dz_{10}}{dx_{10}}=0 \text{ and } \frac{dz_{01}}{dx_{10}}=-\frac{1}{x_{01}}=-z_{10},$$

give the equivalence $$\frac{d}{dx_{10}}=-z_{10}\frac{d}{dz_{01}}-\Omega_1 \quad\text{..........................(12)},$$

where Ω_1 denotes the lineo-linear operator of result (B).

In particular, P denoting a pure function of the doubly suffixed z's, as in the last article,

$$\frac{d}{dx_{10}}\left(\frac{P}{z_{10}^{\mu}}\right)=-\frac{1}{z_{10}^{\mu}}\Omega_1 P \quad\text{..........................(13)},$$

which gives us no new information as to such pure functions P as produce in this manner pure functions of the suffixed x's, since the condition $\Omega_1 P=0$ is one of those previously obtained, but tells us that x_{10} cannot possibly occur in the transform of $z_{10}^{-\mu}P$, where P is pure, without x_{01} occurring also.

6. A very simple symbolical expression for $\frac{d}{dx_{rs}}$ when $r+s>1$, *i.e.*, for any case except those of $r=0$, $s=1$ and $r=1$, $s=0$ already considered, may be obtained as follows. By (7), as in previous cases,

$$\left(\frac{dz_{20}}{dx_{rs}}\xi^2+\frac{dz_{11}}{dx_{rs}}\xi\eta+\frac{dz_{02}}{dx_{rs}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{rs}}\xi^3+\ldots\right)+\ldots$$

$$=-\frac{d\xi}{dx_{rs}}\cdot\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}$$

$$-\frac{1}{x_{01}}\eta^r\zeta^s,$$

which right-hand member may, by the second and first of equations (6), be written $$-\eta^r\zeta^s\left\{\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta)+\frac{1}{x_{01}}\right\},$$

in which inside the bracket ζ stands, for brevity, as meaning its expansion in terms of ξ and η. Now this is merely, since $z_{10} = \frac{1}{x_{01}}$,

$$-\eta^r \zeta^s \frac{d\zeta}{d\xi},$$

i.e.,
$$-\frac{1}{s+1}\eta^r \frac{d}{d\xi}(\zeta^{s+1}) \quad\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

which is to be taken merely as a short representation of

$$-\frac{1}{s+1}\eta^r \frac{d}{d\xi}\left[\{(z_{10}\xi+z_{01}\eta)+(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\ldots)+\ldots\}^{s+1}\right] \quad\ldots\ldots\ldots(14a),$$

the differentiation being strictly partial.

The coefficient of each product $\xi^m\eta^n$, in the final form of this after expansion and differentiation, is then equal to the corresponding

$$\frac{dz_{mn}}{dx_{rs}}.$$

It is to be noticed that, as should obviously be the case, for values of n less than r, and also for values of $m+n$ less than $r+s$, this coefficient vanishes.

Hence, since
$$\frac{d}{dx_{rs}} = \sum_{m+n=1}^{m+n=\infty}\left[\frac{dz_{mn}}{dx_{rs}}\cdot\frac{d}{dz_{mn}}\right],$$

the expression for $\frac{d}{dx_{rs}}$ in terms of the operators $\frac{d}{dz_{mn}}$ is obtained by putting, for all pairs of values of the numbers m, n, $\frac{d}{dz_{mn}}$ instead of $\xi^m\eta^n$ in the final expansion in terms of ξ and η of (14); a fact which is expressed symbolically as before, by saying that

$$\frac{d}{dx_{rs}} = -\frac{1}{s+1}\eta^r \frac{d}{d\xi}(\xi^{s+1}) \quad\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

7. Having drawn conclusions with regard to and by means of the expressions for functions of the doubly suffixed z's, in terms of the doubly suffixed x's, let us now consider their expressions in terms of the doubly suffixed y's. The method is the same as before.

That the third of the equalities (6) is $-y_{10}$ times the first, tells us that
$$\sum_{r+s=2}^{r+s=\infty}\{z_{rs}\xi^r\eta^s\} = -\frac{1}{y_{10}}\sum_{r+s=2}^{r+s=\infty}\{y_{rs}\zeta^r\xi^s\} \quad\ldots\ldots\ldots\ldots(16).$$

Now, in this let y_{10} alone, of all the doubly suffixed y's, receive an infinitesimal change; and of ξ, η, ζ let, for our present purpose, ξ and ζ be kept constant, the consequent infinitesimal change of one of the three being therefore given to η. We obtain, exactly as in the beginning of §4, that, since

$$\frac{d\eta}{dy_{10}} = \zeta = \sum_{r+s=1}^{r+s=\infty} \{z_{rs}\zeta^r\eta^s\},$$

$$\sum_{r+s=2}^{r+s=\infty} \left\{\frac{dz_{rs}}{dy_{10}}\xi^r\eta^s\right\} = -z_{10}\xi\frac{d}{d\eta}\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}$$

$$-z_{01}\left[\eta\frac{d}{d\eta}+1\right]\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta}\left[\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}^2\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

a result corresponding to (8). The equation of coefficients of the various terms $\xi^r\eta^s$ on the two sides of this identity gives expressions for the derivatives $\frac{dz_{rs}}{dy_{10}}$ for all combinations of zero or positive integral values of r and s whose sum exceeds unity; and to these may be added

$$\frac{dz_{10}}{dy_{10}} = \frac{y_{01}}{y_{10}^2} = -z_{10}z_{01}, \quad\text{and}\quad \frac{dz_{01}}{dy_{10}} = -\frac{1}{y_{10}^2} = -z_{01}^2,$$

obtained from $$z_{10} = -\frac{y_{01}}{y_{10}}, \quad z_{01} = \frac{1}{y_{10}}.$$

Hence, just as (10) was obtained, we deduce that, if $\xi^r\eta^s$ is adopted as a symbolical way of writing $\frac{d}{dz_{rs}}$,

$$\frac{d}{dy_{10}} = -z_{10}\left\{(z_{11}\xi^2+2z_{02}\xi\eta)+(z_{21}\xi^3+2z_{12}\xi^2\eta+3z_{03}\xi\eta^2)+\ldots\right\}$$

$$-z_{01}\left\{z_{10}\xi+z_{01}\eta+(z_{20}\xi^2+2z_{11}\xi\eta+3z_{02}\eta^2)\right.$$

$$\left.+(z_{30}\xi^3+2z_{21}\xi^2\eta+3z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]$$
$$\ldots\ldots\ldots\ldots(18),$$

in the last expression the expansion and partial differentiation having to be effected before the substitution $\frac{d}{dz_{rs}}$, &c., for $\xi^r\eta^s$, &c., is made. In the right-hand side of (18) the operation is on any function whatever of all or any of the doubly suffixed z's, and on the left it is supposed to be on the probably unknown equivalent function of the doubly suffixed y's. The operator on the right must then annihilate any function of the z_s where transform does not involve y_{10}, and conversely.

Now choose, for the function of the suffixed z's operated on, an expression of the form $z_{01}^{-\mu}P$, where P is a pure function, as in (11); and express, by means of (18), the conditions that its transform is independent of y_{10}, as will in particular be the case for a proper value of μ (see later) if P is a pure reciprocant. The conclusion is that

$$-\frac{z_{10}}{z_{01}^{\mu}}\left\{\left[z_{11}\frac{d}{dz_{20}}+2z_{02}\frac{d}{dz_{11}}\right]+\left[z_{21}\frac{d}{dz_{30}}+2z_{12}\frac{d}{dz_{21}}+3z_{03}\frac{d}{dz_{12}}\right]+\ldots\right\}P$$

$$+\frac{1}{z_{01}^{\mu-1}}\left\{\mu-\left[z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+3z_{02}\frac{d}{dz_{02}}\right]\right.$$

$$\left.-\left[z_{30}\frac{d}{dz_{30}}+2z_{21}\frac{d}{dz_{21}}+3z_{12}\frac{d}{dz_{12}}+4z_{03}\frac{d}{dz_{03}}\right]-\ldots\right\}P$$

$$-\frac{1}{z_{01}^{\mu}}\cdot\frac{1}{2}\frac{d}{d\eta}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)\right.\right.$$

$$\left.\left.+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]P=0\ldots\ldots(19),$$

the last line being symbolical, as before explained.

Moreover, just as was the case with (11), this is really not one condition only, but three, the terms multiplied by

$$\frac{z_{10}}{z_{01}^{\mu}},\quad \frac{1}{z_{01}^{\mu-1}},\quad \text{and}\quad \frac{1}{z_{01}^{\mu}}$$

having necessarily to vanish separately. It is thus proved that, for the transform of $z_{01}^{-\mu}P$ to be independent of y_{10}, it is necessary, and also sufficient, that P satisfy three linear equations,

$$E_2P=0,\quad \Omega_2P=0,\quad V_2P=0,$$

where the annihilators denoted are

$$E_2\equiv\mu-\left(z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+3z_{02}\frac{d}{dz_{02}}\right)$$

$$-\left(z_{30}\frac{d}{dz_{30}}+2z_{21}\frac{d}{dz_{21}}+3z_{12}\frac{d}{dz_{12}}+4z_{03}\frac{d}{dz_{03}}\right)-\ldots\quad\ldots(\text{D}),$$

$$\Omega_2\equiv\left(z_{11}\frac{d}{dz_{20}}+2z_{02}\frac{d}{dz_{11}}\right)+\left(z_{21}\frac{d}{dz_{30}}+2z_{12}\frac{d}{dz_{21}}+3z_{03}\frac{d}{dz_{12}}\right)+\ldots\quad\ldots\ldots(\text{E}),$$

and

$$V_2 \equiv \frac{1}{2}\frac{d}{d\eta}\left[\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\}^2\right]$$

$$\equiv \frac{1}{2}\frac{d}{d\eta}\{(\zeta-z_{10}\xi-z_{01}\eta)^2\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(F)};$$

of which the last is symbolically written, and is to be interpreted just as was the expression for V_1 in result (C) of § 4.

The interpretation of the first two conditions, $E_2P = 0$, $\Omega_2P = 0$, is exactly similar to that given to the corresponding $E_1P = 0$, $\Omega_1P = 0$, of results (A) and (B), at the end of § 4. The first, $E_2P = 0$, expresses that P, if homogeneous and of degree i, must be isobaric in second suffixes, and that, if w_2 be its weight as measured by the sum of those second suffixes in each of its terms, then $i+w_2 = \mu$; and, again, $\Omega_2P = 0$ asserts that P is an η-seminvariant, or at any rate a sum of such seminvariants, of the system of emanants

$$(z_{20}, z_{11}, z_{02}\between(\xi, \eta)^2, \quad (z_{30}, z_{21}, z_{12}, z_{03}\between(\xi, \eta)^3, \ldots.$$

The remaining annihilator, V_2, is quadro-linear, is a companion to the V_1 of § 4, and a second analogue to Professor Sylvester's annihilator V of pure binary reciprocants.

8. Before considering the application of the above results and those of § 4 to the theory of ternary reciprocants, it will be well to state briefly other results as to the expression of functions of the suffixed z's in terms of the suffixed y's, exactly analogous to those of §§ 5 and 6.

Exactly as in the first of these articles, we obtain the equivalence

$$\frac{d}{dy_{01}} = -z_{01}\frac{d}{dz_{10}} - \Omega_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(20)},$$

where Ω_2 has its meaning in result (E); so that, in particular,

$$\frac{d}{dy_{01}}\left(\frac{P}{z_{01}^{\mu}}\right) = -\frac{1}{z_{01}^{\mu}}\Omega_2P \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(21)},$$

from which, by the results of the last article, we infer that, if the transform of $z_{01}^{-\mu}P$ do not involve y_{10}, it must be also independent of y_{01}.

Once more, just as (14) was proved, it may be established that, when $r+s>1$,

$$\frac{d}{dy_{rs}} = -\frac{1}{r+1}\xi^s\frac{d}{d\eta}\{\zeta^{r+1}\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(22)},$$

where the symbolism on the right means that ζ represents its expansion by Taylor's theorem in terms of ξ and η, that the expansion and partial differentiation for η are to be performed, and that in the result, for all values of m and n, $\xi^m\eta^n$ is to be replaced by $\dfrac{d}{dz_{mn}}$.

9. We proceed now to the theory of pure ternary reciprocants. If $R(z)$ denote such a reciprocant function of the derivatives of z with respect to x and y, and $R(x)$, $R(y)$ respectively denote the cyclically derived functions of the partial derivatives of x and of y, the expression of the property of reciprocance, viz.,

$$\frac{R(z)}{(z_{10}z_{01})^{\frac{1}{2}\mu}} = \omega^k \frac{R(x)}{(x_{10}x_{01})^{\frac{1}{2}\mu}} = \omega^{2k} \frac{R(y)}{(y_{10}y_{01})^{\frac{1}{2}\mu}} \quad\text{..............(23)},$$

may be written (Vol. XVII., p. 184, Note), more suitably for our present purpose,

$$\frac{R(z)}{(-z_{10})^{\mu}} = \omega^k R(x), \quad \frac{R(z)}{(-z_{01})^{\mu}} = \omega^{2k} R(y) \quad\text{...........(24)}.$$

A pure ternary reciprocant is then a function which unites in itself the properties of the pure functions P considered in §§ 4 and 7. It will be seen later that, conversely, all pure functions P which possess these united properties are reciprocants, or at least sums of reciprocants, of like index μ, in case there be more than one such. The facts now obtained with regard to pure ternary reciprocants may be stated as follows:—

(i.) *A pure ternary reciprocant is necessarily a full invariant of the system of emanants*

$$(z_{20}, z_{11}, z_{02})(\xi, \eta)^2, \text{ \&c.}$$

This was proved in my former paper (Vol. XVII., p. 181); and it is to be noticed that the proof there given requires that it be a single invariant, being thus more stringent than that here given of its annihilation by the Ω_1 and Ω_2 of results (B) and (E), which would permit of its being a sum of different invariants of the emanants.

One result of the fact that the reciprocant is a single invariant is that it must be necessarily homogeneous. Thus,

(ii.) *It is homogeneous and doubly isobaric, having each of its terms of the same constant weight in both first and second suffixes.*

The double isobarism, as well as the homogeneity, really follows from the fact of the reciprocant being a single invariant. Granted, however, the homogeneity, the constancy throughout of the two partial weights follows from the fact of annihilation by the E_1 and E_2 of (A) and (D), while the equality of each to the half of the full weight or sum of suffixes arises from the fact that the μ is the same both in E_1 and E_2, by (24). Hence, also,

(iii.) *If i be the degree and w the full weight of a pure ternary reciprocant, the index μ of that reciprocant is given by*

$$\mu = i + \tfrac{1}{2}w.$$

It is, of course, to be remarked that the full weight (sum of suffixes) w is always even.

To these facts we shall be enabled, by the next article, to add—

(iv.) *The mark of character k of a pure ternary reciprocant is always zero.*

But the property of pure ternary reciprocants, to which all these other facts are merely subsidiary, is that,

(v.) *They have two quadro-linear annihilators V_1 and V_2.*

For these symbolical expressions have been found in results (C) and (F). The following will be found to be their expanded forms, written in ordinary notation :—

$$\begin{aligned}
V_1 \equiv 2z_{20}^2 \frac{d}{dz_{30}} &+ 3z_{20}z_{11}\frac{d}{dz_{21}} + 2\left(z_{20}z_{02}+\tfrac{1}{2}z_{11}^2\right)\frac{d}{dz_{12}} + z_{11}z_{02}\frac{d}{dz_{03}} \\
&+ 5z_{20}z_{30}\frac{d}{dz_{40}} + 4\left(z_{20}z_{21}+z_{11}z_{30}\right)\frac{d}{dz_{31}} + 3\left(z_{20}z_{12}+z_{11}z_{21}+z_{02}z_{30}\right)\frac{d}{dz_{22}} \\
&\qquad + 2\left(z_{20}z_{03}+z_{11}z_{12}+z_{02}z_{21}\right)\frac{d}{dz_{13}} + \left(z_{11}z_{03}+z_{02}z_{12}\right)\frac{d}{dz_{04}} \\
&+ 6\left(z_{20}z_{40}+\tfrac{1}{2}z_{30}^2\right)\frac{d}{dz_{50}} + 5\left(z_{20}z_{31}+z_{11}z_{40}+z_{30}z_{21}\right)\frac{d}{dz_{41}} \\
&\qquad + 4\left(z_{20}z_{22}+z_{11}z_{31}+z_{02}z_{40}+\tfrac{1}{2}z_{21}^2+z_{12}z_{30}\right)\frac{d}{dz_{32}} \\
&\qquad + 3\left(z_{20}z_{13}+z_{11}z_{22}+z_{02}z_{31}+z_{21}z_{12}+z_{30}z_{03}\right)\frac{d}{dz_{23}} \\
&\qquad + 2\left(z_{20}z_{04}+z_{11}z_{13}+z_{02}z_{22}+\tfrac{1}{2}z_{12}^2+z_{21}z_{03}\right)\frac{d}{dz_{14}} \\
&\qquad + \left(z_{11}z_{04}+z_{02}z_{13}+z_{12}z_{03}\right)\frac{d}{dz_{05}}
\end{aligned}$$

$$+7\,(z_{20}z_{50}+z_{30}z_{40})\frac{d}{dz_{60}}+6\,(z_{20}z_{41}+z_{11}z_{50}+z_{30}z_{31}+z_{21}z_{40})\frac{d}{dz_{51}}$$

$$+5\,(z_{20}z_{32}+z_{11}z_{41}+z_{02}z_{50}+z_{30}z_{22}+z_{21}z_{31}+z_{12}z_{40})\frac{d}{dz_{42}}$$

$$+4\,(z_{20}z_{23}+z_{11}z_{32}+z_{02}z_{41}+z_{30}z_{13}+z_{21}z_{22}+z_{12}z_{31}+z_{03}z_{40})\frac{d}{dz_{33}}$$

$$+3\,(z_{20}z_{14}+z_{11}z_{23}+z_{02}z_{32}+z_{30}z_{04}+z_{21}z_{13}+z_{12}z_{22}+z_{03}z_{31})\frac{d}{dz_{24}}$$

$$+2\,(z_{20}z_{05}+z_{11}z_{14}+z_{02}z_{23}+z_{21}z_{04}+z_{12}z_{13}+z_{03}z_{22})\frac{d}{dz_{15}}$$

$$+(z_{11}z_{05}+z_{02}z_{14}+z_{12}z_{04}+z_{03}z_{13})\frac{d}{dz_{06}}$$

$$+\ldots\ldots$$

$$+\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C');$$

$$V_2 \equiv z_{11}z_{20}\frac{d}{dz_{30}}+2\,(\tfrac{1}{2}z_{11}^2+z_{02}z_{20})\frac{d}{dz_{21}}+3z_{11}z_{02}\frac{d}{dz_{12}}+2z_{02}^2\frac{d}{dz_{03}}$$

$$+(z_{11}z_{30}+z_{20}z_{21})\frac{d}{dz_{40}}+2\,(z_{02}z_{30}+z_{11}z_{21}+z_{20}z_{12})\frac{d}{dz_{31}}$$

$$+3\,(z_{02}z_{21}+z_{11}z_{12}+z_{20}z_{03})\frac{d}{dz_{22}}+4\,(z_{02}z_{12}+z_{11}z_{03})\frac{d}{dz_{13}}+5z_{02}z_{03}\frac{d}{dz_{04}}$$

$$+(z_{11}z_{40}+z_{20}z_{31}+z_{21}z_{30})\frac{d}{dz_{50}}$$

$$+2\,(z_{02}z_{40}+z_{11}z_{31}+z_{20}z_{22}+\tfrac{1}{2}z_{21}^2+z_{12}z_{30})\frac{d}{dz_{41}}$$

$$+3\,(z_{02}z_{31}+z_{11}z_{22}+z_{20}z_{13}+z_{12}z_{21}+z_{03}z_{30})\frac{d}{dz_{32}}$$

$$+4\,(z_{02}z_{22}+z_{11}z_{13}+z_{20}z_{04}+\tfrac{1}{2}z_{12}^2+z_{21}z_{03})\frac{d}{dz_{23}}$$

$$+5\,(z_{02}z_{13}+z_{11}z_{04}+z_{03}z_{12})\frac{d}{dz_{14}}$$

$$+6\,(z_{02}z_{04}+\tfrac{1}{2}z_{03}^2)\frac{d}{dz_{05}}$$

$$+(z_{11}z_{50}+z_{20}z_{41}+z_{30}z_{31}+z_{21}z_{40})\frac{d}{dz_{60}}$$

$$+2\,(z_{02}z_{50}+z_{11}z_{41}+z_{20}z_{32}+z_{12}z_{40}+z_{21}z_{31}+z_{30}z_{22})\frac{d}{dz_{51}}$$

$$+3\,(z_{02}z_{41}+z_{11}z_{32}+z_{20}z_{23}+z_{03}z_{40}+z_{12}z_{31}+z_{21}z_{22}+z_{30}z_{13})\,\frac{d}{dz_{43}}$$

$$+4\,(z_{02}z_{32}+z_{11}z_{23}+z_{20}z_{14}+z_{03}z_{31}+z_{12}z_{22}+z_{21}z_{13}+z_{30}z_{04})\,\frac{d}{dz_{33}}$$

$$+5\,(z_{02}z_{23}+z_{11}z_{14}+z_{20}z_{05}+z_{03}z_{22}+z_{12}z_{13}+z_{21}z_{04})\,\frac{d}{dz_{24}}$$

$$+6\,(z_{02}z_{14}+z_{11}z_{05}+z_{03}z_{13}+z_{12}z_{04})\,\frac{d}{dz_{15}}$$

$$+7\,(z_{02}z_{05}+z_{03}z_{04})\,\frac{d}{dz_{06}}$$

$$+\ldots\ldots$$

$$+\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(F}'\text{)}.$$

10. Let us now proceed to the proof that the conditions (i.), (ii.), (iii.), (v.) of § 9, which have been found as such that a pure ternary reciprocant must satisfy, are conversely such that functions of the second and higher derivatives which satisfy them are necessarily reciprocants.

The formulæ for transforming a partial differential expression in x, y, z, from a form in which x and y are independent variables to one in which y and z are independent, are

$$z_{10}=\frac{1}{x_{01}},\quad z_{01}=-\frac{x_{10}}{x_{01}},$$

$$\frac{d}{dx}=\frac{1}{x_{01}}\frac{d}{dz},\quad \frac{d}{dy}=\frac{d}{d}-\frac{x_{10}}{x_{01}}\frac{d}{dz}.$$

Thus, if r and s be numbers (inclusive of zero) whose sum exceeds unity,

$$\frac{d^{r+s}z}{dx^r dy^s}=\frac{d^{r+s-1}}{dx^{r-1}dy^s}(z_{10})$$

$$=\left(\frac{1}{x_{01}}\frac{d}{dz}\right)^{r-1}\left(\frac{d}{dy}-\frac{x_{10}}{x_{01}}\frac{d}{dz}\right)^{s}\left(\frac{1}{x_{01}}\right)$$

$$=\frac{1}{x_{01}^{r-1}}\frac{d^{r+s-1}}{dz^{r-1}dy^s}\left(\frac{1}{x_{01}}\right)+\ldots$$

$$=-\frac{1}{x_{01}^{r+1}}\frac{d^{r+s}x}{dy^s dz^r}+\ldots,$$

the terms not written down all involving powers of $\frac{1}{x_{01}}$ surpassing the $(r+1)^{\text{th}}$. We have, then, after multiplication by $\frac{1}{r!\,s!}x_{01}^{r+1}$,

$$\frac{z_{rs}}{z_{10}^{r+1}} = -x_{sr} + \text{terms with } \frac{1}{x_{01}} \text{ as a factor} \quad \ldots\ldots\ldots\ldots(25),$$

in which it is to be noticed that the index of the power of z_{10} in the denominator on the left exceeds r, the first suffix-weight of its numerator, by 1, the degree of that numerator.

Now, take $H(z)$ a homogeneous function of degree i of the derivatives z_{rs}, such as to satisfy condition (ii.) of § 9, *i.e.*, to be isobaric and of the same partial weight $\frac{1}{2}w$ in first and second suffixes separately, and consequently to be of full weight w. Let $H(x)$ be the result of replacing every z_{rs} in it by the corresponding x_{rs}, and let $H'(x)$ be the result of replacing in $H(x)$ every derivative x_{rs} by its conjugate with suffixes interchanged x_{sr}. From (25), we obtain

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^i H'(x) + \text{terms involving } x_{01}.$$

Let, further, $H(z)$ have been so chosen as to be annihilated by Ω_1 and V_1. By § 4, the expression for the left-hand side of the foregoing equality in terms of suffixes x's must, this being so, be independent of x_{01}; and we must have, consequently,

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^i H'(x).$$

But, again, let $H(z)$ be subject to annihilation by Ω_2 as well as by Ω_1. It will then, being homogeneous and doubly isobaric, be a single full invariant of the system of emanants, and must consequently be unaltered (or altered only in sign, if a skew-invariant) when each coefficient z_{rs} in every emanant is replaced by its conjugate coefficient z_{sr}, *i.e.*, when first and second suffixes are interchanged throughout. Thus, writing x instead of z,

$$H'(x) = (-1)^{\frac{1}{2}w} H(x),$$

an invariant being skew or not according as its weight ($\frac{1}{2}w$ in this case) is or is not odd. Hence, finally,

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^{i+\frac{1}{2}w} H(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(26);$$

so that, in virtue of (24), $H(z)$ has the property of a reciprocant, so far as the first transformation from z dependent variable to x dependent is concerned.

In exactly the same way it follows that, if $H(z)$ is annihilated by V_2 as well as by Ω_1 and Ω_2,

$$\frac{H(z)}{z_{01}^{i+\frac{1}{2}w}} = (-1)^{i+\frac{1}{2}w} H(y) \quad \text{.....................(27)};$$

which, by (24), is the only further necessity for $H(z)$ to be a pure ternary reciprocant. The comparison of (26) and (27) with (24) makes evident the accuracy of the statement (iv.) of § 9, that the mark of character k of a pure ternary reciprocant is of necessity zero.

The sufficiency, as well as the necessity, of the conditions expressed in § 9, for a pure function R to be a ternary reciprocant, has then been fully established.

11. That $$z_{20}z_{02} - \tfrac{1}{4}z_{11}^2 \quad \text{............................(28)},$$

the discriminant of the second emanant, is a reciprocant, I have already proved from first principles [Vol. XVII., p. 180, (30)]. That it satisfies the determining conditions arrived at in this paper, is so immediately obvious as only to require statement.

A second pure ternary reciprocant is

$$\begin{vmatrix} z_{30}, & z_{21}, & z_{12}, & z_{03} & \\ & z_{30}, & z_{21}, & z_{12}, & z_{03} \\ z_{20}, & z_{11}, & z_{02} & & \\ & z_{20}, & z_{11}, & z_{02} & \\ & & z_{20}, & z_{11}, & z_{02} \end{vmatrix} \quad \text{.....................(29)},$$

the discriminant of the quadratic and cubic emanants. That it must be one is necessary, if the fact be granted that, to use a convenient nomenclature of Professor Sylvester's, it is the criterion of ruled surfaces, which can, of course, have no special respect to any axes of reference. (A remark akin to this has been made by M. Halphen, at the end of his *Thèse sur les Invariants Différentiels.*) The same may also, without difficulty, be shown from elementary considerations; and it may be here remarked that like reasoning will apply to the eliminant of three emanants in the matter of quaternary reciprocants, that of four of quinary, and generally that of the second to the n^{th} emanants in the matter of n-ary reciprocants.

That the eliminant (29) is annihilated by V_1 and V_2, and so obeys the laws arrived at in this paper, is readily seen. The operation of

V_1 upon it gives

$$\begin{vmatrix} 2z_{20}^2, & 3z_{20}z_{11}, & 2z_{20}z_{02}+z_{11}^2, & z_{11}z_{02} & \\ & z_{30}, & z_{21}, & z_{12}, & z_{03} \\ z_{20}, & z_{11}, & z_{02} & & \\ & z_{20}, & z_{11}, & z_{02} & \\ & & z_{20}, & z_{11}, & z_{02} \end{vmatrix}$$

$$+\begin{vmatrix} z_{30}, & z_{21}, & z_{12}, & z_{03} & \\ & 2z_{20}^2, & 3z_{20}z_{11}, & 2z_{20}z_{02}+z_{11}^2, & z_{11}z_{02} \\ z_{20}, & z_{11}, & z_{02}, & & \\ & z_{20}, & z_{11}, & z_{02} & \\ & & z_{20}, & z_{11}, & z_{02} \end{vmatrix}$$

Now, in the first of these determinants, the first row is the sum of $2z_{20}$ times the third and z_{11} times the fourth; and in the second, the second row is the sum of $2z_{20}$ times the fourth and z_{11} times the fifth; so that both vanish. In like manner, V_2 is an annihilator of (29). That the other laws of § 9 are obeyed, is clear in virtue of the properties of eliminants as invariants.

12. A third pure ternary reciprocant will now be determined. We know (see, for instance, Salmon's *Higher Plane Curves*, 2nd ed., § 193) that the quadratic and quartic emanants have an invariant of degree 3, and partial weights 4, 4,

$$I(a^2c) = 6(z_{02}^2 z_{40} + z_{20}^2 z_{04}) - 3z_{11}(z_{02}z_{31} + z_{20}z_{13}) + z_{22}(z_{11}^2 + 2z_{02}z_{20}),$$

and that the cubic and quadratic emanants have one of like type,

$$I(ab^2) = 2z_{02}(3z_{30}z_{12} - z_{21}^2) - z_{11}(9z_{30}z_{03} - z_{12}z_{21}) + 2z_{20}(3z_{03}z_{21} - z_{12}^2).$$

For any constant value of k, then,

$$I(a^2c) + kI(ab^2)$$

is an invariant of the quadratic cubic and quartic emanants satisfying the laws (i.), (ii.), (iii.) of § 9. Now it is at once found that

$$V.I(a^2c) = 36z_{02}^2 z_{20} z_{30} - 9z_{02}z_{11}^2 z_{30} - 12z_{02}z_{11}z_{20}z_{21} + 12z_{02}z_{20}^2 z_{12}$$
$$+ 3z_{11}^3 z_{21} - 3z_{11}^2 z_{20} z_{12},$$

and that

$$V.I(ab^2) = 12z_{02}^2 z_{20} z_{30} - 3z_{02} z_{11}^2 z_{20} - 4z_{02} z_{11} z_{20} z_{21} + 4z_{02} z_{20}^2 z_{12} + z_{11}^3 z_{21} - z_{11}^2 z_{20} z_{12},$$

the term $z_{11} z_{20}^2 z_{03}$ not occurring in either.

Now, the first of these right-hand members is three times the second. It follows that

$$I(a^2c) - 3\,I(ab^2) \quad \text{.........................(30)}$$

is annihilated by V_1. In like manner, V_2 annihilates it. Consequently it is a reciprocant.

In (28), (29), and (30), we have then three pure ternary reciprocants. From these we know two that are independent and absolute, obtained by dividing either two of the three by proper powers of the third. Hence, by my former paper [Vol. xvii., p. 183, (34)], we have the means of educing an infinite series of pure ternary reciprocants by simple processes of differentiation.

Note on Two Annihilators in the Theory of Elliptic Functions.

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[*Read Dec. 9th*, 1886.]

NOTATION.

$P(x, k)$ and $Q(x, k)$ are taken to be algebraic functions of x involving k-functions—not necessarily rational and integral functions of x. [For example, we may have

$$P = (1+k')\,x\sqrt{1-x^2}, \quad Q = \sqrt{1-k^2x^2}.]$$

$$R^2 = Q^2 - \lambda^2 P^2, \quad S^2 = Q^2 - P^2.$$

$$\partial_k = \frac{d}{dk}, \quad \partial_x = \frac{d}{dx}.$$

$$\Omega = nkk'^2\partial_k + (1-x^2)(1-k^2x^2)\,\phi(x, k)\,\partial_x,$$

$$O = nkk'^2\partial_k + (1-x^2)(1-k^2x^2)\,f(x, k)\,\partial_x,$$