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Notes on the Theory of Groups of Finite Order (continued). *By* W. Burnside.

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The first of the two notes in the present communication deals with certain properties of groups whose order is even. It is shown that if 2^m is the highest power of 2 contained in the order of a group, and if the sub-groups of order 2^m are cyclical, the group cannot be simple; so that, in particular, no group whose order is divisible by 2, but not by 4, can be simple. When the highest power of 2 which divides the order of a group is either 2^2 or 2^3 it is shown that, unless the group contains a smaller number of distinct conjugate sets of operations of orders 2 or 4 than the sub-groups of orders 2^2 and 2^3 respectively contain, the group cannot be simple. In the first case, this condition cannot be satisfied unless 3 is a factor of the order; nor can it be satisfied in the second case unless either 3 or 7 is a factor of the order, and, therefore, no group of even order can be simple unless its order is divisible by 12, 16, or 56. It seems extremely probable that this property may be extended to the more general form that, if the order of a group be

$$N = 2^m n$$

where n is odd, and if N is relatively prime to $2^m-1, 2^{m-1}-1, \dots 2^2-1$, the group cannot be simple; but I have not hitherto succeeded in proving this more general result.

In the second note, Dr. Cole's and Herr Hölder's determination of all simple groups whose orders do not exceed 660 is carried on from 660 to 1092, the order of the next known simple group, with the result of showing that no simple groups exist in the interval.

VIII. *On Groups of Even Order ; and, in particular, those whose Orders are divisible by no higher power of 2 than 2^3.*

Let $$N = 2^m n,$$

where n is odd, be the order of a group; and let the sub-groups of order 2^m be cyclical. Then, if the group contains an operation S of odd order which is not permutable with any operation of order 2, S must be one of a set of $2^m\mu$ conjugate operations, where μ is odd. If the group is simple, it can be represented as a transitive permutation-group arising from the permutations of the $2^m\mu$ conjugate operations among themselves, when they are transformed by the N operations of the group. If the set of conjugate operations be transformed by an operation of order 2^m, the resulting substitution of the permutation-group must consist of μ cycles of 2^m symbols each; for, if any cycle consisted of 2^r $(r<m)$ symbols only, the corresponding 2^r operations conjugate to S would be permutable with a group of order 2^{m-r}, which is supposed not to be the case. Now a substitution consisting of μ (odd) cycles of 2^m symbols each is equivalent to an odd number of transpositions, and a group containing such a substitution cannot be simple.

If, on the other hand, the group contains no operation of odd order which is not permutable with an operation of order 2, an operation of order 2 must itself be contained self-conjugately in the group, which again cannot be simple. Hence a group whose order is $2^m n$ (n odd), in which the sub-groups of order 2^m are cyclical, cannot be simple. In particular, a group whose order is even, but not divisible by 4, cannot be simple.

The number of possible different types of sub-group of order 2^m increases very rapidly with m, but, when m is either 2 or 3, it is not difficult to determine under what limitations it is possible for a group to be simple.

If the order is $$N = 2^2 m,$$

where m is odd, and the sub-groups of order 2^2 are not cyclical, each such sub-group contains 3 operations of order 2. Suppose that a sub-group of order 2^2 is contained self-conjugately in a sub-group of order $2^2 m_1$, where $m = m_1 m_2$. If 3 is a factor of m_1, the 3 operations of order 2 in this sub-group may form a single conjugate set, and then all the operations of order 2 in the main group form a single conjugate set. Suppose now that this is not the case, so that the group contains 3 different conjugate sets of operations of order 2. Every operation of order 2 is certainly self-conjugate in a sub-group of order 2^2m_1, and may be self-conjugate in a more extensive sub-group. Let, then, S, an operation of order 2 be self-conjugate within a group of order $2^2m_1\mu$. An operation of this sub-group whose order is odd, and which is not contained in the sub-group of order 2^2m_1, is permutable with S, and with no operation of order 2 which is not conjugate to S. Hence it must form one of a set of $2r$ conjugate operations, where r is odd. If now the group be represented as a permutation-group, consisting of the permutations of these $2r$ operations among themselves which arise by transforming them by all the operations of the group, any operation of order 2 which is not conjugate to S will give a substitution in the permutation-group, consisting of r transpositions, *i.e.*, an odd substitution, and, therefore, the group cannot be simple. Hence, the group is certainly not simple unless the maximum sub-group, which contains an operation of order 2 self-conjugately, is of order 2^2m_1; and when this condition is satisfied every operation of order 2 is permutable with just 2 other operations of order 2, and with no more.

But, now, if A, B are two operations of order 2 belonging to different conjugate sets, and if

$$(AB)^n = 1,$$

A, B generate a dihedral group of order $2n$. If n were odd, A and B would be conjugate, which is not the case. Hence, n must be even, and then $(AB)^{\frac{1}{2}n}$ is an operation of order 2 which is permutable with n distinct pairs of operations of order 2. But this is in direct contradiction to what has just been proved, so that this case cannot occur. It follows that, if a group whose order is 2^2m (m odd) contains 3 different conjugate sets of operations of order 2, it cannot be simple.

If, next, the order is $$N = 2^3m,$$

where m is odd, and if, as in the case just dealt with, the main group contains the same number of conjugate sets of operations of orders 2 and 4 as are contained in a sub-group of order 2^3, it may again be shown that the group cannot be simple. In this case, however, putting aside the cyclical groups of order 2^3 which have already been dealt with, there are 4 other possible types of sub-groups of order 2^3. These are the groups which may be generated as follows:—

(i) $A^2 = B^2 = C^2 = 1, \quad AB = BA, \quad AC = CA, \quad BC = CB;$

(ii) $A^2 = B^4 = 1, \quad AB = BA;$

(iii) $A^2 = B^4 = 1, \quad AB = B^3A;$

(iv) $A^4 = B^4 = 1, \quad AB = B^3A, \quad A^2 = B^2,$

Of these (i) is an Abelian group containing 7 operations of order 2, each of which is self-conjugate. Group (ii) is again Abelian, and contains 4 operations of order 4, and 3 operations of order 2, each one of the 7 being self-conjugate. In the case of group (iii) there are 2 conjugate operations of order 4, and 5 operations of order 2, one of which is self-conjugate, while the remainder form 2 conjugate sets of 2 each. Group (iv) contains a single self-conjugate operation of order 2, and 6 operations of order 4 forming 3 conjugate sets of 2 each.

If, now, in the group of order

$$N = 2^3m \quad (m \text{ odd})$$

the sub-groups of order 2^3 are of type (i), and if the main group contains 7 conjugate sets of operations of order 2, let A, B be two such operations chosen from different sets. The group generated by A and B must be a dihedral group of order $4n$, where n is odd. If this sub-group is not self-conjugate within a sub-group of order 2^3n, it must form one of a set of $2r$ conjugate sub-groups, where r is odd, and, when these are transformed among themselves by operations of order 2 of conjugate sets other than those contained in the dihedral group, the corresponding substitution of the permutation-group will consist of r transpositions, which involves that the group is composite.

If the dihedral group is contained self-conjugately in a group of order 2^3n, the cyclical sub-group of order n which it contains must be transformed into itself by the operations of a group of order 2^3. Let S be the operation of order n generating the cyclical sub-group, and let A, B, C be the generating operations of the group of order 2^3, A and B belonging to the dihedral group. Then

$$ASA = S^{-1} \quad \text{and} \quad BSB = S^{-1}.$$

If, now, $\qquad (\alpha) \; CSC = S^{-1},$

S is permutable with the sub-group formed by 1, BC, CA, AB and if $\qquad (\beta) \; CSC = S,$

S is permutable with the sub-group formed by 1, C, AB, ABC. Hence, in either case, S is one of a set of $2s$ conjugate operations, where s is odd; and it follows as before that the group is composite.

If next, the sub-groups of order 2^3 are of type (ii), and if the main group contains 3 distinct sets of conjugate operations of order 2, one of these sets contains exclusively operations which are the squares of operations of order 4, and the other two sets those that are not.

Let, now, A be an operation of order 2 which is the square of an operation of order 4, and let B be an operation of order 2 belonging to a different conjugate set from A. Then A and B must generate a dihedral group of order $4n$, where n is odd. Suppose that AB is an operation of this group of order $2n$, and write

$$(AB)^n = C, \quad (AB)^2 = S_n,$$

so that C is an operation of order 2, and S_n an operation of order n. The operation C must clearly belong to a different conjugate set from both A and B. Now

$$AS_nA = S_n^{-1}, \quad BS_nB = S_n^{-1}, \quad CS_nC = S_n.$$

If A^1 is any operation contained in the sub-group within which the cyclical sub-group generated by S_n is self-conjugate, and belonging to the same conjugate set as A, then

$$A^1S_nA^1 = S_n^{-1},$$

and, therefore, S_n cannot certainly be permutable with any operation of order 4, since it is not permutable with the square of any such operation. The operation S_n therefore forms one of a set of $4r$ conjugate operations, where r is odd; and, when these are transformed by any operation of order 4, the resulting substitution of the permutation-group consists of r cycles of 4 symbols each. This is an odd substitution, and therefore, again, in this case, the group cannot be simple.

A group of order 8 of type (iii), generated by A and B, where

$$A^2 = 1, \quad B^4 = 1, \quad AB = B^3A,$$

contains 5 operations of order 2, viz.,

$$A, \quad B^2, \quad AB, \quad AB^2, \quad AB^3,$$

of which B^2 is self-conjugate, while A, AB^2 and AB, AB^3 form conjugate sets. From these 5 operations and identity 2 groups of order 4 may be formed, viz.,

$$1, \quad B^2, \quad A, \quad AB^2$$

and

$$1, \quad B^2, \quad AB, \quad AB^3.$$

If, now, the sub-groups of order 2^3 contained in a group of order 2^3m (m odd) are of this type, and if the main group contains 3 distinct sets of conjugate sub-groups of order 2, one of these sets consists of the squares of operations of order 4, and the other two sets of operations of order 2, which are not such squares. Moreover, the above analysis of the operations of such a group of order 2^3 shows that no operation of one of the two latter sets can be permutable with any operation of the other. Now each set contains $2r$ conjugate operations, where r is odd, and, if one set is transformed by an operation of the other set, the resulting substitution consists of r transpositions, and is therefore an odd substitution. Once, again, in this case, the group, then, cannot be simple.

Finally, when the group of order 2^3 is of type (iv), and the main group contains 3 distinct sets of conjugate operations of order 4, the number of operations contained in each set must be of the form $4\mu+2$.

If such a set is transformed by one of its own operations, the resulting substitution will keep 2 symbols unchanged, and interchange the remainder in $2r$ cycles of 2 and $(\mu-r)$ cycles of 4 each, where r is some number less than μ. If the set is transformed by an operation of order 4 belonging to another conjugate set, the resulting substitution will consist of $2r^1+1$ cycles of 2 and $(\mu-r^1)$ cycles of 4 each. Now, since there is only a simple conjugate set of operations of order 2 in this case, the squares of these two substitutions must be of the same type, and therefore $r=r^1$. Hence, one of the two substitutions is necessarily odd, and it follows again in this last case that the group must be composite.

The conditions under which it has been shown that groups of order 2^2m and 2^3m, m being odd, cannot be simple may now be shown to hold necessarily if in the one case 3, and in the other 3 and 7, are not factors of m. For this purpose I prove the following theorem.

If, p^m being the highest power of a prime p which divides the order of a group G, a sub-group h of order p^m is Abelian, and if H be the greatest sub-group that contains h self-conjugately, the number of distinct sets of conjugate operations whose orders are powers of p in G is the same as the number in H.

Let P be any operation of h, and let it be one of x conjugate operations of H. Then P is permutable in a sub-group of H of order $\frac{n_H}{x}$, n_H being the order of H. Hence, if the order of the greatest sub-group within which P is permutable is $\frac{n_H x'}{x}$, P must belong to x'

different groups of order p^m. Hence, summing for the distinct sets of conjugate operations whose orders are powers of p contained in G,

$$\Sigma \frac{n_G x}{n_H x'} x' = (p^m - 1) \frac{n_G}{n_H},$$

n_G being the order of G, or

$$\Sigma x = p^m - 1,$$

which proves the theorem.

Now, if the order of H is relatively prime to $p^m - 1, p^{m-1} - 1, \dots p - 1$, every operation of h is self-conjugate in H, and G contains $p^m - 1$ distinct sets of conjugate operations whose orders are powers of p.

When $p^m = 2^2$, this condition will be satisfied if the order of H does not contain 3; and, when $p^m = 2^3$, it will be satisfied if the order of H contains neither 3 nor 7 as a factor.

A group of order $2^2 m$ is therefore certainly composite if the odd number m is not divisible by 3; and a group of order $2^3 m$, in which the sub-groups of order 2^3 are Abelian, is certainly composite if m is divisible by neither 3 nor 7.

Suppose, next, that in a group of order $2^3 m$ the sub-groups of order 2^3 are of type (iii), given by

$$A^2 = 1, \quad B^4 = 1, \quad AB = B^3 A;$$

and suppose that A and B^2 are conjugate operations in the group. Then A must be the square of some operation B' of order 4, and the sub-group formed by

$$1, \quad B^2, \quad B'^2, \quad B^2 B'^2$$

occurs in the two sub-groups of order 2^3 which contain B and B'. In the first B'^2 and $B^2 B'^2$ are conjugate operations, and in the second B^2 and $B^2 B'^2$ are conjugate. Hence $B^2, B'^2, B^2 B'^2$ form a single conjugate set in the sub-group that contains the group

$$1, \quad B^2, \quad B'^2, \quad B^2 B'^2$$

self-conjugately. The order of this sub-group is therefore divisible by 3; and hence, unless m is divisible by 3, A and B^2 cannot be conjugate operations.

The operation A must enter into an odd number n' of sub-groups of order 2^3. If, then, A and AB belong to the same conjugate set, each operation of the set enters into n' sub-groups of order 2^3; while the number of operations in the set is $2n''$, n'' being odd.

Hence $2n''n'$ is the total number of these operations, distinct or not, which enter in the conjugate set of sub-groups of order 2^3. But, since 4 enter into each sub-group of order 2^3, this is impossible; and therefore A and AB cannot be conjugate.

Suppose, now, lastly, that in a group of order 2^3m the sub-groups of order 2^3 are of type (iv), given by

$$A^4 = 1, \quad B^4 = 1, \quad A^2 = B^2, \quad AB = B^3A;$$

and suppose that A and B are conjugate, so that

$$S^{-1}AS = B,$$

$$S^{-1}A^2S = B^2 = A^2,$$

and S occurs in the group g within which A^2 is permutable. Let the order of this sub-group be $2^3n_1n_2$, and let it contain n_2 sub-groups of order 2^3. Since, within g, A and B are conjugate, it cannot contain 3 distinct conjugate sets of cyclical sub-groups of order 4. Suppose, now, that the sub-group of g of order 2^3n_1 which contains a sub-group of order 2^3 self-conjugately also contains each of its 3 sub-groups of order 4 self-conjugately. Then any sub-group of order 4 will be self-conjugate within a sub-group of g of order $2^3n_1n_2'$, and will form one of n_2'' conjugate sub-groups within g, and each of these will enter in n_2' of the $n_2 (= n_2'n_2'')$ sub-groups of g of order 2^3.

Hence $$3n_2 = \Sigma n_2'n_2'',$$

where the summation is extended to the different distinct sets of conjugate sub-groups of order 4 contained in g. This is impossible, since the number of these sets does not exceed 2; and therefore the 3 sub-groups of order 4 contained in the sub-group of g of order 2^3n_1 are, in this sub-group, conjugate to each other. Hence n_1 must be divisible by 3; and, unless this condition obtains, A and B cannot be conjugate.

Hence a group whose order is 2^3m (m odd) in which the sub-groups of order 2^3 are not Abelian cannot be simple unless m is divisible by 3.

Combining now all the results, they give the theorem that a group whose order is even cannot be simple unless the order contains either 12, 16, or 56 as a factor.

IX. *On the non-Existence of Simple Groups whose Orders lie between 660 and 1092.*

In Vol. xv of the *American Journal of Mathematics*, Dr. Cole has carried on from 201 to 660 a discussion of the possibility of a simple group corresponding to a given order, which was begun and taken as far as 200 by Herr Hölder (*Math. Ann.*, Vol. XLII). The simple group of next smallest order to 660 that is known to exist is a group of order 1092; and it appears a not uninteresting application of the tests for the simplicity of a group, which depend on its order, that have been given in these notes and elsewhere, to determine how many of the 432 numbers from 661 to 1092 inclusive are at once shown to have no simple group corresponding to them. These tests may now be stated as follows. There are no simple groups whose orders are

(i) the power of a prime,

(ii) the product of two or three prime factors,

(iii) the product of four prime factors (with the exception of the order $2^2 . 3 . 5$),

(iv) the product of five prime factors (with the exceptions of the orders $2^3 . 3 . 7$, $2^2 . 3 . 5 . 11$, $2^2 . 3 . 7 . 13$),

(v) of the forms $p_1^m p_2$ (p_1, p_2 primes in ascending order),

(vi) even, but not divisible by 12, 16, or 56.

These tests imply that, if there are simple groups whose orders are odd, none can be of smaller order than $3^4 . 5^2$ or 2025, so that in the interval in question there can be no simple groups of odd order. One further test that may be given here for the sake of completeness is that there are no simple groups whose orders are $p_1^2 p_2^m$ or $p_1^3 p_2^m$.†

These tests applied to the 432 orders from 661 to 1092 dispose of all cases except the following sixteen, viz.:—

$*672 = 2^5 . 3 . 7$,	$800 = 2^5 . 5^2$,	$*880 = 2^4 . 5 . 11$,	$960 = 2^6 . 3 . 5$,
$720 = 2^4 . 3^2 . 5$,	$*816 = 2^4 . 3 . 17$,	$900 = 2^2 . 3^2 . 5^2$,	$1040 = 2^4 . 5 . 13$,
$756 = 2^2 . 3^3 . 7$,	$840 = 2^3 . 3 . 5 . 7$,	$*912 = 2^4 . 3 . 19$,	$1056 = 2^5 . 3 . 11$,
$*784 = 2^4 . 7^2$,	$864 = 2^5 . 3^3$,	$*936 = 2^3 . 3^2 . 13$,	$1080 = 2^3 . 3^3 . 5$.

Of these the six that are marked with a star are immediately shown,

† Unless $p_2 = 3$, a group of order $p_1^3 p_2^m$, if simple, would necessarily contain p_1^3 conjugate sub-groups of order p_2^m. If the operations of these were all distinct, the sub-group of order p_1^3 would be self-conjugate. If, on the other hand, two sub-

each by a simple application of Sylow's theorem, not to correspond to a simple group. That none of the remaining ten correspond to a simple group may be shown by considering them individually.

$$N = 720 = 2^4 . 3^2 . 5.$$

A simple group of this order would contain either 16 or 36 conjugate sub-groups of order 5. If there are 16, each is self-conjugate in a group of order $3^2 . 5$. Such a group is necessarily Abelian, and cannot be expressed in 15 symbols. There must therefore be 36 conjugate sub-groups of order 5, and each is then self-conjugate in a sub-group of order $2^2 . 5$. If this sub-group contains an operation of order 4, it must, when expressed in 36 symbols, consist of either 5, 7 or 8 cycles. If it has 5 or 7, it is an odd substitution, and the group cannot be simple. If it has 8, it is one of 45 conjugate cyclical sub-groups of order 4 whose squares are all distinct. No one of these can transform another, or the square of another, into itself, and therefore, when expressed in 45 symbols, these operations of order 4 consist of 11 cycles, and are odd operations, making the group composite. If the sub-group of order $2^2 . 5$ contains no operation of order 4, it must contain an operation of order 10. The corresponding operation of order 2, which is permutable with an operation of order 5, must, if it is an even substitution of 36 symbols, consist of 10 transpositions. Such an operation is permutable with 16 distinct sub-groups of order 5, and is therefore permutable in a sub-group of order $2^4 . 5$ at least, which makes the group composite.

$$N = 756 = 2^2 . 3^3 . 7.$$

If simple, the group must contain $3^3 . 7$ operations of even order and 6 . 36 operations of order 7, leaving 350 operations whose orders are powers of 3. There are 28 sub-groups of order 3^3, and, if any two of these have a common sub-group of order 3^2, the group is certainly composite. The sub-groups of order 3 which are common to two sub-groups of order 3^3 form a single conjugate set; and when the group is expressed in 28 symbols a simple calculation will show that each sub-group of order 3^3 must contain 8 operations keeping 1 symbol unchanged and 18 keeping 4 symbols unchanged. There are therefore 63 sub-groups of order 3 which occur in more than one sub-group of order 3^3. On the other hand, such a sub-group of

groups of order p_2^m had a maximum common sub-group of order p_2^r, this would (see Note VI) be self-conjugate in a group of order $p_1^3 p_2^r$, and therefore in the main group. See also the recent investigations of Herr Frobenius in the *Berliner Sitzungsberichte*.

order 3 must (Note VI) be permutable in a sub-group of order $2^2.3^2$ at least; and must therefore be one of a set of 21 conjugate sub-groups at most. The supposition that the group is simple thus leads to a contradiction.

$$N = 800 = 2^5.5^2.$$

A simple group of this order must contain 16 conjugate sub-groups of order 5^2, each self-conjugate in a group of order 2.5^2. Expressed in 16 symbols, the sub-group of order 5^2 must contain 3 sub-groups of order 5, each of which consists of 3 cycles, and 3 each of which consist of 2 cycles. Hence, in the sub-group of order 2.5^2 an operation of order 2 must be permutable with an operation of order 5, which consists of 3 cycles. The operation of order 2 therefore must consist of 5 transpositions, and, this being an odd substitution, the group cannot be simple.

$$N = 840 = 2^3.3.5.7.$$

There must be 8, 15, or 120 conjugate sub-groups of order 7. That there should be 8 is clearly impossible if the group is simple; while, if there are 120, there can only be $2^3.3.5$ operations whose orders are not 7. Now (method of Note V), the group contains at least $3.5.7$ operations of even order, so that in this case there would only remain 15 operations of orders 1, 3, and 5. This is clearly impossible if the group be simple. Hence, there must be 15 conjugate sub-groups of order 7, each contained self-conjugately in a sub-group of order $2^3.7$. Such a sub-group necessarily contains an Abelian sub-group of order $2^2.7$, and this cannot be represented in 14 symbols. The group is therefore composite.

Before dealing with the next case, it will be convenient to prove the following lemma:—

If p^m is the highest power of a prime p, which divides the order of a group, and if h is a sub-group of order p^m, the number of sub-groups conjugate to h that have a sub-group of order p^r, but no sub-group of order p^{r+1}, in common with h is zero or a multiple of p^{m-r}.

If there are any such sub-groups, let h' be one, and let

$$P_1\,(=1),\ P_2,\ \dots\ P_{p^m},$$

be the operations of h. Then, of the sub-groups

$$P_1^{-1}h'P_1,\ \ P_2^{-1}h'P_2,\ \dots\ P_{p^m}^{-1}h'P_{p^m},$$

just p^{m-r} are distinct, and each has in common with h a sub-group of order p^r, and none of higher order. If these do not exhaust the sub-groups conjugate to h which have in common with it a sub-group of order p^r, and none of higher order, let h'' be another such sub-group. Then, of the sub-groups

$$P_1^{-2}h''P_1,\quad P_2^{-2}h''P_2,\ \dots\ P_{p^m}^{-1}h''P_{p^m},$$

no one can be the identical with any one of the previous set, and just p^{m-r} are distinct. This process can be continued till the set is exhausted, and the lemma is thus proved.* A theorem which is equivalent to the above is given without proof in a note by M. E. Maillet (*Comptes Rendus*, CXVIII, pp, 1187, 1188).

$$N = 864 = 2^5 . 3^3.$$

There is no transitive group of 9 symbols of this order. (*Cf.* Dr. Cole, *Bull. New York Math. Soc.*, Vol. II, No. 10.) Hence, if the group is simple, there must be 27 conjugate sub-groups of order 2^5. Let h be one of them; then there must be $2x_1$, $4x_2$, $8x_3$, and $16x_4$ groups conjugate to h, and having in common with it sub-groups of order 2^4, 2^3, 2^2, 2 respectively, and no sub-groups of respectively higher orders. Hence,

$$1+2x_1+4x_2+8x_3+16x_4 = 27,$$

and therefore x_1 must be different from zero.

But a sub-group of order 2^4 which is common to the sub-groups of order 2^5 must (Note VI) be self-conjugate in a sub-group of order $2^5 . 3$ at least. Hence the group must be isomorphous to a transitive group of 9 symbols; and, therefore, since the isomorphism must be merihedric, the group cannot be simple.

$$N = 900 = 2^2 . 3^2 . 5^2.$$

There must be 36 sub-groups of order 5^2. If a sub-group of order 5 were contained self-conjugately in a more extensive sub-group than one of order 5^2, it must necessarily be in one of order $6 . 5^2$, and the sub-group would then be one of 6 conjugate sub-groups, which would make the group composite. If this is not the case, all the operations of the 36 groups of order 5^2, except identity, are distinct; so that there are only $2^2 . 3^2$ operations whose orders are not powers of 5. But the group must contain at least $3^2 . 5^2$ operations of even order; so that this latter supposition is impossible.

* From this lemma it follows at once that a group of order $p_1^m p_2^2$ cannot in any case be simple unless $p_2 \equiv 1 \pmod{p_1^2}$. *Cf.* Note VI.

$N = 960 = 2^6 . 3 . 5.$

The group must contain 15 conjugate sub-groups of order 2^6. Let T be an operation of order 2 which is contained self-conjugately in a sub-group of order 2^6. If the group is simple, T, when expressed in 15 symbols, must consist of 6 or 4 transpositions. Represent the symbols by 1, 2, ... 14, 15, and consider that sub-group of order 2^6 which keeps 15 unchanged. Let T keep 13, 14, and 15 unchanged. Then, if T is self-conjugate in the sub-group, every one of its operations must either keep 13 and 14 unchanged, or must interchange them. Now T belongs to 3 different sub-groups of order 2^6, and therefore the sub-group that keeps 15 unchanged must contain 3 operations of the conjugate set to which T belongs. These 3 operations must all keep 13, 14, and 15 unchanged, as otherwise T would be self-conjugate in a group of greater order than 2^6. Hence the 15 conjugate operations consist of 5 sets of 3 each, each set keeping 3 of the 15 symbols unchanged. The group is therefore imprimitive in 5 sets of 3 symbols each, and, if simple, must be expressible as a transitive group of 5 symbols. This is impossible for a group whose order contains the factor 2^6. The case in which T consists of 4 transpositions may be treated in a similar manner.

$N = 1040 = 2^4 . 5 . 13.$

If simple, the group must have 26 sub-groups of order 5, each contained self-conjugately in a sub-group of order $2^3 . 5$. Such a sub-group necessarily contains an operation of order 10; and the corresponding operation of order 2, which is permutable with an operation of order 5, must, if expressed as an even substitution of 26 symbols, consist of 10 transpositions. It must therefore occur in 6 sub-groups of order $2^3 . 5$, and be permutable with 6 sub-groups of order 5. But, since 6 is not a factor of the order of the group, this is impossible.

$N = 1056 = 2^5 . 3 . 11.$

There must be 12 conjugate sub-groups of order 11, each self-conjugate in a sub-group of order $2^2 . 11$. But such a sub-group necessarily contains an operation of order 22, and this cannot be expressed in 12 symbols.

$$N = 1080 = 2^3 . 3^3 . 5.$$

There must be 6, 36, or 216 sub-groups of order 5; and the first supposition is clearly impossible for a simple group. If there were 36, each would be self-conjugate in a group of order $2 . 3 . 5$. This group would contain a sub-group of order 15, and a sub-group of order 3 self-conjugately. The latter would necessarily be self-conjugate in a sub-group of order $2 . 3^2 . 5$, and would be therefore one of at most 12 conjugate sub-groups. But in a group of degree 12 an operation of order 15 would contain a single cycle of 5 symbols, so that this case cannot occur. There must therefore be 216 sub-groups of order 5; leaving only $2^3 . 3^3$ operations whose orders are not 5. Now, since 7 is not a factor of the order of the group, there must be more than one conjugate set of operations whose orders are 2, or powers of 2, and corresponding to each there must be a distinct set of either $3^3 . 5$ or $2 . 3^2 . 5$ operations of even order in the group (method of Note V). Hence this case certainly cannot occur, and this group must be composite.

$$N = 1092 = 2^2 . 3 . 7 . 13.$$

If simple, a group of this order must contain 14 sub-groups of order 13, each being self-conjugate in a group of order $6 . 13$. Since a group of degree 14 cannot contain operations of order 26 or 39, this latter sub-group must be metacyclical in type. Again, there must be 78 sub-groups of order 7, each self-conjugate in a sub-group of order $2 . 7$; and this must be dihedral in type, as otherwise the 78 sub-groups would contain $78 . 12$ distinct operations. Hence the distribution of the operations of the group in conjugate sets is necessarily identical with that of the known simple group of this order.