Connexion of Quadratic Forms.

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1. *Quadratic Forms.*—The function

$$f(m, n) = (mX^2 + nY^2)$$

is styled a *Quadratic Form* or sometimes (for shortness) simply a *Form*; the numbers m, n are the distinctive *characters* of the form itself, the numbers X, Y merely determining the particular number N expressed in the form. When either of $m, n = 1$, the form is styled *simple*; when neither of $m, n = 1$, it is styled *non-simple*.

2. *Object of this Paper.*—In this paper a method is developed (Art. 24) whereby from two given *distinct* forms of the same number a new and *distinct* form may be derived under certain conditions. Each of the three so "allied forms" will be shown to be in certain cases similarly *derivable* from the other two. The process depends on two known processes here styled *conformal multiplication* and *conformal division*; to render it clear, it has been necessary to develop these in considerable detail, especially with regard to the conditions of possibility of the latter (the conformal division); this occupies Art. 6–23. The new process of "derivation" occupies Art. 24–35, followed by examples Art. 36–41.

3. *Notation.*—The following notation is used uniformly throughout this paper. All symbols used denote *integers*.

N = a *positive* integer; p = a *prime*;

f, F, ϕ, Φ denote quadratic forms (usually*) of *positive* numbers;

$$f(m, n) = mt^2 + nu^2; \quad f(1, mn) = t^2 + mnu^2;$$

$$F(m, n) = mT^2 + nU^2; \quad F(1, mn) = T^2 + mnU^2;$$

$$\phi(m, n) = mx^2 + ny^2; \quad \phi(1, mn) = x^2 + mny^2;$$

$$\Phi(m, n) = mX^2 + nY^2; \quad \Phi(1, mn) = X^2 + mnY^2.$$

* It is often convenient to use the *negative* unit-form (Art. 9), $f(m, -n) = -1$, instead of the *positive* form $f(n, -m) = +1$.

Thus $f(m, -n)$, $f(n, -m)$, $f(1, -mn)$, &c., usually denote *positive* numbers; also $f(m, \sim n)$ denotes the *ambiguous* $(mt^2 \sim nu^2)$, *i.e.*, the *positive* number $(mt^2 - nu^2)$ or $(nu^2 - mt^2)$.

4. *Isomorphs, Antimorphs, Symmorphs.*—Forms $f_1(m, n)$, $f_2(m, n)$, &c., containing the *same* m, n, are styled *Isomorphs*, (as being of *same* form); the two forms $f(m, n)$, $f(n, m)$ are also *isomorphs* when m, n are *both positive.* Two forms $f(m, -n)$, $f(n, -m)$ in which the m, n carry *opposite* signs, and are *interchanged in position*, are styled *Antimorphs;* thus $f(1, -n)$, $f(n, -1)$ are *simple antimorphs.* Two forms $f(m, n)$, $f(1, mn)$, wherein the mn of the simple form = the product of the m, n of the non-simple form, are styled *Symmorphs*, (as being closely *related*, as will appear later). When m, n carry opposite signs, the antimorphs $f(m, -n)$, $f(n, -m)$ are each *symmorphs* of both the *antimorphs* $f(1, -mn)$, $f(mn, -1)$.

5. *Conformals, Non-conformals.*—The three forms *Iso-*, *Anti-*, and *Sym-morphs* are so closely related (as will appear later on), that it is convenient to include them under a single name, viz., *Conformals.* Forms not so related are styled *Non-conformal.* The *simple* form $f(1, mn)$ may be a symmorph of several *non-simple* forms $f(m_1, n_1)$, $f(m_2, n_2)$, &c., which are non-conformal among each other, provided $mn = m_1 n_1 = m_2 n_2 =$ &c.

Note.—When either of m, n contains square factors, these factors may—if convenient—be cancelled from m, n, and *absorbed* into the adjacent squares X^2, Y^2. The properties of *iso-*, *anti-*, and *sym-morphism*, and of *conformality* are not affected by the presence of square factors (even though these factors be different) in the m, n of some of the forms; these properties are defined by the various m, n after *excluding* square factors from them.

6. *Conformal Multiplication.*—The following* process, whereby the product of two conformals (f, F) is expressed as a conformal (Φ), is styled *Conformal Multiplication;* there are two principal cases (marked i., ii.)

i. $$f(m, n)\,F(m, n) = (mt^2 + nu^2)(mT^2 + nU^2)$$
$$= X^2 + mnY^2 = \Phi(1, mn), \qquad (1)$$

where $X = mtT \pm nuU$, $Y = tU \mp uT$, (opp. signs in X, Y). (1a)

ii. $$f(m, n)\,F(1, mn) = (mt^2 + nu^2)(T^2 + mnU^2)$$
$$= mX^2 + nY^2 = \Phi(m, n), \qquad (2)$$

* These results are well known: they are here given *in detail*, for reference further on.

where $X = tT \pm nuU$, $Y = mtU \mp uT$, (opp. signs in X, Y). (2a)

The results when the coefficients m, n carry the same, or opposite, sign are worth recording.

i. $f(m, n)F(m, n) = \Phi(1, mn)$; $f(m, -n)F(m, -n) = \Phi(1, -mn)$, (3)

ii. $f(m,n)F(1,mn) = \Phi(m,n)$; $f(m, -n)F(1, -mn) = \Phi(m, -n)$; (4)

with their sub-cases

i*a*. $f(m, -n)F(n, -m) = \Phi(mn, -1)$, (5)

ii*a*. $f(m, -n)F(mn, -1) = \Phi(n, -m)$. (6)

The simple cases, when either m or $n = \pm 1$, are of special importance:

i*b*, ii*b*. $f(1,n)F(1,n) = \Phi(1,n)$; $f(1, -n)F(1, -n) = \Phi(1, -n)$; (7)

with their sub-cases

i*c*. $f(m, -1)F(m, -1) = \Phi(1, -m)$, (8)

ii*c*. $f(m, -1)F(1, -m) = \Phi(m, -1)$. (9)

Thus, in all cases, the product of two conformals (f, F) can be expressed as a *conformal;* also, when the two factors f, F are *unequal*, it is expressible in *two* isomorphic ways, in consequence of the double sign in X, Y; and the signs used in X, Y are *opposite* or *similar*, according as the coefficients m, n carry the *same* or opposite signs.

[If m, n contain a common factor κ, so that $m = \kappa\mu$, $n = \kappa\nu$, then its square appears in the product $mn = \kappa^2 . \mu\nu$ in Results 1, 3, &c., and may be absorbed into the adjacent square (Y^2), if found convenient.]

Product of Non-conformals.—The product of two unequal non-conformals (f, F) is *sometimes* (arithmetically) expressible *in one way* as a conformal of *one* of the factors; but it does not seem possible to formulate this *algebraically*.

Ex.—Let $f = 17$, $F = 33 = 5 . 1^2 + 7 . 2^2$; here f cannot be expressed conformally with the given form of F; but the product $fF = 561 = 1^2 + 35 . 4^2$, which is a *conformal* (symmorph) of F.

7. *Degraded Conformals.*—Since the three numbers m, n, mn may be written

$$m = m . 1^2 + n . 0^2; \quad n = m . 0^2 + n . 1^2; \quad mn = 0^2 + mn . 1^2;$$

they may be looked on as *degraded conformals* of $f(m, n)$ and $f(1, mn)$. The intimate connexion of symmorphs is now seen, as they are interchanged by multiplication by m, n; for

$$m(mt^2+nu^2) = (mt)^2+mnu^2; \quad n(mt^2+nu^2) = (nu)^2+mnt^2;$$

$$m(t^2+mnu^2) = mt^2+n(mu)^2; \quad n(t^2+mnu^2) = m(nu)^2+nt^2.$$

In four cases *one* of the two product-forms Φ also degenerates, by one of its terms X, Y becoming zero,

Case i., when $mtT = nuU$; or $tU = uT$.

Case ii., when $tT = nuU$; or $mtU = uT$.

8. *Continued Conformal Multiplication.*—The above results are easily extended to the continued product of conformals; the formulæ are inconveniently heavy, but the results expressed in words are simple.

"The product of several conformals is a conformal." (10)

"That of several (r) non-simple isomorphs is an isomorph or symmorph, according as r is *odd* or *even*." (10a)

"That of several simple isomorphs is an isomorph." (10b)

In each case the final product-form Φ is—when the factors are *unequal*—expressible in *four* isomorphic ways when there are *three* factors, and generally in 2^{r-1} isomorphic ways when there are r factors, on account of the double sign involved in X, Y; and the signs used in X, Y are opposite or similar, according as the coefficients m, n carry the same or opposite signs.

9. *Unit-Forms.*—When m, n are such that values of τ, v can be found, so that

$$f_0(m, n) = m\tau_0^2+nv_0^2 = \pm 1, \text{ (m, n must be of opp. signs)}, \quad (11)$$

then the powers of f_0 are all $= \pm 1$, and—being products of conformals—are expressible (Art. 6, 8) as conformals of f_0; thus (ω, ϵ denoting *odd* and *even* integers),

$$\{f_0(m, n)\}^{\epsilon} = \tau_\epsilon^2+mn\,.\,v_\epsilon^2 = f_\epsilon(1, mn) = (\pm 1)^\epsilon = +1, \quad (11a)$$

$$\{f_0(m, n)\}^{\omega} = m\tau_\omega^2+nv_\omega^2 = f_\omega(m, n) = (\pm 1)^\omega = \pm 1. \quad (11b)$$

Such quantities $f_0(m, n)$, $f_0(1, mn)$, and their powers are called *Unit-forms* of $f(m, n)$, $f(1, mn)$.

[Note that $f_0(1, -n) = \tau_0^2-nv_0^2 = +1$ is always possible;
but $f_0(1, -n) = \tau_0^2-nv_0^2 = -1$, $f(m, n) = +1$ or -1 are only possible under certain conditions.]

10. *Automorphs.*—The product of a number N in a given form $N = f(m, n)$ by any of the unit-forms $N_0 = f_0(m, n)$ or $f_0(1, mn)$ thereof, or by their powers, is unchanged in value, and—being a product of conformals—is expressible as a conformal. All such forms of the *same number*, N, derivable from one another by conformal multiplication by a unit-form (N_0) thereof, are styled *Automorphs* (of that form); and are considered *equivalent forms.* Forms of the same number—not so derivable from one another—are styled *Anautomorphs*, even though they be conformal (as conformals of composite numbers are not necessarily automorphs).

10*a*. *Use of unit-forms.*—Unit-forms are of great practical use (in computation) in modifying the *form* of a number N into an automorph thereof (*i.e.*, without changing the value of N), and chiefly in two ways :—

(1) In *changing the form* $f(m, n)$ or $f(1, mn)$ into the *symmorph* $f(1, mn)$ or $f(m, n)$; or $f(m, -n)$ into its *antimorph* $f(n, -m)$.

(2) In *reducing* a form $(mt^2 - nu^2)$ in which t, u are *large* numbers into an *isomorph* $(mt'^2 - nu'^2)$ in which t, u are smaller numbers.

Examples of both cases are given, Art. 39–41.

11. *Monomorphs, Dimorphs, Polymorphs.*—A form $f(m, n)$ of a particular number (N) is styled a *Monomorph*, *Dimorph*, or *Polymorph*; when N is expressible (anautomorphically) in *only one* way, in *two* ways, or in *several* ways in that form, or conformally therewith; automorphs (of that form) being here reckoned, as above stated, as *equivalent* forms.

Examples.—(1) Primes are always *monomorphs* of every form in which they are expressible: thus conformal forms of a prime are necessarily automorphs. (2) Products of two *unequal* conformals $f(m, n)$ and $F(m, n)$ or $F(1, mn)$ are always *dimorphs* (of that form). (3) Products of three *unequal* conformals are always *tetramorphs* (of that form). (4) Products of several (r) *unequal* conformals are always *polymorphs* of order 2^{r-1} (of that form). (5) Composite numbers (N) may be *monomorphs* of certain forms, and are always monomorphs of any form wherein m, n are both $+$, and either m or n $> \frac{1}{4}N$. (6) Thus dimorphism marks a composite number; whilst monomorphism is a necessary (but not a sufficient) mark of a prime.

12. *Powers of Forms.*—The r^{th} power of a form $N = f(m, n)$—being a product of conformals—is expressible conformally (Art. 6, 8), with $f(m, n)$, noting that, by (10*a*, *b*),

"The r^{th} power of a non-simple form $f(m, n)$ is an isomorph $\Phi(m, n)$, or symmorph $\Phi(1, mn)$ of $f(m, n)$, according as r is *odd* or *even*." (12)

But there is an apparent exception to the rules of Art. 11, as to the *poly*-morphism of the result (which are there given only for *unequal* factors). Thus, by the rules of conformal multiplication of

$$N = f(m, n) = (mt^2 + nu^2)$$

by itself, the two power-forms arising at each step from the double sign in X, Y are—writing $mt^2 + nu^2 = f$ (for shortness)—

$$N^2 = (mt^2 - nu^2)^2 + mn\,(2tu)^2 = (mt^2 + nu^2)^2 + mn\,.\,0^2. \quad (13a)$$

$$N^3 = m\{t(mt^2 - 3nu^2)\}^2 + n\{u(3mt^2 - nu^2)\}^2 = m\,(tf)^2 + n\,(uf)^2. \quad (13b)$$

$$N^4 = (m^2t^4 - 6mnt^2u^2 + n^2u^4)^2 + mn\{4tu\,(mt^2 - nu^2)\}^2$$

$$= \{(mt^2 - nu^2)f\}^2 + mn\{2tu\,.f\}^2 = (mt^2 + nu^2)^4 + mn\,.\,0^4. \quad (13c)$$

$N^5 =$ &c., &c., &c.

It will be seen that there is only one new, really distinct, form at each step; the remaining forms of each step degenerating to simpler forms, including the original form $f(m, n)$ as a factor of both X, Y, and ending with

$$N^r = (f^r + mn\,.\,0^r), \text{ when } n \text{ is } even;$$

or $$= m\,(tf^{(r-1)/2})^2 + n\,(uf^{(r-1)/2})^2, \text{ when } n \text{ is } odd. \quad (13d)$$

In a certain sense, therefore, the powers of a form $f(m, n)$ may be said to be *monomorphs* of that form.

13. *Product of Automorphs.*—The continued product of several (say r) automorphs $f_1, f_2, f_3, \ldots f_r$ of any form $f(m, n)$ of a number N is of course expressible conformally with $f(m, n)$, since automorphs are only a special kind of conformals. Now, if f_0 be any unit-form of $f(m, n)$, then (Art. 10) each automorph of f differs from f only by some power of f_0 being included in it, so that

$$f_1 = f\cdot f_0^{\alpha}, \quad f_2 = f\cdot f_0^{\beta}, \quad f_3 = f\cdot f_0^{\gamma}, \text{ \&c.}$$

Hence $$f_1\cdot f_2\cdot f_3 \ldots f_r = f^r\cdot f_0^{\alpha+\beta+\gamma+\ldots+\rho} = N^r. \quad (14)$$

Thus the continued product of r automorphs of f differs from f^r only by some power of the unit-form f_0 being included, and all the forms in which N^r is thus expressible are automorphs of f^r itself, and therefore comprise no new distinct forms other than those arising from the involution of $f(m, n)$ by itself, and may therefore be said to be (in a certain sense) *monomorphic*, (Art. 12).

14. *Ratio of Conformals.*—The ratio of two conformals $F \div f$, &c., is generally expressible in an infinite number of ways as a ratio of two numbers $\Phi \div \phi$, each conformal with F, f by merely multiplying both numerator and denominator by the *same* number χ conformal with both. Thus

$$N = \frac{F(m, n)}{f(m, n)} = \frac{F(m, n)}{f(m, n)} \frac{\chi(m, n)}{\chi(m, n)} = \frac{\Phi(1, mn)}{\phi(1, mn)} \tag{15a}$$

$$= \frac{F(m, n)}{f(m, n)} \frac{\chi(1, mn)}{\chi(1, mn)} = \frac{\Phi(m, n)}{\phi(m, n)}. \tag{15b}$$

The simplest change of this sort is by use of the degraded (Art. 7) conformals m, n, or mn; thus

$$N = \frac{mT^2 + nU^2}{mt^2 + nu^2} = \frac{(mT)^2 + mnU^2}{(mt)^2 + mnu^2} = \frac{(nU)^2 + mnT^2}{(nu)^2 + mnt^2} = \frac{m(nU)^2 + n(mT)^2}{m(nt)^2 + n(mu)^2}. \tag{15c}$$

Similarly, the ratios of conformals

$$F(m, n) \div f(1, mn), \quad F(1, mn) \div f(m, n), \quad \text{\&c., \&c.,}$$

admit of similar automorphic changes.

15. *Conformal Division.*—The following* process, whereby the quotient (N) of two conformals ($F \div f$) is—under certain conditions—expressed as a conformal (ϕ), is styled *Conformal Division.* The conditions are

F to be a multiple of f, or $N = F \div f =$ integer. (16)

F to be dimorph, and f to be monomorph (of their form). (17)

The squares (t, u) in f, and the squares (T, U) in F, to be in each case prime-pairs. (18)

Then, dividing the product-identities $f \cdot F = \Phi$, [(1), (2) of Art. 6], by f^2, so that $N = F \div f = \Phi \div f^2$, and writing out at length as below, it will be shown (below) that the X and Y of *one* of the two pairs X, Y in each of the three results (19, 20, 21) below—arising from the double signs in X, Y in the three formulæ (19*a*, 20*a*, 21*a*) are always both (arithmetically) divisible by their denominator (f),

* These results are already known: they are given here in detail, for use further on; it has been necessary to investigate the *conditions of possibility* of this division pretty fully here, as the new process of "derivation" (the object of this paper) is strictly limited by these.

so that the $X \div f$, $Y \div f$ used below are integers; and, finally, the quotient N becomes in each case expressed *conformally* with F, f; thus three principal cases arise.

[In forming Case iii. from Result (2), an obvious interchange of f with F, and of t, u with T, U of that result has had to be made.]

$$\text{i. } N = \frac{F(m, n)}{f(m, n)} = \frac{mT^2+nU^2}{mt^2+nu^2} = \frac{X^2+mnY^2}{(mt^2+nu^2)^2}, \text{ [by (1)]}$$

$$= \left\{\frac{X}{f(m, n)}\right\}^2 + mn\left\{\frac{Y}{f(m, n)}\right\}^2$$

$$= x^2+mn\,.\,y^2 = \phi(1, mn); \tag{19}$$

where $X = mtT \pm nuU$, $Y = tU \mp uT$; (opp. signs in X, Y). (19a)

$$\text{ii. } N = \frac{F(1, mn)}{f(m, n)} = \frac{T^2+mnU^2}{mt^2+nu^2} = \frac{mX^2+nY^2}{(mt^2+nu^2)^2}, \text{ [by (2)]}$$

$$= m\left(\frac{X}{f(m, n)}\right)^2 + n\left(\frac{Y}{f(m, n)}\right)^2$$

$$= mx^2+ny^2 = \phi(m, n); \tag{20}$$

where $X = tT \pm nuU$, $Y = mtU \mp uT$; (opp. signs in X, Y). (20a)

$$\text{iii. } N = \frac{F(m, n)}{f(1, mn)} = \frac{mT^2+nU^2}{t^2+mnu^2} = \frac{mX^2+nY^2}{(t^2+mnu^2)^2}, \text{ [by (2)]}$$

$$= m\left(\frac{X}{f(1, mn)}\right)^2 + n\left(\frac{Y}{f(1, mn)}\right)^2$$

$$= mx^2+ny^2 = \phi(m, n); \tag{21}$$

where $X = tT \pm nuU$, $Y = mTu \mp Ut$; (opp. signs in X, Y). (21a)

Argument.—The argument* (as to the divisibility of X and Y by f) is that unless one pair (X, Y) of the two pairs (X, Y) given by the double sign be arithmetically divisible by the denominator (f), the double sign would make the number of ways in which the product $f.F$ could be expressed in the form $\Phi(m, n)$ or $\Phi(1, mn)$ double of that in which the numerator F itself could be so expressed (when the X, Y of Φ are mutual primes, and the t, u of f are also mutual primes);

* Case i., with its proof, is adapted from Mr. Bickmore's paper, "On the Numerical Factors of a^n-1," Art. 8, in *Messenger of Mathematics*, Vol. xxv., p. 10. The proof of Case ii. was communicated to the author by Mr. Bickmore. The proofs of the three cases, being quite similar, have been combined into one, as above, by the author.

whereas, when f is a monomorph factor of the dimorph conformal F, this number is unchanged.

The cases when the coefficients m, n carry the *same*, or *opposite, sign* are worth writing down side by side for contrast and reference:

i. $F(m,n)\div f(m,n)=\phi(1,mn)$; $F(m,-n)\div f(m,-n)=\phi(1,-mn)$, (22a)

ii. $F(1,mn)\div f(m,n)=\phi(m,n)$; $F(1,-mn)\div f(m,-n)=\phi(m,-n)$, (22b)

iii. $F(m,n)\div f(1,mn)=\phi(m,n)$; $F(m,-n)\div f(1,-mn)=\phi(m,-n)$; (22c)

with their sub-cases

i*a*. $F(m,-n)\div f(n,-m)=\phi(mn,-1)$, (23a)

ii*a*. $F(mn,-1)\div f(m,-n)=\phi(n,-m)$, (23b)

iii*a*. $F(m,-n)\div f(mn,-1)=\phi(n,-m)$. (23c)

The simple cases when either m or $n=\pm 1$ are of special importance;

i., ii., or iii*b*. $F(1,n)\div f(1,n)=\phi(1,n)$; $F(1,-n)\div f(1,-n)=\phi(1,-n)$, (24)

with their sub-cases

i*c*. $F(m,-1)\div f(m,-1)=\phi(1,-m)$, (24a)

ii*c*. $F(1,-m)\div f(m,-1)=\phi(m,-1)$, (24b)

iii*c*. $F(m,-1)\div f(1,-m)=\phi(m,-1)$. (24c)

Thus, in all cases [under the conditions (16) to (18)], the quotient (N) of two conformals ($F\div f$) can be expressed as a *conformal* (ϕ). The quotient-results are exactly analogous with the product-results, and are worth comparing with them * in detail; but in the division process (except when the denominator $f=\pm 1$, Art. 16), one of the signs ($\pm$) involved in the X, Y leads to no result (as in every such case only one of the pairs of X, Y is divisible by its denominator f), so that the quotient-forms are all *unique* for the particular values of m, n, t, u, T, U in F, f.

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* Note that the single product, Case ii., with its Sub-cases ii. *a*, *b*, *c*, gives rise to *two* quotient Cases ii., iii., with corresponding Sub-cases ii. *a*, *b*, *c*, and iii. *a*, *b*, *c*.

16. *Conformal Division by a Unit-form.*—When the divisor f is a unit-form f_0 (Art. 9) of the numerator F, then both pairs of the X, Y are of course divisible by the unit-denominator ($f_0 = \pm 1$), so that *two* conformal quotients are here yielded, but these are merely automorphs or equivalent forms; this requires no formal proof.

17. *Conformal Division by an Automorph.*—When the divisor f is an automorph of the numerator F, the conformal quotient is *unique* and is always a unit-form of F, f; this requires no formal proof.

18. *Conditions of Conformal Division.*—The conditions (16, 17, 18) of conformal division need careful attention. The condition (16) of (arithmetical) divisibility of F by f is obviously necessary to any sort of division. The remaining conditions are needed to the argument (Art. 15) that one pair of the X, Y should be (arithmetically) divisible by f; these will be considerably relaxed in Art. 19–21.

The condition (17) that F should be dimorph when f is monomorph is the most important; it is needed from the fact that, in all the quotient-theorems of Art. 15, the numerator (F) = the product of two conformals (f, ϕ), and must therefore be dimorph (of their form), by Art. 6.

Ex.: $$N = 17 = \frac{561}{33} = \frac{1^2+35.4^2}{5.1^2+7.2^2} = \frac{F(1, mn)}{f(m, n)}; \quad (m = 5,\ n = 7).$$

Here $F \div f$ is an integer; F, f are conformals (symmorphs), and f is a *monomorph* of form $f(m, n)$; also the t, u of f and the T, U of F are mutual prime-pairs. Thus all the conditions of conformal division are satisfied, except that F is a *monomorph* (instead of dimorph); so that the conformal division of $F \div f$ is impossible, and N cannot be expressed in the form $\phi(5, 7)$.

Note.—Unfortunately the investigation of these conditions (dimorphism of F, *monomorphism* of f) is laborious when F or f respectively is a high number. In that case the best plan in practice is to examine the monomorphism of f only; and (when f is a monomorph) to proceed to the conformal division at once: if one pair of the X, Y turn out to be divisible by f, this shows that the conformal division is possible; whilst, if neither pair of X, Y is divisible by f, this shows that some condition of conformal division is violated. When f is not a monomorph, proceed as in Art. 20.

Exception.—In two cases the dimorphism of F is unnecessary, viz., (1) when f is a unit form of F, and (2) when f, F are automorphs; in both these cases F may be a monomorph (when f is monomorph); this is sufficiently evident from Art. 16, 17.

19. *Extension of Conditions of Conformal Division.*—One of the main conditions of conformal division of Art. 15 was that f should be a *monomorph.* This condition is, however, not essential.

In every case the numerator (F) is—by hypoth.—the product of two conformals ($f.\phi$); hence, when f itself is a polymorph, it follows that, for every distinct (*i.e.*, anautomorph) *conformal* partition of f, there are *two* distinct conformal partitions of F; and thus, if f is capable of r distinct conformal partitions, F has $2r$ distinct conformal partitions, only *two* of which *correspond* to any particular partition of f. Hence, when f is not a monomorph, the proper condition as to the form of F is—excluding the two exceptional cases of Art. 16, 17—

"The form of the numerator (F) should be one of the two (anautomorphic) conformals *corresponding* to the particular form (f) of the divisor." (25)

Note.—When f is a high number, the *a priori* investigation as to whether this condition is satisfied would be very laborious. The simplest plan in practice would be to attempt the conformal division direct: if successful, this would be sufficient proof of its possibility; whilst, if neither of the X, Y are divisible by f, this would show that some condition of conformal division was violated.

If, however, the quotient N be really expressible* conformally with F, f, then the preceding analysis shows that the numerator must, in general—(omitting the exceptional cases of Art. 16, 17)—be expressible in *two* forms satisfying the above condition, but the analysis does not show how these two *new* forms of F are to be found from the given F, f. The labour involved would probably be prohibitory when F is a high number. This difficulty may, however, be entirely avoided by the process of the next article.

20. *Practical Conformal Division*, (f polymorph).—When f is not a monomorph (of the given form), this indicates that it is a *composite* number (Art. 11).

Let f be resolved into its *prime* factors, say $f_1, f_2, f_3, \dots f_r$; next, let each of these, or as many of them as possible, be expressed† in quadratic forms, conformal with f; these will necessarily be monomorphs. Any, not so expressible, should be combined into products of two or three, &c., at a time, and these products should then be expressed† in forms conformal with f; these will also be monomorphs. Thus finally f will have been resolved into *monomorphic conformal* factors, say $f = f'_1 . f'_2 . f'_3 . \dots f'_r$.

The process of conformal division may now be applied to forming the following conformal quotients in succession:—

$$F_1 = \frac{F}{f'_1}, \quad F_2 = \frac{F_1}{f'_2}, \quad F_3 = \frac{F_2}{f'_3}, \quad \text{\&c.; lastly,} \quad \phi = \frac{F_{r-1}}{f'_r}. \quad (26)$$

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* In many cases this is known to be possible *a priori*: *e.g.*, if

$$(T^2 + nU^2) \div (t^2 + nu^2) = p \text{ (a prime)},$$

then p is known to be of form $x^2 + ny^2$, when $n = 1, \pm 2, \pm 3, -5$, &c.

† This should not be laborious, because the factors are supposed to be not large numbers.

The denominators all satisfy the necessary condition; if the conformal division of the *given* F by f be possible, the process *will succeed* at every step; but, if at any step the conformal division fail—*i.e.*, if it turn out that neither pair of the X_{s-1}, Y_{s-1} are (arithmetically) divisible by their denominator f_s—then the conformal division proposed of F by f is *impossible.*

Note.—The process, thus* conducted, is direct and certain—(effecting the conformal division when possible, and showing its impossibility when impossible)—involving no ambiguity beyond that due to the double sign in X, Y, which is removed at each step before proceeding to the next, so that the ambiguity is not cumulative.

21. *Common Factors of F, f.*—The arithmetical operations become of course easier, when the numbers m, n, t, u, T, U to be operated on can be reduced in magnitude. When F, f contain common factors, these can sometimes be cancelled out without destroying the conformality of F, f, and also without violating the conditions of conformal division. This is especially the case when some two or more of m, n, t, u, T, U contain common factors; such factors are easily recognised. The chief cases are:

(1) Factors common to t, u, T, U may be cancelled.

(2) Factors common to m, n may be cancelled in Case i.

(3) m may be cancelled when common to u, U in Case i.; or when common to u, T in Case ii.; or when common to t, U in Case iii.

(4) n may be cancelled when common to t, T in Cases i., ii., iii.

(5) Factors common to T, U, but not contained in t, u may be removed from F temporarily, and the conformal division will take effect upon the numerator so reduced; the factors removed are to be restored to the quotient after the operation.

Note that the cancelling of m, n in (3), (4) changes F, f into their symmorphs.

Exception.—When f is a divisor of both T^2, U^2 (in which case f must be a square) the mere cancelling of f out of both T, U yields at once a conformal quotient, and this is in general the only mode of procedure: if, however, this quotient (say F') be still divisible by f, the conformal division may be performed on the quantity $F' \div f$; and the result when multiplied by f will be a new conformal quotient-form of $F \div f$.

22. *Case when mn is —.*—When m, n carry *opposite signs*, the form being then $f(m, -n)$, $f(1, -mn)$, &c., it may happen that the con-

* An example (in very high numbers) of the process conducted *direct* by Result (25) and another conducted as here described are given at end of Art. 41.

formal division indicated in Results (22*a*, *b*, *c*; 23*a*, *b*, *c*; 24*a*, *b*, *c*) will *fail*, *i.e.*, it may turn out that neither pair of the X, Y of formulæ 19*a*, 20*a*, 21*a* is arithmetically divisible by the denominator (f): this would show—provided that the precautions of Art. 18, 19, 20 have been attended to—that the forms of ϕ in question were *not possible forms.* In such a case the conformal division may still be possible if *either* the numerator (F), *or* denominator (f), can be converted into their *antimorphs*, say F' or f': by this conversion the following formulæ are interchanged (22*a*, 23*a*), (22*b*, 23*b*), (22*c*, 23*c*); (24*a*, 24*b* or *c*), and the quotients are of course the *antimorphs* of the original quotient-forms. The above conversion can always be done if either F or f possess an *antimorphic* unit-form F_0' or f_0' (by the conformal multiplication $F' = F.\,F_0'$, $f' = f.\,f_0'$, Art. 6), and also in other* cases not readily recognisable *a priori.*

Ex.:
$$N = 137 = \frac{2055}{15} = \frac{F}{f} = \frac{17\,.\,11^2-2\,.\,1^2}{17\,.\,1^2-2\,.\,1^2}.$$

Here $f = 15$ is a *monomorph*† of the form $(17t^2-2u^2)$, so that Result (22*a*) should be *directly* applicable; and yet the conformal division (22*a*) *fails* (for it will be found that neither pair of the X, Y of formula 19*a* is divisible by 15).

But, converting f into its *antimorph* $f = 15 = 2\,.\,4^2-17\,.\,1^2$, then

$$N = \frac{F}{f} = \frac{17\,.\,11^2-2\,.\,1^2}{2\,.\,4^2-17\,.\,1^2}; \text{ which falls under (23}a\text{), with (19}a\text{),}$$
$$= 34\left(\frac{11\,.\,4+1\,.\,1}{15}\right)^2 - \left(\frac{17\,.\,11\,.\,1+2\,.\,4\,.\,1}{15}\right)^2$$
$$= 34\,.\,3^2-13^2 \text{ (the result required).}$$

23. *Conformal Division, Uses of.*—This process is of great use in effecting *directly* the quadratic partition‡ $\phi(m, n)$ of a *large* number N when given—as often happens in higher arithmetic—in the form of a quotient $F \div f$, when the corresponding partition $F(m, n)$ of the numerator is also *given*: in most cases the resolution of the divisor f into factors, and the expression of these in quadratic forms conformal with F, would be comparatively easy.

[Examples, some very large, will be given later (Art. 41).]

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* *E.g.*, when $-mn$ = 34, 146, 194, 205, 221, 305, 377, 386, 410, 466, &c.

† Because its prime factors 3, 5 cannot be expressed in the form $(17t^2-2u^2)$.

‡ The problem is, in general, one of considerable labour, when the number N is large.

24. *Derivation.*—A method* will now be developed whereby, under certain conditions, from two given non-conformals of the same number N, a third form of that number, non-conformal with both the given forms, may be *derived:* the process will be called *Derivation*, the two given forms will be called *Base-forms*, and the new form will be called the *Derived form;* the three forms together will be called *Allied forms.*

Let N be a number expressed in the two non-conformal base-forms

$$N = \theta v^2 + mw^2 = \theta x^2 + ny^2, \quad (m \neq n). \tag{27}$$

Then N can always be reduced to the form

$$N = \theta \cdot \frac{F(m, -n)}{f(m, -n)}, \text{ (where } F, f \text{ are isomorphs).} \tag{28}$$

For
$$N = \frac{\theta v^2 \cdot ny^2 + mw^2 \cdot ny^2}{ny^2} = \frac{\theta x^2 \cdot mw^2 + ny^2 \cdot mw^2}{mw^2}$$

$$= \theta \cdot \frac{m(wx)^2 - n(yv)^2}{mw^2 - ny^2}, \text{ which is of required form. (28)}$$

25. *Conditions of Derivability.*—When certain relations exist between θ, m, n, the expression $(\theta . F \div f)$ just found for N may be (algebraically) reduced to a *new* quadratic form which will be found to be conformal with F, f, but non-conformal with both base-forms. These conditions take four principal forms, with two cases in each (corresponding to a $\pm$ sign of θ), viz.,

i. $\theta = 1$; ia. $\theta = -1$;

$\begin{cases} \text{ii. } \theta v_0^2 + \mu w_0^2 = 1, & \textit{and} \quad \theta x_0^2 + ny_0^2 = 1; \\ \text{iia. } \theta v_0^2 + \mu w_0^2 = -1, & \textit{and} \quad \theta x_0^2 + ny_0^2 = -1; \end{cases}$

iii. $\theta\sigma^2 = m\tau^2 - nv^2$; iiia. $\theta\sigma^2 = nv^2 - m\tau^2$;

iv. $\theta\sigma^2 = \tau^2 - mnv^2$; iva. $\theta\sigma^2 = mnv^2 - \tau^2$.

As the process involves a conformal division, the conditions of such division (Art. 18) must also be satisfied; the satisfaction of these conditions seems difficult to recognise *a priori*, but in any actual numerical case this will be sufficiently revealed—as explained in Art. 18–20—by the success or failure of the attempt at the con-

* This method—believed to be new—is the object of this paper; the methods of multiplication and division of a form f by any of its unit-forms, above developed, provides the means of finding the *automorphs* thereof, but not of finding *new* or *distinct* forms.

formal division itself. Moreover, as the result of conformal division is *unique* (Art. 15), only *one* new form can be "derived" from a given pair of base-forms.

26. *Quadratic Triads.*—In certain cases the *allied forms* (Art. 24) are such that each one of the three can be similarly *derived* from the other two considered as base-forms. Such "allied forms" are called a *Quadratic Triad.* It will be found that the allied forms of Cases i., ii., iii., i*a*., ii*a*., iii*a*. are always quadratic triads; but that those of Cases iv., iv*a*. are a quadratic triad only under a certain further condition.

27. *Derivation.*—Cases i., i*a*. ($\theta = 1$, or -1; $m \neq n$).

In i., $$N = v^2 + mw^2 = x^2 + ny^2. \qquad (29)$$

In i*a*., $$N = mw^2 - v^2 = ny^2 - x^2. \qquad (29a)$$

Here (28) reduces to

$$\text{i. } N = \frac{m(wx)^2 - n(yv)^2}{mw^2 - ny^2}; \quad \text{i}a.\ N = \frac{n(yv)^2 - m(wx)^2}{mw^2 - ny^2}, \qquad (30)$$

and, by* conformal division (when possible), these yield

either $$\text{i. } N = t^2 - mnu^2; \quad \text{i}a.\ N = mnu^2 - t^2; \qquad (31)$$

or (Art. 22), $$\text{i}'.\ N = mnu^2 - t^2; \quad \text{i}a'.\ N = t^2 - mnu^2. \qquad (31')$$

Thus, from any two single non-conformal base-forms (29, 29*a*) of the same number N, the third form (31) is *directly* derivable when it is a *possible form*; or the alternative form (31′) is *indirectly* derivable when the first form (31) is impossible (see Art. 22).

Further, the derived form (31), but not the alternative (31′), is identifiable with $(t^2 + lu^2)$ in Case i., by taking $l = -mn$; or with $(lu^2 - t^2)$ in Case i*a*., by taking $l = mn$. The symmetry of the three forms shows that, if N be actually expressible in the three simple non-conformals

$$\text{i. } N = t^2 + lu^2 = v^2 + mw^2 = x^2 + ny^2, \qquad (32)$$

or $$\text{i}a.\ N = lu^2 - t^2 = mw^2 - v^2 = ny^2 - x^2, \qquad (32a)$$

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* If m, n contain a common factor, say $m = \kappa\mu$, $n = \kappa\nu$; then the product mn may in all such formulæ be replaced (Art. 5, Note) by $\mu\nu$, the square (κ^2) being absorbed into u^2.

where l, m, n are connected* by *any one* of the relations

$$\text{i. } l = -mn, \quad m = -nl, \quad n = -lm, \tag{33}$$

$$\text{i}a.\ l = mn, \quad m = nl, \quad n = lm, \tag{33a}$$

then each one of the three forms is *directly derivable* from the other two (as base-forms), so that the three forms (32) or (32*a*) are a *quadratic triad*.

[But when the two base-forms (29) or (29*a*) of N lead to the *alternative* derived form i.′ or i*a*.′ of (31′)—(which can only happen when the forms i., i*a*. of (31) are impossible)—then the three forms are merely allied forms, but *not quadratic triads;* as neither of the forms (29), (29*a*) would be *derivable* from the other pair taken as base-forms.]

Ex.—One of the most useful examples of Case i. is when $n = -m$, giving—when $(x^2 - my^2)$ is a possible form of N—the *quadratic triad*

$$N = \mathrm{a}^2 + \mathrm{b}^2 = v^2 + mw^2 = x^2 - my^2. \tag{34}$$

This includes all primes of form $p = 4m\varpi + 1$,

when $m > 1$, but < 11 (and also many cases when $m > 11$).

When $(x^2 - my^2)$ is not a possible form of N, then the two forms

$$\mathrm{a}^2 + \mathrm{b}^2 = v^2 + mw^2 \tag{34a}$$

would lead *indirectly* (Art. 22) to $(my^2 - x^2)$ as derived form, provided m be of form $m = \alpha^2 + \beta^2$ (a necessary condition of the coexistence of the three forms), and they then form a quadratic triad; this form is discussed in the Example at end of Cases iii., iii*a*., Art. 30.

28. *Cases* ii., ii*a*.

$$N = \theta v^2 + mw^2 = \theta x^2 + ny^2 \quad (\theta \neq 1;\ m \neq n), \tag{35}$$

where each of the base-forms admits of a unit-form (both $+1$, or both -1), *i.e.*,

$$\textit{Case}\ \text{ii.}\quad \theta v_0^2 + mw_0^2 = 1, \quad \text{and} \quad \theta x_0^2 + ny_0^2 = 1, \tag{36}$$

$$\textit{Case}\ \text{ii}a.\quad \theta v_0^2 + mw_0^2 = -1, \quad \text{and} \quad \theta x_0^2 + ny_0^2 = -1 \tag{36a}$$

(both involving "m, n of same sign, opposite to that of θ").

By conformal multiplication of each base-form by its own unit-

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* See previous note: Square factors in mn, nl, lm may be cancelled from the values of l, m, n respectively, and absorbed into u^2, w^2, y^2 respectively.

form, each is reduced to a *simple* form (its own symmorph), so that ii., ii*a*. become

ii. $N = V^2 - (-\theta m)\, W^2 = X^2 - (-\theta n)\, Y^2$ $(-\theta m,\ -\theta n$ are both $+)$; (37)

ii*a*. $N = (-\theta m)\, W^2 - V^2 = (-\theta n)\, Y^2 - X^2$ $(-\theta m,\ -\theta n$ are both $+)$, (37*a*)

which now fall under Cases i., i*a*.; so that a third form, non-conformal with both base-forms (as below), is directly derivable in each case (when conformal division is possible); θ^2 being here absorbed under u^2,

either ii. $N = t^2 - mnu^2$; ii*a*. $N = mnu^2 - t^2$, (38)

or* (Art. 22) ii′. $N = mnu^2 - t^2$; ii*a*′. $N = t^2 - mnu^2$. (38′)

Hence, reasoning as in Cases i., i*a*. [noting that the base-forms are considered *equivalent* to their automorphs (Art. 10)], it follows that, if N be expressible in the three non-conformals below (subject to the conditions accompanying), then each set is a† *quadratic triad.*

ii. $N = t^2 + lu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (39)

where *one* of $l = -mn,\ m = -nl,\ n = -lm$, is satisfied, (40)

and *each* form admits of a unit-form $(= +1)$. (41)

ii*a*. $N = lu^2 - t^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (39*a*)

where *one* of $l = mn,\ m = nl,\ n = lm$, is satisfied, (40*a*)

and *each* form admits of a unit-form $(= -1)$. (40*a*)

[Note that the first form $(lu^2 \pm t^2)$ of each triad necessarily admits of the unit-form $\{= +1$ in (39), and $= -1$ in (39*a*)$\}$, when *one* of the conditions $l = \mp mn$, &c. is satisfied, since $mn,\ -\theta m,\ -\theta n$ are all $+$.]

29. *Cases* iii., iii*a*., iv., iv*a*. $(\theta \neq 1,\ m \neq n.)$

Here $N = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (41)

* As in Case i., conformal division leads *directly* to the forms (38) when these are *possible*, and only leads *indirectly* to the *alternative* forms (38′) when the former are impossible.

† The alternative forms (38′) are excluded from (39), (39*a*), as they would not form a *quadratic triad* with (35).

subject to *one* of the conditions

Case iii. $\theta\sigma^2 = m\tau^2 - nv^2$; iii*a*. $\theta\sigma^2 = nv^2 - m\tau^2$, (42, 42*a*)

Case iv. $\theta\sigma^2 = \tau^2 - mnv^2$; iv*a*. $\theta\sigma^2 = mnv^2 - \tau^2$. (43, 43*a*)

Now, in each case, the reduced form (28) of N may be written

$$N = (\theta\sigma^2)\frac{m\,(wx)^2 - n\,(yv)^2}{m\,(\sigma w)^2 - n\,(\sigma y)^2} = \Theta \cdot \frac{F\,(m,\,n)}{f\,(m,\,n)}, \tag{44}$$

wherein f, F, and $\Theta = \theta\sigma^2$ are *isomorphs* by virtue of the conditions (42, 42*a*, 43, 43*a*).

By conformal multiplication of Θ, F, the numerator can be in each case reduced to a conformal of f, F, viz.,

$$\text{iii. } \Theta \cdot F = (m\tau^2 - nv^2)\{m\,(wx)^2 - n\,(yv)^2\} = T^2 - mnU^2,$$

$$\text{iii}a.\ \Theta \cdot F = (nv^2 - m\tau^2)\{m\,(wx)^2 - n\,(yv)^2\} = mnU^2 - T^2,$$

$$\text{iv. } \Theta \cdot F = (\tau^2 - mnv^2)\{m\,(wx)^2 - n\,(yv)^2\} = mT^2 - nU^2,$$

$$\text{iv}a.\ \Theta \cdot F = (mnv^2 - \tau^2)\{m\,(wx)^2 - n\,(yv)^2\} = nU^2 - mT^2.$$

Next, by conformal division of the reduced ΘF by f (when the conditions of such division are satisfied), the quotient-form is obtained conformal with f, F; thus

$$\text{iii. } N = \frac{\Theta \cdot F}{f} = \frac{T^2 - mnU^2}{m\,(\sigma w)^2 - n\,(\sigma y)^2} = mt^2 - nu^2, \tag{45}$$

$$\text{or} \quad = nu^2 - mt^2 \text{ (Art. 22)}, \tag{45'}$$

$$\text{iii}a.\ N = \frac{\Theta \cdot F}{f} = \frac{mnU^2 - T^2}{m\,(\sigma w)^2 - n\,(\sigma y)^2} = nu^2 - mt^2, \tag{45a}$$

$$\text{or} \quad = mt^2 - nu^2 \text{ (Art. 22)}, \tag{45a'}$$

$$\text{iv. } N = \frac{\Theta \cdot F}{f} = \frac{mT^2 - nU^2}{m(\sigma w)^2 - n\,(\sigma y)^2} = t^2 - mnu^2, \tag{46}$$

$$\text{or} \quad = mnu^2 - t^2 \text{ (Art. 22)}, \tag{46a'}$$

$$\text{iv}a.\ N = \frac{\Theta \cdot F}{f} = \frac{nU^2 - mT^2}{m\,(\sigma w)^2 - n\,(\sigma y)^2} = mnu^2 - t^2, \tag{46a}$$

$$\text{or} \quad = t^2 - mnu^2 \text{ (Art. 22)}. \tag{46a'}$$

Thus, in each case, a third* form is derivable from the two base-forms, non-conformal with both.

* By Art. 22, the first form (45, 45*a*, 46, 46*a*) of the above pairs of *alternative* forms is the correct form, whenever it is a possible form; the second form (45′, 45*a*′, 46′, 46*a*′) being used only when the former is an impossible form.

30. *Cases* iii., iii*a.*; *Quadratic Triads.*—It will now be shown that the two forms (41) along with their derived form (45) or (45*a*) [not (45′) or (45′*a*)] are in each case a *quadratic triad.* Taking Case iii. first:

Case iii.—It has been shown that, when N is *actually expressible* in the three *allied forms*

$$\left.\begin{array}{ll} & N = mt^2 - nu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2 \quad (\theta \neq 1,\ m \neq n)\,; \\ \text{or} & N = f_1(m, -n) = f_2(\theta, m) = f_3(\theta, n), \end{array}\right\} \quad (47)$$

then $f_1(m, -n)$ is *directly derivable* from $f_2(\theta, m)$, $f_3(\theta, n)$ provided

$$\theta\sigma^2 = m\tau^2 - nv^2, \quad \text{or} \quad m\tau^2 = \theta\sigma^2 - (-n)\,\tau^2, \qquad (48)$$

and this latter is seen to be the necessary and sufficient condition that $f_3(\theta, n)$ should be *directly derivable* from $f_1(m, -n)$, $f_2(\theta, m)$.

The derivability of the form $f_2(\theta, m)$ from $f_1(m, -n)$, $f_3(\theta, n)$ is not, however, so obvious. The two forms

$$N = mt^2 - nu^2 = \theta x^2 + my^2,$$

combined as in Art. 24, give

$$N = \frac{\theta\,(ux)^2 + m\,(yt)^2}{u^2 + y^2} = \frac{F}{f} \text{ (suppose).}$$

Here $\qquad f = u^2 + y^2 = \dfrac{1}{n}\,(mt^2 - \theta x^2)$, by the data,

$$= \frac{v^2}{m\tau^2 - \theta\sigma^2}\,(mt^2 - \theta x^2), \text{ by (48)},$$

$$= t_1^2 - \theta m\,.\,u_1^2, \quad \text{(by conformal division)}$$

$$= t_2^2 + \theta m\,.\,u_2^2, \text{ (by Case i.)},$$

because the two forms $f = f(1, 1) = f(1, -\theta m)$ involve $f = f(1, \theta m)$, by Case i. Hence

$$N = \frac{\theta\,(ux)^2 + m\,(yt)^2}{t_2^2 + \theta m\,.\,u_2^2}$$

$$= \theta v^2 + mw^2 = f_2(\theta, m), \text{ (by conformal division).}$$

Thus the three forms (47) of Case iii., with the relation (48), are a quadratic triad.

Case iii*a.*—Interchanging m, n of Case iii., a similar procedure shows that, when N is actually expressible in the three allied forms

$$N = nu^2 - mt^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2, \tag{47a}$$

subject to the condition $\theta\sigma^2 = nv^2 - m\tau^2$, (48a)

then the three forms (47*a*) are a *quadratic triad.*

[*Note.*—The preliminary hypothesis that N is *actually expressible* in the three forms (47, 47*a*) involves the *possibility* of all the conformal divisions required above, and also excludes the chance of any of the conformal quotients taking the *antimorphic* form of Art. 22.]

Ex.—One of the most useful examples of above is when $m = \pm 1 = -n$. Cases iii., iii*a.* coalesce, so that, when N is *actually expressible* in the three forms

$$N = \mathrm{a}^2 + \mathrm{b}^2 = \theta v^2 + w^2 = \theta x^2 - y^2, \tag{49}$$

the coexistence of which involves

$$\theta = \tau^2 + v^2, \tag{49a}$$

then the three forms (49) are *a quadratic triad.* [Compare (34*a*), Art. 26.]

Ex. 2.—When $m+n = 0$, the systems (47), (47*a*) are each divisible throughout by m (because the first form is): this case is then reducible to the last (49, 49*a*).

31. *Cases* iv., iv*a. may be quadratic triads.*—Let N be a number *actually expressible* in the three allied forms

Case iv. $N = t^2 - mnu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (50)

wherein $\theta\sigma^2 = \tau^2 - mnv^2$, (51)

or *Case* iv*a.* $N = mnu^2 - t^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (50*a*)

wherein $\theta\sigma^2 = mnv^2 - \tau^2$; (51*a*)

then it has been shown (Art. 29) that in each case the first form $\{\pm(t^2 - mnu^2)\}$ is *directly derivable* from the other two, as base-forms.

Now, if the first form in each case admits *either* of the symmorphic unit-forms

$$m\tau_0^2 - nv_0^2 = 1, \quad \text{or} \quad nv_0^2 - m\tau_0^2 = 1, \tag{52}$$

then Cases iv., iv*a.*, reduce to Cases iii., iii*a.* respectively. For N and $\theta\sigma^2$ are thereby changed as follows :—

iv. $m\tau_0^2 - nv_0^2 = 1$ changes $N = t^2 - mnu^2$ to $(mt'^2 - nu'^2)$,

and $\theta\sigma^2 = \tau^2 - mnv^2$ to $(m\tau'^2 - nv'^2)$;

iv*a.* $m\tau_0^2 - nv_0^2 = 1$ changes $N = mnu^2 - t^2$ to $(nu'^2 - mt'^2)$,

and $\theta\sigma^2 = mnv^2 - \tau^2$ to $(nv'^2 - m\tau'^2)$;

iv. $nv_0^2 - mr_0^2 = 1$ changes $N = t^2 - mnu^2$ to $(nu'^2 - mt'^2)$,

and $\theta\sigma^2 = r^2 - mnv^2$ to $(nv'^2 - mr'^2)$;

iv*a*. $nv_0^2 - mr_0^2 = 1$ changes $N = mnu^2 - t^2$ to $(mt'^2 - nu'^2)$,

and $\theta\sigma^2 = mnv^2 - r^2$ to $(mr'^2 - nv'^2)$.

Thus $mr_0^2 - nv_0^2 = 1$ changes Case iv. into iii., and iv*a*. into iii*a*.,

$nv_0^2 - mr_0^2 = 1$ changes Case iv. into iii*a*., and iv*a*. into iii.

Hence, with the added condition $(mr_0^2 - nv_0^2) = \pm 1$, the three allied forms (50, 50*a*) of Cases iv., iv*a*. are *quadratic triads*, but *not otherwise*.

Ex.—An interesting example of Case iv. is when $\sigma^2 = 1$, $n = -m$, giving

$$N = t^2 + (mu)^2 = \theta v^2 + mw^2 = \theta x^2 - my^2; \quad \text{and } \theta = r^2 + (mv)^2. \tag{53}$$

But these allied forms do not form a quadratic triad, except when $m = \pm 1$, as this is the only way in which the condition $mr_0^2 - nv_0^2 = \pm 1$ can be satisfied (viz., by $m = -n = \pm 1$, $r_0 = 1$, $v_0 = 0$), giving the quadratic triad (49) of Case iii.

32. *Conditions of Derivability.*—The four Cases (i., ii., iii., iv.) investigated are not the only cases in which a new form is derivable from two given base-forms of the same number (N). For, writing the reduced expression (28) to which these lead,

$$N = \theta \cdot \frac{m(wx)^2 - n(yv)^2}{mw^2 - ny^2} = \theta \frac{F}{f}, \tag{54}$$

then—since F, f are conformal—it follows that, if F contain f as a factor, and if the conditions of conformal division are also satisfied, the quotient $F \div f$ is itself expressible as a conformal, say F', of F, f; excluding now the case of θ being conformal with F', and therefore with F, f, as being really a particular case of Case iii. (making $\sigma = 1$), there remains the case of the product (N) of the two non-conformals (θ, F') being conformally expressible with F' (*i.e.*, with F, f); this case is *sometimes* (arithmetically) possible, (Art. 6); but it seems difficult to formulate it algebraically, and it is doubtful whether the derived form in question would, along with the two base-forms, yield a quadratic triad.

33. *Practical Procedure.*—The procedure in Cases iii., iii*a*., iv., iv*a*. merits further examination. After reducing N to the form $\Theta f \div f$, it consists of two steps:—I., conformal multiplication of Θ, F'; II., conformal division of $(\Theta . F)$ by f. This procedure is general, and certain of success when θ has the requisite form, and when the conformal division required is possible.

For practical computation, however, it is often much more convenient to invert the order of the above steps, performing the conformal divisions first (whenever possible), and following with the conformal multiplication, as by this procedure much lower numbers have to be dealt with than by the previous order. This may be done in the following cases (whenever the conformal divisions are possible):—

(1) $f = \Theta f'$, (2) $F = fF'$, (3) $f = f_1 f_2$, $\Theta = \Theta' f_1$, $F = f_2 F'$. (55)

The order and nature of the operations are then as follows:—

Step.	Conditions.		
	(1) $f = \Theta f'$	(2) $F = fF'$	(3) $f = f_1 f_2$, $\Theta = \Theta' f_1$, $F = f_2 F'$
I. Conform. Divn. ...	$f \div \Theta = f'$	$F \div f = F'$	$\Theta \div f_1 = \Theta'$, $F \div f_2 = F'$
II. Conform. Multn....	... ...	$N = \Theta F'$	$N = \Theta' F$
II. Conform. Divn. ...	$N = F \div f'$	... ...	... ...

The simplest case is, of course, when $\Theta = f$, for then N obviously reduces to

$$N = F = m(wx)^2 - n(yv)^2.$$

Note.—The condition $f = \Theta f'$ of Case (1) above is *always satisfied*, for the data [(27) Art. 24] give

$$mw^2 - ny^2 = \theta(x^2 - v^2),$$

whence

$$f = \Theta(x^2 - v^2);$$

but, as the conformal division may in this case be impossible, it is often convenient to use the other cases.

The denominator f being thus always composite (often highly composite), the procedure of Art. 20, viz., dissecting f into its monomorphic factors, should be adopted; and, to shorten the numerical work, it is important to strike out the obvious common factors of ΘF, f as explained in Art. 21: one case is especially simple, viz., when either of the pairs (m, n), (w, y) have common factors; thus, if

$$\frac{m}{n} = \frac{\mu}{\nu}; \quad \frac{w}{y} = \frac{\omega}{\eta}, \tag{56}$$

then the expression (28) for N is at once reduced to

$$N = \theta \cdot \frac{\mu(\omega x)^2 - \nu(\eta v)^2}{\mu\omega^2 - \nu\eta^2}, \tag{57}$$

which has the practical advantage of containing lower numbers than (28).

34. *Repeated Derivation.*—The process of derivation, above applied to *two* non-conformals, may be extended to *three* or more non-conformals, in Cases i., i*a*., ii., ii*a*., say,

$$N = \theta x_1^2 + l_1 y_1^2 = \theta x_2^2 + l_2 y_2^2 = \theta x_3^2 + l_3 y_3^2 = \&c., \tag{58}$$

wherein θ is the same throughout, and $l_1 \neq l_2 \neq l_3 \neq \&c.$, (58*a*)

when the conformal divisions requisite are all possible. It will suffice to show this for *three* forms of Case i.

Case i. ($\theta = 1$).—Let N be expressed in the three non-conformals

$$N = x_1^2 + l_1 y_1^2 = x_2^2 + l_2 y_2^2 = x_3^2 + l_3 y_3^2. \tag{59}$$

By combining the second and third, the third and first, and the first and second forms in succession, the three new derived forms are obtained, when the conformal divisions are possible (excluding the alternative case of Art. 22),

$$N = x_{23}^2 - l_2 l_3 y_{23}^2 = x_{31}^2 - l_3 l_1 y_{31}^2 = x_{12}^2 - l_1 l_2 y_{12}^2, \tag{59a}$$

and, by combining any one of these with the non-corresponding original form, a new derived form results,

$$N = x_{123}^2 - l_1 l_2 l_3 y_{123}^2, \quad \text{or} \quad l_1 l_2 l_3 y_{123}^2 - x_{123}^2. \tag{59b}$$

Case i*a*. ($\theta = -1$).—This follows by merely slight alterations.

Cases ii., ii*a*.—Also follow by reduction to Cases i., i*a*., as in Art. 28.

Cases iii., iii*a*., iv., iv*a*.—Do not appear to be susceptible of this extension, without some additional conditions.

35. *Generalization.*—The process of derivation may be extended* to the more general quadratic forms

$$N = kv^2 + avw + bw^2 = kx^2 + cxy + dy^2, \tag{60}$$

wherein *one* of the coefficients (k) is common to both forms. Here

$$4kN = (2kv + aw)^2 + (4kb - a^2)\,w^2 = (2kx + cy)^2 + (4kd - c^2)\,y^2$$
$$= v'^2 + mw^2 = x'^2 + ny^2, \tag{60a}$$

by obvious substitutions. Hence, provided $m \neq n$ (so that these

* This extension is due to Mr. Bickmore.

forms may be non-conformal), these yield (when the conformal division required is possible) the new form of $4kN$,

$$4kN = t^2 - mn \,.\, u^2, \quad \text{or} \quad mn \,.\, u^2 - t^2, \qquad (60b)$$

and, when $4k$ is conformal with $f(1, -mn)$, N itself is expressible [by conformal division of (60*b*) by $4k$] in the three forms (60*a*, *b*), which then form a quadratic triad.

36. *Numerical Examples.*—The power of the process of derivation is best seen with *very large* numbers; as with small numbers the partitions can often be more easily effected by direct trial. But it is well to begin with small numbers.

37. *Ex.*—The small prime $p = 241$ affords, from its numerous partitions (shown below), many easy examples; the antimorphs of the second and third lines are connected by the (negative) unit form shown in the fifth line

$$\begin{array}{rlllll}
p = 15^2+4^2 = & 13^2+2.6^2 = & 7^2+3.8^2 = & 5^2+6.6^2 = & 14^2+5.3^2 = & 9^2+10.4^2 \\
= & 21^2-2.10^2 = & 17^2-3.4^2 = & 25^2-6.8^2 = & 31^2-5.12^2 = & 41^2-10.12^2 \\
= & 2.11^2-1^2 = & \dots\ \dots = & \dots\ \dots = & 5.7^2-2^2 = & 10.5^2-3^2 \\
= & \dots\ \dots = & \dots\ \dots = & 3.9^2-2.1^2 = & \dots\ \dots = & \dots\ \dots \\
\left.\begin{array}{l}\text{Unit-}\\ \text{form}\end{array}\right\}, -1 = & 1^2-2.1^2 = & \dots\ \dots = & \dots\ \dots = & 2^2-5.1^2 = & 3^2-10.1^2 \\
 & & & = & 1^2+15.4^2 = & 11^2+30.2^2 \\
 & & & = & 16^2-15.1^2 = & 19^2-30.2^2 \\
 & & & = & \dots\ \dots = & \dots\ \dots \\
 & & & = & \dots\ \dots = & 6.9^2-5.7^2.
\end{array}$$

Thus there are seven quadratic triads of the *simplest* form (34) of Case i.,

$$p = a^2 + b^2 = v^2 + mw^2 = x^2 - my^2,$$

given by $m = -n = 2, 3, 6, 5, 10, 15, 30$.

Of Case i. ($\theta = 1$), and also of Case i*a*. ($\theta = -1$), there are twenty-eight quadratic triads (Art. 27) given by the four interchanges of sign ($\pm m$, $\pm n$) in the scheme

$$\begin{array}{rl}
\pm m = & 2,\ \ 2,\ \ 2,\ \ 3,\ \ 3,\ \ 6,\ \ 6, \\
\pm n = & 3,\ \ 5,\ 15,\ \ 5,\ 10,\ \ 5,\ 10, \\
l = \pm mn \text{ or } \pm\mu\nu = & 6,\ 10,\ 30,\ 15,\ 30,\ 30,\ 15.
\end{array}$$

Of Case ii. there are two quadratic triads given by

$$(\theta, m, n) = (-2, 1, 3) \quad \text{and} \quad (-5, 1, 6).$$

Of Cases iii., iv. there are four quadratic triads, given by

$$(\theta, m, n) = (3, 1, -2),\ (-2, 1, 3),\ (6, 1, -5),\ (-5, 1, 6),$$

these all falling under *both* Cases iii., iv., because $m = 1$ in each instance.

There are no instances of Cases ii*a*., iii*a*., iv*a*. in the above scheme (the conditions not being satisfied).

38. *Case* i.—Any two non-conformal simple forms of the small prime $p = 241$ will serve to prove the actual *working* of the process, *e.g.*,

$$p = 5^2+6 \cdot 6^2 = 14^2+5 \cdot 3^2;$$

here $\theta = 1$, $m = 6$, $n = 5$. Then, by (30),

$$p = \frac{6\,(6 \cdot 14)^2-5\,(5 \cdot 3)^2}{6 \cdot 6^2-5 \cdot 3^2} = \frac{F}{f}.$$

Here the factor 9 may be cancelled out of both F, f; leaving the reduced divisor $f \div 9 = (6 \cdot 2^2-5 \cdot 1^2) = 19$, a prime, and therefore a *monomorph* (see Art. 11, 21).

Hence $$p = \frac{6 \cdot 28^2-5 \cdot 5^2}{6 \cdot 2^2-5 \cdot 1^2};$$

whence, by conformal division,

$$p = \left(\frac{6 \cdot 2 \cdot 28+5 \cdot 5}{19}\right)^2-30\left(\frac{28+10}{19}\right)^2$$
$$= 19^2-30 \cdot 2^2,$$

the required *derived* form.

[To illustrate the *working* of the cases when $\theta \neq \pm 1$, it is better to take larger numbers, giving $m \neq \pm 1$, $n \neq \pm 1$.]

39. *Case* ii.—Given $N = 42{,}641$ in the three allied forms

$$N = 2 \cdot 542^2-7 \cdot 279^2 = 11 \cdot 277^2-2 \cdot 633^2 = 11 \cdot 384^2-7 \cdot 475^2,$$

with the corresponding unit-forms (determining the derivability)

$$1 = 2 \cdot 2^2-7 \cdot 1^2 = 11 \cdot 3^2-2 \cdot 7^2 = 11 \cdot 4^2-7 \cdot 5^2.$$

By conformal multiplication of each form of N by its own unit-form, there follow

$$N = 215^2-14 \cdot 16^2 = 279^2-22 \cdot 40^2 = 271^2-77 \cdot 20^2,$$

which now fall under Case i. Thus either set by itself, or any one form of either set, together with the non-conformal pair of the other set, is a *quadratic triad*.

40. *Cases* iii., iii*a*.—Take $N = 3317$.

$$\left.\begin{aligned} N &= 5 \cdot 91^2-2 \cdot 138^2 \\ &= 2 \cdot 41^2-5 \cdot 3^2 \end{aligned}\right\} = 13 \cdot 12^2+5 \cdot 17^2 = 13 \cdot 15^2+2 \cdot 14^2 \left\{\begin{aligned} &\text{Case iii.} \\ &\text{Case iii}a. \end{aligned}\right.$$

Here $\theta = 13$, $m = 5$, $n = 2$; and taking $\sigma = 1$, the condition of derivability of both Cases iii., iii*a*. is fulfilled, viz.,

$$\theta\sigma^2 = 13 \cdot 1^2 = 5 \cdot 3^2-2 \cdot 4^2 = 2 \cdot 3^2-5 \cdot 1^2,$$

the two cases being connected by the unit-forms

$$19^2-10 \cdot 6^2 = 1 = 10 \cdot 1^2-3^2,$$

by which the two leading forms are seen to be antimorphs, and the conditions themselves (48), (48*a*) are also antimorphic. Hence the two sets of "allied forms" are *quadratic triads*.

To show the *working* of the process of derivation, let N be given in the second and third forms, to derive the first form

$$N = 13 \cdot 12^2+5 \cdot 17^2 = 13 \cdot 15^2+2 \cdot 14^2,$$

giving $\theta = 13, \quad m = 5, \quad n = 2,$
with the condition $\theta\sigma^2 = 13 . 1^2 = 5 . 3^2 - 2 . 4^2.$

Here, by (28), $N = 13 . \frac{5(17 . 15)^2 - 2(12 . 14)^2}{5 . 17^2 - 2 . 14^2} = \theta . \frac{F}{f}.$

Here $f = 1053 = 13 . 81,$

so that θ divides out of f; whilst 9, being a factor of f and of *both* members of F, may be cancelled out of both F, f: the reduced denominator

$$f \div 13 . 9 = 9 = (7^2 - 10 . 2^2),$$

which is a *monomorph* (symmorphic with the reduced numerator, $F \div 9$). This reduces N to

$$N = \frac{5 . 85^2 - 2 . 56^2}{7^2 - 10 . 2^2};$$

whence, by conformal division,

$$N = 5\left(\frac{7 . 85 + 2 . 2 . 56}{9}\right)^2 - 2\left(\frac{5 . 2 . 85 + 7 . 56}{9}\right)^2$$
$$= 5 . 91^2 - 2 . 138^2,$$

as the *derived* form.

41. *Partition of Large Primes.*—A frequently recurring necessity in the theory of residues is the partition of a *large* prime (p) into the simple form $p = t^2 + lu^2$, where l is some (given) *small* number ($l = +1, \pm 2, \pm 3$, &c.). Unless one of t, u be a *small* number (in which case the partition—by direct trial—is easy), this is often a laborious work. The properties of quadratic triads afford considerable help in this: for, if *any* two (non-conformal) partitions

$$p = v^2 + mw^2 = x^2 + ny^2 \qquad (61)$$

can be discovered, forming a *quadratic triad* with the required form ($t^2 + lu^2$), the latter is at once* *derivable.*

Base-Forms.—Suitable base-forms (61) may often be discovered by the following tentative process. Subtract from the given p a number of† squares (say v^2, x^2) each $< p$; also subtract the given p from a number of† squares (say v^2, x^2) each $> p$. Resolve‡ the differences ($p \sim v^2$), ($p \sim x^2$) into their factors: some of these differences

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* It is assumed that the required form ($t^2 + lu^2$) is a *possible* form of p (this would be known *a priori*): this being so appears to involve the possibility of the conformal division required (when conducted properly; see Art. 20).

† A *large* Table of squares is required for this part of the process: the extent of the Table available is a *practical* limit to the utility of the process. Barlow's Tables extend only to $10{,}000^2$: Ludolf's and Kulik's Tables both extend to $100{,}000^2$, but are both out of print, and are rare books now.

‡ The trial squares (v^2, x^2) should be so chosen that these differences should be resolvable into factors *at sight* from a factor-table (otherwise the labour of factorization is considerable). Barlow's *New Mathematical Tables* (1814) are very suitable, as they give the factors of *all* numbers < 10,000: they are unfortunately out of print.

will be found to contain square factors, *i.e.*, will be of form mw^2, ny^2; these will give a *number* of partitions (61) with various values of m, n. If *any* pair (m, n) of these are such that *one* of the conditions

$$l = -mn, \quad m = -nl, \quad n = -lm \tag{62}$$

is satisfied, then that pair will be suitable base-forms.

The advantage of this is that in that part of it which is *tentative* any two forms (61) which will satisfy *any one* of the conditions (62) will suffice; so that there is a good chance of a suitable pair being found with less labour than would be requisite for *directly* effecting the partition $(t^2 + lu^2)$. It sometimes happens that one or more forms of a given prime are determinable *algebraically*, and that in the course of the above tentative work a new form may be found suitable for use with it as a base-form.

Ex.—Let $p = 10{,}567{,}201$ (the large factor of $2^{75}-1$). Find the partitions

$$p = \mathrm{a}^2 + \mathrm{b}^2 = \mathrm{c}^2 + 2\mathrm{d}^2.$$

As the last two figures (01) of p are the ending of the squares of four classes of numbers only, viz., of such as end in 01, 49, 51, 99, the trial values of v, x are conveniently assumed of these forms, because the differences $(p \sim v^2)$, $(p \sim x^2)$ will then all end in 00, and will be therefore more easily factorized. Starting from the critical value $\sqrt{p} = 4516$ nearly, and trying all possible values of v, x of above forms, both $< \sqrt{p}$ and $> \sqrt{p}$, forming the differences $(p \sim v^2)$, $(p \sim x^2)$, and factorizing these differences, there result (after a good many trials) the two suitable forms (for the a, b partition)

$$p = 2351^2 + 14 \,.\, 600^2 =^* 76{,}399^2 - 14 \,.\, 20400^2,$$

which fall under (34), Case i. Hence, by (28),

$$p = \frac{(76{,}399 \,.\, 600)^2 + (2351 \,.\, 20400)^2}{600^2 + 20400^2} = \frac{76{,}399^2 + (2351 \,.\, 34)^2}{1^2 + 34^2} = \frac{F}{f}$$

$$= \left(\frac{76{,}399 \,.\, 1 + 2351 \,.\, 34 \,.\, 34}{1157}\right)^2 + \left(\frac{76399 \,.\, 34 - 2351 \,.\, 34}{1157}\right)^2, \text{ (by 19)}$$

$$= 2415^2 + (34 \,.\, 64)^2 = \mathrm{a}^2 + \mathrm{b}^2 \text{ (as required).}$$

[*Note.*—The divisor $f = 1157 = 13 \,.\, 89 = 1^2 + 34^2 = 31^2 + 14^2$ is *not a monomorph*. The success of the conformal division in this instance is due to the fact that the forms of F, f here used chanced to be *corresponding* forms (Art. 19): the work has been here done with the non-monomorphic divisor only to show that it may sometimes succeed: the *safer* plan would be to proceed as in Art. 20.]

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* A much smaller solution of this form $(x^2 - 14y^2)$ can be found from this one by conformal multiplication by the unit-form $15^2 - 14 \,.\, 4^2 = 1$; thus

$$p = (76{,}399^2 - 14 \,.\, 20{,}400^2)(15^2 - 14 \,.\, 4^2),$$

which, on reduction, gives $p = 3585^2 - 14 \,.\, 404^2$. This might have been used as the $(x^2 - 14y^2)$ form above. [This was pointed out by Mr. Bickmore.]

Next, to effect the partition $p = c^2+2d^2$. In the course of the tentative work above, the form

$$p = 3251^2 - 2\,.\,30^2$$

was obtained (at the start): this, and the now known form (a^2+b^2), are suitable base-forms (since a^2+b^2, c^2+2d^2, e^2-2f^2 are a quadratic triad), so that

$$p = 2415^2+2176^2 = 3251^2-2\,.\,30^2.$$

This falls under Case i. Hence, by (28),

$$p = \frac{(2176\,.\,3251)^2+2\,(2415\,.\,30)^2}{2176^2+2\,.\,30^2} = \frac{(2415\,.\,15)^2+2\,(544\,.\,3251)^2}{15^2+2\,.\,544^2} = \frac{F}{f}$$

(the factor 8 having been cancelled out of F, f).

Here $f = 592{,}097 = 11\,.\,19\,.\,2833 = (3^2+2\,.\,1^2)(1^2+2\,.\,3^2)(41^2+2\,.\,24^2)$,

so that the *safe* procedure is to divide conformally by each factor separately (writing $f_1 = 11$, $f_2 = 19$, $f_3 = 2833$), as in Art. 20. Omitting details, the steps are

$$F_1 = \frac{F}{f_1} = \frac{(2415\,.\,15)^2+2\,(544\,.\,3251)^2}{11 = (3^2+2\,.\,1^2)} = 331{,}433^2+2\,.\,479{,}037^2,$$

$$F_2 = \frac{F_1}{f_2} = \frac{331{,}433^2+2\,.\,479{,}037^2}{19 = (1^2+2\,.\,3^2)} = 133{,}831^2+2\,.\,77{,}544^2,$$

$$F_3 = \frac{F_2}{f_3} = \frac{133{,}831^2+2\,.\,77{,}544^2}{833 = (41^2+2\,.\,24^2)} = 623^2+2\,.\,2256^2 = p,$$

the required partition.

On a Series of Cotrinodal Quartics. *By* H. M. Taylor, M.A.

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1. Many well-known theorems in the geometry of the triangle relate to straight lines drawn at right angles to the sides of the triangle.

In attempting to generalize these theorems we are led to consider in what cases and in what circumstances it is possible by orthogonal projection to project a triangle and a given point in its plane into a triangle and its orthocentre.

2. A well-known theorem given by Lazare Carnot states that, if the angular points A, B, C of a triangle be joined to any two points O, O' in its plane by straight lines which cut the opposite sides of the triangle in the points D, D', E, E', F, F', then these six points lie on a conic. From the well-known properties of the nine-point circle it