



On a formula for the transformation of mortality tables

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On a formula for the transformation of mortality tables.

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1. Given a mortality table and its corresponding financial tables, calculated at a certain rate of interest.

The principal elements of the table are

$$d_x, l_x, \Sigma l_x \text{ etc.}$$

Its principal values are

$$q_x, p_x, e_x = \frac{\Sigma l_{x+1}}{l_x}, f_x = \frac{\Sigma^2 l_{x+1}}{l_x}.$$

The financial elements of the table are

$$C_x = d_x v^{x+1}, M_x = \Sigma C_x, R_x = \Sigma^2 C_x \text{ etc.}$$

$$D_x = l_x v^x, N_x = \Sigma D_x, S_x = \Sigma^2 D_x \text{ etc.}$$

Its financial values are

$$a_x = \frac{N_x}{D_x}, a_x = \frac{N_{x+1}}{D_x}, a_{x:n} = \frac{N_x - N_{x+n}}{D_x} \text{ etc.}$$

We propose to construct a new mortality table in such a manner that its principal and financial elements can be calculated directly from the elements of the given table.

In the simplest manner this may be obtained by a linear transformation of the elements of the given table, and we then get two transformations, according as we start with the principal elements (transformation I), or with the financial elements (transformation II).

2. *Transformation I.* We put

$$(1) \quad \begin{aligned} l'_x &= (\alpha + \beta x) l_x + \gamma \Sigma l_{x+1} \\ &= (\alpha + \beta x + \gamma e_x) l_x, \end{aligned}$$

where α, β, γ are constants.

Hence

$$d'_x = (\alpha + \beta x) d_x + (\gamma - \beta) l_{x+1}$$

and

$$p'_x = \frac{(\alpha + \beta x + \beta + \gamma e_{x+1}) p_x}{\alpha + \beta x + \gamma e_x}$$

or

$$(2) \quad \frac{p'_x - p_x}{p_x} = \frac{\beta - \gamma(e_x - e_{x+1})}{\alpha + \beta x + \gamma e_x}.$$

For the financial elements we have

$$\begin{aligned} D'_x &= v^x l'_x = (\alpha + \beta x) D_x + \gamma v^x \Sigma l_{x+1} \\ &= (\alpha + \beta x + \gamma e_x) D_x \\ C'_x &= v^{x+1} d'_x = (\alpha + \beta x) C_x + (\gamma - \beta) D_{x+1}, \end{aligned}$$

whence by summation

$$\begin{aligned} N'_x &= (\alpha + \beta x) N_x + \beta S_{x+1} + \gamma \Sigma [v^x \Sigma l_{x+1}] \\ &= (\alpha + \beta x) N_x + \beta S_{x+1} + \gamma \frac{1+i}{i} (e_x D_x - N_{x+1}) \\ M'_x &= (\alpha + \beta x) M_x + \beta R_{x+1} + (\gamma - \beta) N_{x+1}. \end{aligned}$$

We may also find N'_x from the formula

$$N'_x = \frac{1+i}{i} (D'_x - M'_x).$$

3. Transformation II. Putting

$$(3) \quad \begin{aligned} D'_x &= (\alpha + \beta x) D_x + \gamma N_{x+1} \\ &= (\alpha + \beta x + \gamma a_x) D_x \end{aligned}$$

we find

$$\begin{aligned} C'_x &= v D'_x - D'_{x+1} = \\ &(\alpha + \beta x) C_x - \beta D_{x+1} + \gamma M_{x+1}; \end{aligned}$$

hence by summation

$$\begin{aligned} N'_x &= (\alpha + \beta x) N_x + (\beta + \gamma) S_{x+1} \\ M'_x &= (\alpha + \beta x) M_x - \beta N_{x+1} + (\beta + \gamma) R_{x+1}; \end{aligned}$$

further

$$p'_x = \frac{1}{v} \frac{D'_{x+1}}{D'_x} = \frac{(\alpha + \beta x + \beta + \gamma a_{x+1}) p_x}{(\alpha + \beta x + \gamma a_x)}$$

or

$$(4) \quad \frac{p'_x - p_x}{p_x} = \frac{\beta - \gamma(a_x - a_{x+1})}{\alpha + \beta x + \gamma a_x}.$$

4. The transformations contain three constants (α, β, γ) , two of which are effective (independent). The transformed mortality curve (q'_x) can then generally be made to pass through given points for two given ages. Denoting these ages by x and y , and putting

$$\begin{aligned} p'_x &= h p_x \\ p'_y &= k p_y \end{aligned}$$

we have for the determination of the constants

$$(5) \quad \begin{aligned} \alpha(1-h) + \beta(1+x-xh) - \gamma(q_x h - q_{x+1}) &= 0 \\ \alpha(1-k) + \beta(1+y-yk) - \gamma(q_y k - q_{y+1}) &= 0, \end{aligned}$$

where $q_x = e_x$ or a_x , according to the transformation I or II.

One of the constants (if not $= 0$) may have an arbitrary value. Generally it will be convenient to put $\gamma = \pm 1$, or $\pm 0,1$ etc.

From (2) or (4) it is seen that

$$\alpha + \beta x + \gamma q_x = 0, \quad q_x = e_x \text{ or } a_x$$

generally makes

$$p'_x = \infty,$$

and the transformation, therefore, must be limited to an age interval within which the expression $\alpha + \beta x + \gamma q_x$ does not alter its sign. On the whole it must be provided that p'_x satisfies the conditions

$$0 < p'_x < 1.$$

5. From (2) or (4) it is seen that

$$p'_z = p_z \text{ or } q'_z = q_z$$

at an age z , determined by

$$(6) \quad q_z - q_{z+1} = \beta : \gamma, \quad q_z = e_z \text{ or } a_z.$$

As $q_z - q_{z+1}$ may assume the same value for different values of z , the transformed mortality curve (q'_x) may intersect the given curve (q_x) in several points. At the age ω , where

$$p_\omega = 0 \text{ or } q_\omega = 1$$

intersection always takes place; in fact we have

$$p'_\omega = p_\omega = 0 \text{ or } q'_\omega = q_\omega = 1$$

for all allowable values of the constants.

6. If we have simultaneously

$$\alpha + \beta x + \gamma q_x = 0$$

and

$$\beta - \gamma(q_x - q_{x+1}) = 0,$$

then we get

$$\alpha + \beta x + \beta + \gamma q_{x+1} = 0,$$

whence

$$\frac{p'_x}{p_x} = \frac{0}{0}$$

and further

$$p'_{x-1} = \frac{(\alpha + \beta x + \gamma q_x) p_{x-1}}{\alpha + \beta x - \beta + \gamma q_{x-1}} = 0$$

$$l'_x = (\alpha + \beta x + \gamma q_x) l_x = 0,$$

i. e. for the ages below x the transformed mortality table runs out at the age x . For higher ages the transformed table (according to article 5) runs out at the same age as the given table does; however, at the ages next above x the p' generally will be > 1 .

7. If the transformed mortality curve (q'_x) intersects the given curve (q_x) at an age z , situated between x and $x + n$, we have from (6) and (2) or (4), writing for brevity $q_x - q_{x+1} = \Delta q_x$, $q_x = e_x$ or a_x ,

$$(7) \quad \frac{p'_{x+n} - p_{x+n}}{p'_x - p_x} = \frac{p_{x+n}}{p_x} \cdot \frac{\Delta q_z - \Delta q_{x+n}}{\Delta q_z - \Delta q_x} \cdot \frac{\alpha + \beta x + \gamma q_x}{\alpha + \beta x + \beta n + \gamma q_{x+n}};$$

$\alpha = \infty$ gives the identical transformation, and high values of α then give small variations of p ; for such values of α the last factor of the right-hand member is $= 1$ approximately. Within a limited age interval Δq may be considered, with a rough approximation, as a linear function of x , and we then have approximately (k being a factor of proportionality)

$$\Delta q_z - \Delta q_{x+n} = k(z - x - n)$$

$$\Delta q_z - \Delta q_x = k(z - x),$$

whence

$$\frac{p'_{x+n} - p_{x+n}}{p'_x - p_x} = \frac{p_{x+n}}{p_x} \cdot \frac{z - x - n}{z - x},$$

i. e. within a limited age interval the transformation may be considered as a kind of rotation of the curve q about a centre z , the variations $(p'_x - p_x)$ at the age x being approximately proportionate to the distance from z to x . The factor $\frac{p_{x+n}}{p_x}$ does not affect the result materially, not being very different from 1 within the age intervals which may be here considered.

8. *To find the transformations I under which the expectation of life at the age x is invariant ($e'_x = e_x$).*

We have

$$\begin{aligned} l'_x &= (\alpha + \beta x) l_x + \gamma \Sigma l_{x+1} \\ \Sigma l'_{x+1} &= (\alpha + \beta x + \beta) \Sigma l_{x+1} + (\beta + \gamma) \Sigma^2 l_{x+2}; \end{aligned}$$

putting

$$\frac{\Sigma^2 l_{x+2}}{l_x} = \frac{\Sigma^2 l_{x+1} - \Sigma l_{x+1}}{l_x} = f_x - e_x$$

we get

$$e'_x = \frac{(\alpha + \beta x + \beta) e_x + (\beta + \gamma) (f_x - e_x)}{\alpha + \beta x + \gamma e_x}$$

whence

$$e'_x - e_x = \frac{\beta f_x + \gamma (f_x - e_x - e_x^2)}{\alpha + \beta x + \gamma e_x}.$$

We consequently shall have

$$e'_x = e_x$$

if

$$\frac{\beta}{\gamma} = \frac{e_x^2 + e_x - f_x}{f_x}.$$

It will be seen that the constant α is not contained in the condition, and remembering now that by (6) in article 5

$$e_x - e_{x+1} = \beta : \gamma$$

gives

$$q'_x = q_x,$$

we have

$$e'_x = e_x$$

for all the transformed mortality curves that intersect the given curve at the age z , determined by

$$(8) \quad e_x - e_{x+1} = \frac{e_x^2 + e_x - f_x}{f_x}.$$

The equation (8) may have more than one solution.

We give below (table 1) specimen values of z , according to HM . If (8) has more than one solution, the value of z immediately above x is given.

Table 1.

HM , trf. I. The solutions z of the equation (8)

x	z	x	z	x	z
20	46	35	52	50	60
25	47	40	55	55	63
30	49	45	57	60	67

9. To find the transformations I and II under which $a'_{x\bar{n}} = a_{x\bar{n}}$.

We find in a similar way as above, for the transformation I:

$$\frac{\beta}{\gamma} = - \frac{G_{x+1} - G_{x+n+1} - e_x D_x a_{x\bar{n}}}{S_{x+1} - S_{x+n+1} - n N_{x+n}},$$

where

$$G_{x+1} = \Sigma(v^x \Sigma l_{x+1}) = \frac{1+i}{i} (e_x D_x - N_{x+1}),$$

and for the transformation II:

$$\frac{\beta}{\gamma} = - \frac{S_{x+1} - S_{x+n+1} - N_{x+1} a_{x\overline{n}}}{S_{x+1} - S_{x+n+1} - n N_{x+n}}.$$

Thus we have that

$$a'_{x\overline{n}} = a_{x\overline{n}}$$

for all the transformed mortality curves that intersect the given curve at the age z , determined by

$$(9) \quad \varrho_x - \varrho_{x+1} = \beta : \gamma$$

where $\varrho_x = e_x$ or a_x according to the transformation I or II.

The equation (9) may have more than one solution.

We give below (table 2) specimen values of z , according to the transformation II of H^M , 4 %. If z has more than one value, the value immediately above x is given. z has been calculated by first-difference interpolation in a_x .

Table 2.

H^M , 4 %, trf. II. The solutions z of the equation (9).

$x+n$	97	70	65	60	55	50
x	z	z	z	z	z	z
20	33,1	32,4	31,8	30,9	29,7	27,7
30	40,4	39,9	39,4	38,6	37,4	35,7
40	48,0	47,2	46,2	44,6	43,6	42,2
50	53,5	53,6	53,1	52,1	50,9	
60	70,8	63,3	61,2			
70	73,4					

It should be noticed that there are some irregularities in the situation of z , especially for $x+n=97$. This is owing to the fact that the equation $a_x - a_{x+1} = \text{constant}$ may have more than one solution, and that $a_x - a_{x+1}$ within a certain interval varies but slowly. For certain values of $\beta:\gamma$ the transformed mortality curve will then intersect the given curve at several points, or approach to it over a certain space.

It will be seen that for $x+n$ up to 70 years

$$(10) \quad z - x = 0.26n$$

approximately, the deviations being less than one year (except for $x=50, n=20$).

10. Putting $\gamma=0$ we find for both transformations

$$\frac{p'_x - p_x}{p_x} = \frac{\beta}{\alpha + \beta x}$$

and

$$(11) \quad a'_{x+n} - a_{x+n} = \frac{S_{x+1} - S_{x+n+1} - nN_{x+n}}{D_x} \cdot \frac{\beta}{\alpha + \beta x},$$

where β may be supposed to be ± 1 , and $\alpha + \beta x > 0$.

We write for brevity

$$\frac{S_{x+1} - S_{x+n+1} - nN_{x+n}}{D_x} = s(i, x)$$

and

$$\frac{p'_x - p_x}{p_x} = \frac{\beta}{\alpha + \beta x} = k_x$$

and have

$$(12) \quad a'_{x+n} - a_{x+n} = s(i, x) k_x.$$

For high values of α k_x will be about a constant for a certain space of ages, and accordingly

$$(13) \quad k_x p_x = p'_x - p_x = q_x - q'_x$$

also will be about a constant, p_x not differing very much from 1. Table 3 gives the values of $10^5 k_x$ for some values of α ($\beta = \pm 1$).

We also give some values of $s(i, x)$, according to H^M , 4 % (table 4).

Table 3.

Values of $10^5 k_x = 10^5 \cdot \frac{\beta}{\alpha + \beta x}$

$\beta = +1$					$\beta = -1$			
x	20	40	60	80	20	40	60	80
α	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$	$10^5 k_x$
400	238	227	217	208	-263	-278	-294	-313
600	161	156	152	147	-172	-179	-185	-192
800	122	119	116	114	-128	-132	-135	-139
1000	98	96	94	93	-102	-104	-106	-109
1500	66	65	64	63	-68	-68	-69	-70
2000	50	49	49	48	-51	-51	-52	-52

Table 4.

H^M , $i = 0,04$. Values of $s(i, x)$

$x + n$	97	70	65	60	55	50
x	$s(i, x)$	$s(i, x)$	$s(i, x)$	$s(i, x)$	$s(i, x)$	$s(i, x)$
20	307	285	267	244	215	182
30	252	223	201	173	140	104
40	191	155	129	98	65	33
50	130	86	60	31	8	
60	76	28	8			

Thus a variation by a unit in the fifth decimal place of q gives, for H^M , 4 %, a variation = 0,00267 in $a_{20,45}$, and = 0,0001 in $A_{20,45}$.

11. In the function $a_{x\overline{n}|}^i$ let the rate of interest, i , increase by $\mathcal{A}i$, the increment being so small that in an expansion by TAYLOR's formula powers above the first may be neglected. We shall then have

$$a_{x\overline{n}|}^{i+\mathcal{A}i} - a_{x\overline{n}|}^i = \frac{da_{x\overline{n}|}^i}{di} \mathcal{A}i.$$

Comparing this formula with the formulas (12) and (13), and remembering that

$$\frac{da_{x\overline{n}|}^i}{di} = -v \frac{S_{x+1} - S_{x+n+1} - nN_{x+n}}{D_x} = -vs(i, x),$$

we see that $a_{x\overline{n}|}$ remains nearly invariant when q and i , within certain limits, vary so as to make

$$\begin{aligned} q - q' &= v\mathcal{A}i = v(i' - i) \\ \text{or} \quad q' + i' &= q + i + i(q - q'), \end{aligned}$$

which for the limits here considered may be replaced by

$$q' + i' = q + i.$$

As to the continuous annuity ($\bar{a}_{x\overline{n}|}$) the corresponding proposition

$$\mu' + \delta' = \mu + \delta,$$

as is well known, generally makes the annuity invariant.

For $a_{x\overline{n}|}$ a sufficient condition of invariability is easily found to be

$$(14) \quad q' + i' = (q + i) \left(1 + \frac{i' - i}{1 + i} \right).$$

By this formula a transformation of the mortality can be changed into a transformation of the rate of interest. This will be shown in article 18. However, q' (or p') in the formula (21) of that article belongs to i , and q to i' (accordingly to the problem there dealt with).

Geometrical representation of the transformation (with application to trf. II).

12. We put

$$\frac{p'_x - p_x}{p_x} = \lambda = \frac{\beta - \gamma(a_x - a_{x+1})}{\alpha + \beta x + \gamma a_x}$$

and have

$$(15) \quad \begin{aligned} (\alpha + \beta x + \gamma a_x)\lambda - (\beta - \gamma(a_x - a_{x+1})) &= 0, \text{ or} \\ \alpha\lambda + \beta(\lambda x - 1) + \gamma(\lambda a_x + a_x - a_{x+1}) &= 0. \end{aligned}$$

Here the age x is a constant, λ is a parameter, α, β, γ are variables, and may be considered as homogeneous coordinates in a tri-linear coordinate system. Later on we shall pass to a bi-linear coordinate system, but it will be convenient to begin with the more general case.

13. Considering α, β, γ as point coordinates we have:

The equations (15) represent a pencil with its vertex in the point of intersection of the straight lines

$$\begin{aligned} \alpha + \beta x + \gamma a_x &= 0 \\ \beta - \gamma(a_x - a_{x+1}) &= 0. \end{aligned}$$

To each age x a pencil corresponds, the vertex of which is the geometrical representation of the age, and may be called the x -point. To each value of λ there corresponds a straight line (λ -line) through the x -point and vice versa; this line is the geometrical representation of the transformation $p'_x = p_x(1 + \lambda)$. The coordinates α, β, γ of the point of intersection of a λ_1 -line from the x -point and a λ_2 -line from the y -point indicate the constants of the transformation

$$\begin{aligned} p'_x &= p_x(1 + \lambda_1) \\ p'_y &= p_y(1 + \lambda_2), \end{aligned}$$

and hereby the transformation is fully determined. The λ -lines from the other age points to the point α, β, γ indicate the λ -values which determine p' for these ages. Thus

the point α, β, γ (or the pencil through this point and the age points) becomes the geometrical representation of the new (transformed) mortality curve.

The coordinates of a line λ from the x -point are

$$(16) \quad \begin{aligned} \sigma u_1 &= \lambda \\ \sigma u_2 &= \lambda x - 1 \\ \sigma u_3 &= \lambda a_x + a_x - a_{x+1}, \end{aligned}$$

where u_1, u_2, u_3 are the line coordinates of λ , and σ is a factor of proportionality. By the equations (16) the system can be constructed, for instance with a λ -line for every thousandth part: $\lambda = 0$, $\lambda = \pm 0,001$, $\lambda = \pm 0,002$ etc.

14. Considering α, β, γ as line coordinates we have:

The equations (15) represent a straight line (range) through the points

$$\begin{aligned} \alpha + \beta x + \gamma a_x &= 0 \\ \beta - \gamma(a_x - a_{x+1}) &= 0. \end{aligned}$$

To each age x a straight line corresponds, which is the geometrical representation of the age x , and may be called the x -line. To each value of λ there corresponds a point (λ -point) on the x -line and vice versa; this point is the geometrical representation of the transformation $p'_x = p_x(1 + \lambda)$. The coordinates α, β, γ of the straight line between a λ_1 -point on the x -line and a λ_2 -point on the y -line indicate the constants of the transformation

$$\begin{aligned} p'_x &= p_x(1 + \lambda_1) \\ p'_y &= p_y(1 + \lambda_2), \end{aligned}$$

and hereby the transformation is fully determined. The λ -points in which the other age lines intersect the straight line α, β, γ indicate the λ -values which determine p' for these ages. Thus the straight line α, β, γ (or the range of intersection points of this line and the age lines) becomes the geometrical representation of the new (transformed) mortality curve.

The coordinates of a point λ on the x -line are:

$$(17) \quad \begin{aligned} qx_1 &= \lambda \\ qx_2 &= \lambda x - 1 \\ qx_3 &= \lambda a_x + a_x - a_{x+1}, \end{aligned}$$

where x_1, x_2, x_3 are the point coordinates of λ , and q is a factor of proportionality. By the equations (17) the system may be constructed, for instance with a λ -point for every thousandth part: $\lambda = 0$, $\lambda = \pm 0,001$, $\lambda = \pm 0,002$ etc.

15. The geometrical representation of the transformation becomes more convenient, when α, β, γ are dealt with as line coordinates. In this system the transformed mortality curve (q') is represented by a straight line (i. e. a range of points on the line), in the other system by a point (i. e. a pencil through the point), and in this case the drawing paper might easily be inconveniently crowded.

We then pass to a bi-linear right-angled system in line coordinates. The transition to this system is, as is known, made by letting the hypotenuse of a right-angled coordinate triangle pass into infinity, whereby the corresponding line coordinate becomes $= 1$, and the two other line coordinates become $=$ the negative inverse value of the distances from origo to the points of intersection of the line and the two coordinate axes.

From the equations (17) it is seen that on the lines

$$qx_1 = 0, \quad qx_2 = 0, \quad qx_3 = 0$$

respectively, are situated the λ -points

$$\lambda = 0, \quad \lambda = \frac{1}{x}, \quad \lambda = -\frac{a_x - a_{x+1}}{a_x};$$

$\lambda = \frac{1}{x}$ represents a decrease in mortality (q_x) at rates to be mentioned only at the highest ages, and we then let the corresponding coordinate line $qx_2 = 0$ pass into infinity. The two other coordinate lines

$$\varrho x_1 = 0, \varrho x_3 = 0$$

then become

$$X = 0, Y = 0,$$

X and Y being point coordinates in the new coordinate system.

The equations (17) change into

$$(18) \quad \begin{aligned} X = x_1 : x_2 &= \frac{\lambda}{\lambda x - 1} \\ Y = x_3 : x_2 &= \frac{\lambda a_x + a_x - a_{x+1}}{\lambda x - 1}, \end{aligned}$$

whence the equation of the age line for the age x

$$(19) \quad Y - X(a_x + x a_x - x a_{x+1}) + (a_x - a_{x+1}) = 0.$$

By the equation (19) and the first equation (18) the age lines and their λ -points can be plotted. Hereby a convenient scale is to be applied, and generally the scale must be different for the two coordinates, i. e. X and Y are to be plotted with the values mX and nY , where m and n are constants.

The mortality curve, determined by the transformation α, β, γ , is represented by the straight line

$$\alpha X + \gamma Y + \beta = 0$$

where $\beta = 1$, and

$$\alpha = -\frac{1}{a}$$

$$\gamma = -\frac{1}{b},$$

a and b being the distances from origo to the points of intersection of the line and the coordinate axes. If X and Y in the drawing paper are replaced by mX and nY , then a and b are to be replaced by $\frac{a}{m}$ and $\frac{b}{n}$, and we have

$$\alpha = -\frac{m}{a}, \gamma = -\frac{n}{b}, \beta = 1$$

or

$$\alpha = \frac{m}{n} \cdot \frac{b}{a}, \beta = -\frac{b}{n}, \gamma = 1.$$

16. As every straight line in the plane represents a transformation, the age lines also must represent transformations. For the age line x we then have

$$\frac{p'_x - p_x}{p_x} = \lambda = \frac{\beta - \gamma(a_x - a_{x+1})}{\alpha + \beta x + \gamma a_x}$$

for all values of λ , and λ accordingly must be of the form $\frac{0}{0}$. In fact, by (19) we have for the age line x

$$\alpha = -\frac{a_x + x(a_x - a_{x+1})}{a_x - a_{x+1}}$$

$$\gamma = \frac{1}{a_x - a_{x+1}}$$

$$\beta = 1,$$

whence

$$\beta - \gamma(a_x - a_{x+1}) = 0$$

$$\alpha + \beta x + \gamma a_x = 0.$$

This is the case dealt with in article 6.

17. The appended diagram shows a representation of the transformation II of H^M , 4 %. The age lines have been drawn for $x = 20, 30, 40, 50$ and 60 years, the λ -points have been plotted for $\lambda = 0, \lambda = \pm 0,001, \lambda = \pm 0,002$ etc., and are indicated by circles with the centre in the λ -point; every fifth value ($\lambda = 0, \lambda = \pm 0,005, \lambda = \pm 0,010$ etc.) is emphasized by larger circles. The Y -axis represents $\lambda = 0$ (the identical transformation); the X -axis represents $\lambda = -\frac{a_x - a_{x+1}}{a_x}$. The positive λ 's ($p' > p, q' < q$), accordingly, will be found on the

left side of the Y -axis, the negative λ 's ($p' < p$, $q' > q$) on the right side.

The coordinates were plotted in centimetres¹, after being multiplied by

$$\begin{aligned} m &= 1500 \text{ for the } X\text{-coordinates} \\ n &= 25 \quad \gg \quad Y- \quad \gg \end{aligned}$$

The dotted lines represent transformations (transformed mortality curves). The line (a) cuts off from the X -axis a length $a = 12$ cm; it passes through the point $X = 13,2$ cm, $Y = -10$ cm, accordingly cutting off from the Y -axis a length $b = 100$ cm. The constants of the transformation then become

$$\alpha = -\frac{m}{a} = -\frac{1500}{12} = -125$$

$$\gamma = -\frac{n}{b} = -\frac{25}{100} = -\frac{1}{4}$$

$$\beta = 1$$

or

$$\alpha = 0,5, \quad \beta = -0,004, \quad \gamma = 0,001.$$

We give below (table 5 a) for some ages the values of $q'_x D'_x N'_x M'_x$. The values q''_x have been drawn from a mortality table, applied by a Norwegian insurance company to a certain type of substandard life (the tuberculosis table); this table, too, has been derived from H^M , by adding to the mortality (q_x) a percentage, diminishing as x increases.

The line (b) represents the transformation dealt with in article 19. The constants of the transformation are $\alpha = 2,7$, $\beta = -0,0122$, $\gamma = -0,01$. We give below (table 5 b) for some ages the values of D'_x and N'_x .

Transformation of $a_{x|\overline{n}|}^i$ into $a_{x|\overline{n}|}^{i'}$.

18. We transform $a_{x|\overline{n}|}$, corresponding to p, i , into $a'_{x|\overline{n}|}$, corresponding to p', i' , and try to determine the transformation so that $a'_{x|\overline{n}|}$ corresponds also to p, i' .

¹ The diagram (page 19) has been reproduced on a reduced scale.

² — 2057. *Skandinavisk Aktuarietidskrift*. 1920.

Table 5 a. Diagram (page 19). Line (a).

 H^M , 4 %, trf. II. $\alpha = 0,5$, $\beta = -0,004$, $\gamma = 0,001$.

x	q_x	q'_x	D'_x	N'_x	M'_x	q''_x
20	0,0063	0,0157	19263	321878	6883,1	0,0174
25	0,0066	0,0165	14591	235397	5536,8	0,0172
30	0,0077	0,0181	11003	169982	4465,6	0,0189
35	0,0088	0,0199	8225,5	120813	3578,9	0,0202
40	0,0103	0,0221	6086,6	84186,5	2848,7	0,0222
45	0,0122	0,0248*	4454,1	57197,5	2254,2	0,0244
50	0,0160	0,0295	3198,3	37567,5	1753,4	0,0296
55	0,0210	0,0355	2238,9	23599,5	1331,2	0,0358
60	0,0297	0,0452	1507,9	13947,4	971,41	0,0460
65	0,0434	0,0600	954,7	7578,2	663,25	0,0608

Table 5 b. Diagram (page 19). Line (b).

 H^M , 4 %, trf. II. $\alpha = 2,7$, $\beta = -0,0122$, $\gamma = -0,01$.

x	D'_x	N'_x	x	D'_x	N'_x
20	99867,6	1819444	50	20105,0	259992
30	59921,9	1017744	60	10483,3	105786
40	35317,3	538915	70	4365,49	31003,5

Putting

$$(20) \quad \frac{D'_{x+1}}{D'_x} = v p'_x = v' p_x = \frac{D''_{x+1}}{D''_x}$$

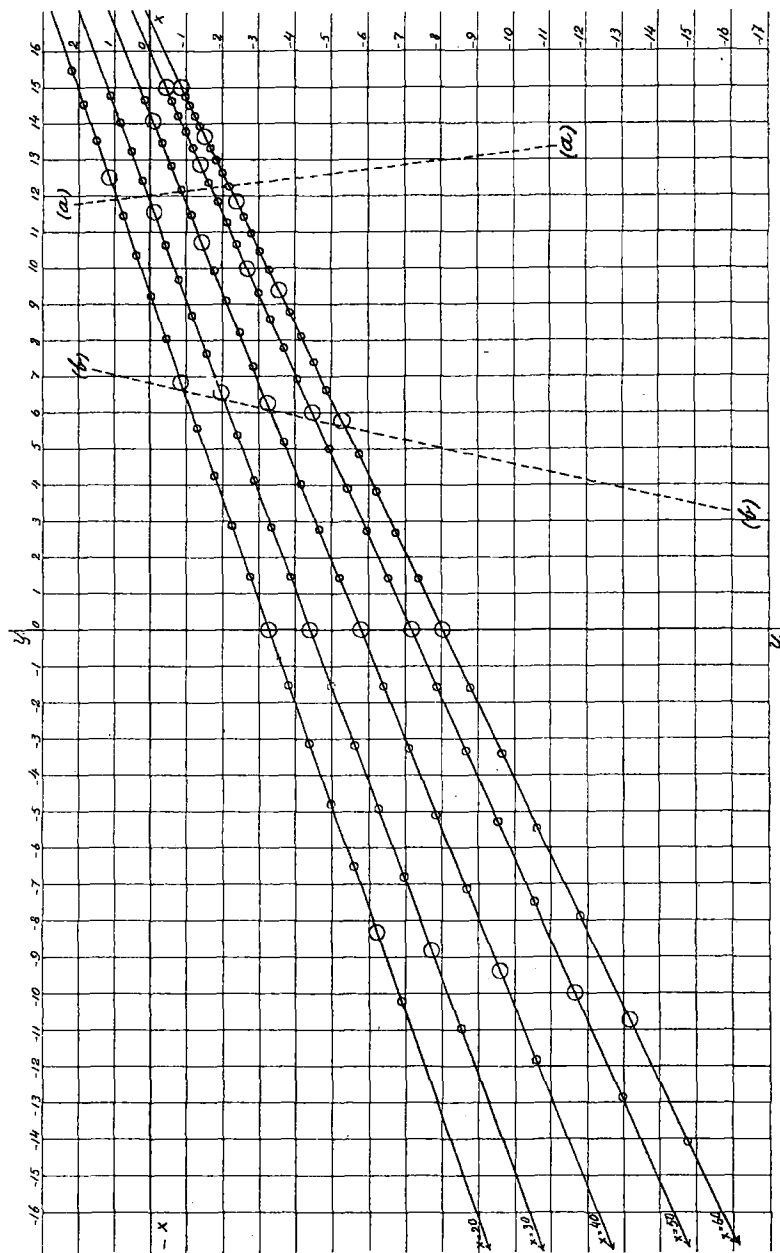
we have

$$(21) \quad \frac{p'_x}{p_x} = \frac{v'}{v} = \frac{1+i}{1+i'} = \frac{q'+i}{q+i'}$$

or

$$(21a) \quad \frac{p'_x - p_x}{p_x} = -\frac{i' - i}{1 + i'} = \lambda,$$

and if this condition holds for all ages between x and $x + n$ (more strictly: $x + n - 2$), we have



$$(22) \quad a'_{x\bar{n}} = a''_{x\bar{n}},$$

a' corresponding to p', i , a'' corresponding to p, i' .

In this investigation we consider the age of entrance and the duration as constants, and to distinguish the age current from the age of entrance, we denote the former by x , the latter by x_0 . Further it will be necessary to express $a_{x_0\bar{n}}$ as a function of the mortality and the rate of interest, and omitting the constant suffix $x_0\bar{n}$ we write $a(p, i)$, $a(p, i')$ etc. The equation (22) then takes the form

$$(22a) \quad a(p', i) = a(p, i').$$

If now we can find a transformation α, β, γ , giving

$$(23) \quad \frac{p'_x - p_x}{p_x} = \frac{\beta - \gamma(a_x - a_{x+1})}{\alpha + \beta x + \gamma a_x} = \lambda = -\frac{i' - i}{1 + i'}$$

for all ages between x_0 and $x_0 + n$, then the condition in (21) or (21a) is fulfilled, and $a(p, i')$ is found.

19. In article 10 it has been shown that a transformation $\gamma = 0$, $\beta = \pm 1$, $\alpha > 2000$ (whence $-0,0005 < \lambda < 0,0005$) gives very nearly

$$\frac{p'_x - p_x}{p_x} = \lambda = \text{constant}$$

for a space of ages of 60 years (or more).

From the diagram of the transformation II of H^M , 4 % (page 19) it is seen that a transformation $\gamma \geq 0$, within a limited space of ages, approximately complies with the above mentioned condition for λ -values considerably greater (in numerical value) than 0,0005, the λ -points for λ up to $\pm 0,005$ still very nearly lying on a straight line.

In the diagram, where $i = 0,04$, the λ -points for $i' = 0,045$ (whence $\lambda = -0,0048$) happen to be nearly coincident with the lefthand intersection points of the x -lines with the λ -circles for $\lambda = -0,005$. A straight line, cutting off from the X -axis a length $a = 6,78$ cm and from the Y -axis a length $b = -30,5$ cm, passes very nearly through the said points

and accordingly gives, for the space of ages of the diagram, an approximate transformation of $a(p, i)$ into $a(p, i')$.

We find

$$\begin{aligned} \alpha &= -221, & \beta &= 1, & \gamma &= 0,820 \text{ or} \\ \alpha &= 2,7, & \beta &= -0,0122, & \gamma &= -0,01. \end{aligned}$$

This gives for some entrance ages and durations the following values of $a'_{x_0 \overline{n}}$ (table 6). The values of $a_{x_0 \overline{n}}$ at $i = 0,045$ are given for comparison.

Table 6.

x_0	$x_0 + n = 70$		$x_0 + n = 60$		$x_0 + n = 50$	
	$a'_{x_0 \overline{n}}$	$\frac{4^{1/2}}{a_{x_0 \overline{n}}} \%$	$a'_{x_0 \overline{n}}$	$\frac{4^{1/2}}{a_{x_0 \overline{n}}} \%$	$a'_{x_0 \overline{n}}$	$\frac{4^{1/2}}{a_{x_0 \overline{n}}} \%$
20	17,944	17,949	17,194	17,199	15,647	15,651
30	16,467	16,470	15,219	15,222	12,646	12,648
40	14,381	14,378	12,264	12,262	7,898	7,897
50	11,390	11,388	7,670	7,669		

D'_x and N'_x have been given in article 17 (table 5 b).

This method, however, can not be expected, in general, to give so good approximations, and besides it has the disadvantage that it does not admit of a safe and easy test of accuracy.

20. A better method can be obtained by using the transformation $\gamma = 0$, already dealt with in article 10. The formula (12) of that article in the above notations may be written

$$(24) \quad a(p', i) - a(p, i) = -\frac{1}{v} \frac{da(p, i)}{di} k_{x_0},$$

where

$$(24a) \quad k_{x_0} = \frac{p'_{x_0} - p_{x_0}}{p_{x_0}} = \frac{\beta}{\alpha + \beta x_0}.$$

A transformation $\gamma = 0$ or

$$(25) \quad \frac{p'_x - p_x}{p_x} = \frac{\beta}{\alpha + \beta x}, \quad \beta = \pm 1, \alpha + \beta x > 0$$

obviously can not comply with the conditions in (21 a) or (23) for more than one age z ; but this, in fact, is sufficient, it being possible to determine the age z , or what amounts to the same, the constant α so as to make

$$a(p', i) = a(p, i')$$

with any required degree of accuracy.

In (21 a) we write π for p' and have

$$(26) \quad a(\pi, i) = a(p, i')$$

if

$$(27) \quad \frac{\pi_x - p_x}{p_x} = \lambda = -\frac{i' - i}{1 + i'}$$

for all values of x between x_0 and $x_0 + n$.

In the transformation (25) we may suppose $\alpha + \beta x > 0$. Comparing this transformation with the equation (27) we may let

$$p'_x \gtrless p_x$$

correspond to

$$\pi_x \gtrless p'_x$$

i. e.

$$\lambda > 0, i' < i$$

corresponds to

$$\beta = 1,$$

and

$$\lambda < 0, i' > i$$

corresponds to

$$\beta = -1.$$

21. We deal with the case

$$\beta = 1, \lambda > 0, i' < i.$$

Putting in (25)

$$\alpha = \frac{1}{\lambda} - z$$

where

$$x_0 \leq z \leq x_0 + n,$$

we have

$$(28) \quad \frac{p'_x - p_x}{p_x} = \frac{1}{\frac{1}{\lambda} - z + x} \leq \lambda \text{ for } z \leq x.$$

We now put

$$z = x_0, \alpha = \frac{1}{\lambda} - x_0$$

and find from (27) and (28)

$$\begin{aligned} p'_x &= \pi_x \text{ for } x = x_0 \\ p'_x &< \pi_x \text{ for } x > x_0, \end{aligned}$$

whence

$$a(p', i) < a(\pi, i)$$

and by (26)

$$a(p', i) < a(p, i').$$

Then we put

$$z = x_0 + n, \alpha = \frac{1}{\lambda} - (x_0 + n)$$

and find in a similar way

$$\begin{aligned} p'_x &= \pi_x \text{ for } x = x_0 + n \\ p'_x &> \pi_x \text{ for } x < x_0 + n, \end{aligned}$$

whence

$$a(p', i) > a(\pi, i)$$

or by (26)

$$a(p', i) > a(p, i').$$

But $a(p', i)$ is a continuous function of α , and accordingly there exists a value α , and a corresponding value $z = \frac{1}{\lambda} - \alpha$, so as to make

$$a(p', i) = a(\pi, i) = a(p, i').$$

Substituting this value of z in (24 a) we have

$$k_{x_0} = \frac{1}{\frac{1}{\lambda} - (z - x_0)} = \frac{-1}{\frac{1+i}{\mathcal{A}i} + 1 + z - x_0}$$

where $\mathcal{A}i = i' - i$; and from (24) we then get

$$a(p', i) - a(p, i) = a(p, i') - a(p, i) = \frac{\frac{1}{v} \frac{da(p, i)}{di}}{\frac{1+i}{\mathcal{A}i} + 1 + z - x_0}.$$

In the two right-hand members p is the same for both the a -functions, and we can then more conveniently write

$$(29) \quad a(i') - a(i) = \frac{\frac{1}{v} \frac{da(i)}{di}}{\frac{1+i}{\mathcal{A}i} + 1 + z - x_0},$$

where $a = a_{x_0 \overline{n}}$.

For

$$\beta = -1, \lambda < 0, i' > i$$

we find the same formula.

22. *Determination of z in formula (29).* By expansion of $a(i') - a(i)$ in powers of i z can be determined with any required degree of accuracy. But we shall at first, by some

examples, show that z , for small values of $\mathcal{A}i$, is very nearly independent of the mortality, and may be expressed as a simple function of the rate of interest and the duration. Thus, for $\mathcal{A}i$ up to ± 0.005 , very good approximations are obtained by the formulae

$$(30) \quad \begin{aligned} z - x_0 &= (0.30 - i)n \text{ for } \mathcal{A}i > 0 \\ z - x_0 &= (0.29 - i)n \text{ for } \mathcal{A}i < 0 \end{aligned}$$

for temporary annuities, and

$$(31) \quad \begin{aligned} z - x_0 &= (0.28 - 2i)(85 - x_0) \text{ for } \mathcal{A}i > 0 \\ z - x_0 &= (0.264 - 1.6i)(85 - x_0) \text{ for } \mathcal{A}i < 0 \end{aligned}$$

for whole-life annuities.

We give below specimen values for $\mathcal{A}i = \pm 0.005$, according to various tables.

The values of $\log s(i, x) = \log \left(-\frac{1}{v} \frac{da(i)}{di} \right)$ are given, in Table 7 for temporary annuities, and in Table 9 for whole-life annuities.

In Table 8 and 10 we give the differences between the true values of a and the values determined from (29) by means of z from (30) and (31).

23. *Closer determination of z in formula (29).* From (29) we have, replacing x_0 by x ,

$$\frac{1+i}{\mathcal{A}i} + 1 + z - x = \frac{\frac{1}{v} \frac{da(i)}{di}}{a(i') - a(i)},$$

whence

$$(32) \quad 1 + z - x = - \frac{\frac{1}{2v} \frac{d^2a(i)}{di^2} + \frac{1}{6v} \frac{d^3a(i)}{di^3} \mathcal{A}i + \dots}{\frac{da(i)}{di} + \frac{1}{2} \frac{d^2a(i)}{di^2} \mathcal{A}i + \dots}.$$

$\log s(i, x) = \log \left(-\frac{1}{v} \frac{da_{x\pi}}{di} \right)$

Table 7.

Table		HM					HF		OM			$O \frac{[M]}{[x]}$
$x+n$	x	$i=0,03$	$i=0,04$	$i=0,05$	$i=0,06$	$i=0,03$	$i=0,04$	$i=0,02$	$i=0,025$	$i=0,04$	$i=0,03$	$\log s(i, x)$
		$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$
70	20	2,5609	2,4542	2,3545	2,2614	2,5400	2,4331	2,6922	2,0341	2,4705		2,5684
	30	2,4365	2,3476	2,2634	2,1836	2,4265	2,3377	2,5411	2,4935	2,3578		2,4465
	40	2,2594	2,1899	2,1233	2,0594	2,2522	2,1962	2,3378	2,3012	2,1957		2,2711
	50	1,9848	1,9366	1,8898	1,8444	1,9790	1,9553	2,0385	2,0134	1,9405		2,0015
	60	1,4714	1,4466	1,4221	1,3981	1,4886	1,4629	1,4999	1,4872	1,4497		1,4979
60	20	2,4795	2,3874	2,3001	2,2174	2,4604	2,3640	2,5924	2,5430	2,4023		2,4863
	30	2,3099	2,2379	2,1688	2,1023	2,3018	2,2233	2,3939	2,3560	2,2467		2,3189
	40	2,0422	1,9924	1,9441	1,8971	2,0373	1,9905	2,0976	2,0718	1,9965		2,0524
	50	1,5220	1,4966	1,4717	1,4472	1,5198	1,5028	1,5492	1,5367	1,4984		1,5351
50	20	2,3332	2,2602	2,1900	2,1227	2,3163	2,2383	2,4227	2,3843	2,2732		2,3393
	30	2,0658	2,0154	1,9665	1,9189	2,0600	2,0007	2,1247	2,0986	2,0224		2,0738
	40	1,5422	1,5167	1,4915	1,4668	1,5401	1,5119	1,5705	1,5575	1,5188		1,5505
40	20	2,0774	2,0288	1,9776	1,9297	2,0634	2,0110	2,1399	2,1137	2,0369		2,0827
	30	1,5504	1,5247	1,4995	1,4747	1,5472	1,5186	1,5807	1,5676	1,5289		1,5568
30	20	1,5549	1,5292	1,5039	1,4791	1,5461	1,5232	1,5870	1,5738	1,5350		1,5594

Table 8.

$10^3 \delta^*$ for $\mathcal{A}i = i' - i = \pm 0.005$

Table		HM						HM(s)		HF		OM				$O_{(x)}^{(M)}$			Table
i	i'	3 %	4 %	5 %	6 %	3 %	4 %	3 %	4 %	2 %	2 %	2 %	2 %	2 %	2 %	3 %	3 %	3 %	i
$x+n$	x	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	$10^{3/2} \delta$	x
70	20	-6	-6	-6	-5	-6	-6	-6	-6	-5	-5	-5	-5	-5	-6	-6	-6	-6	20
	30	-3	-1	-1	-1	-2	-1	-2	-2	-2	-1	-1	-1	-1	-2	-2	-2	-2	30
	40	-1	-1	0	0	-1	0	-1	0	-1	0	0	0	0	0	0	0	0	40
	50	0	0	0	1	0	0	0	0	-1	1	1	1	1	-1	0	0	0	50
	60	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	60
60	20	1	0	-1	1	0	0	0	0	1	3	1	3	1	0	1	0	1	20
	30	1	1	1	2	1	1	1	1	1	2	1	2	1	1	2	1	2	30
	40	0	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	1	40
	50	0	0	0	1	0	0	0	0	-1	-1	0	-1	0	0	1	0	1	50
50	20	1	1	1	-1	1	1	1	1	1	3	3	3	3	1	2	1	2	20
	30	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	30
	40	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	40
40	20	1	1	1	1	1	1	1	1	0	1	1	1	1	0	0	0	0	20
	30	-1	0	0	1	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1	30
30	20	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	20

* δ is the difference between the true value of $a(i')$ and the value determined from (29) by means of z , calculated from (30).

$$\log s(i, x) = \log \left(-\frac{1}{v} \frac{da_x^i}{di} \right)$$

Table 9.

Table	HM					HF			OM		$OM(s)$	$O[a/x]$
	$i = 0,03$	$i = 0,04$	$i = 0,05$	$i = 0,06$		$i = 0,03$	$i = 0,04$	$i = 0,05$	$i = 0,02$	$i = 0,03$	$i = 0,025$	$i = 0,03$
x	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$		$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$	$\log s(i, x)$
10	2,6896	2,5583	2,4387	2,3298		2,6670	2,5355	2,4162	2,8513	2,7058	2,7528	—
20	2,6042	2,4871	2,3791	2,2796		2,5887	2,4701	2,3612	2,7510	2,6229	2,6707	2,6308
30	2,5026	2,4007	2,3057	2,2172		2,5020	2,3978	2,3008	2,6258	2,5156	2,5623	2,5406
40	2,3674	2,2816	2,2008	2,1247		2,3875	2,2989	2,2156	2,4687	2,3768	2,4183	2,4237
50	2,1830	2,1137	2,0478	1,9852		2,2215	2,1496	2,0813	2,2658	2,1921	2,2262	2,2725
60	1,9317	1,8787	1,8279	1,7792		1,9772	1,9218	1,8688	1,9995	1,9434	1,9700	2,0614
70	1,5944	1,5565	1,5199	1,4846		1,6451	1,6045	1,5655	1,6521	1,6122	1,6318	1,7555
80	1,1676	1,1425	1,1180	1,0942		1,2721	1,2440	1,2169	1,2081	1,1821	1,1950	1,3594
90	0,5780	0,5657	0,5536	0,5417		0,9142	0,8947	0,8757	0,6581	0,6428	0,6504	0,8902

$10^3 \delta^*$ for $Ji = i' - i = \pm 0.005$ Table 10.

Table	HM						HF						OM				$OM(s)$				$O(\frac{a}{x})$			
	3%	4%	5%	4%	5%	6%	3%	4%	4%	3%	4%	5%	2%	3%	2%	3%	$2\frac{1}{2}\%$	3%	$2\frac{1}{2}\%$	2%	3%	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$	3%
i'	$3\frac{1}{2}\%$	$4\frac{1}{2}\%$	$5\frac{1}{2}\%$	$3\frac{1}{2}\%$	$4\frac{1}{2}\%$	$5\frac{1}{2}\%$	$3\frac{1}{2}\%$	$4\frac{1}{2}\%$	$4\frac{1}{2}\%$	$3\frac{1}{2}\%$	$4\frac{1}{2}\%$	$5\frac{1}{2}\%$	$2\frac{1}{2}\%$	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$	2%	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$	$3\frac{1}{2}\%$	$2\frac{1}{2}\%$
x	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$	$10^3 \hat{\gamma}$
10	-4	-4	-1	-7	-5	-3	-5	-2	-6	-5	-4	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
20	-1	0	2	-1	1	1	0	0	-1	0	-3	-2	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
30	1	2	3	2	2	3	2	3	3	3	-1	1	1	1	1	1	1	1	1	1	1	1	1	1
40	1	2	3	2	3	2	4	4	4	4	-1	1	1	1	1	1	1	1	1	1	1	1	1	1
50	1	1	2	1	2	3	3	2	2	3	0	2	1	1	1	1	1	1	1	1	1	1	1	1
60	1	1	2	2	2	2	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
70	0	1	0	0	0	1	1	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
80	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

* $\hat{\gamma}$ is the difference between the true value of $a(i')$ and the value determined from (29) by means of z , calculated from (31).

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Putting

$$S(x+1, x+n) = \sum_1^{n-1} n D_{x+n}$$

$$T(x+1, x+n) = \sum_1^{n-1} \frac{n(n+1)}{2} D_{x+n}$$

$$U(x+1, x+n) = \sum_1^{n-1} \frac{n(n+1)(n+2)}{6} D_{x+n}$$

etc.

we get

$$\frac{da(i)}{di} = -v \frac{S(x+1, x+n)}{D_x}$$

$$\frac{1}{2} \frac{d^2 a(i)}{di^2} = \frac{v^2 T(x+1, x+n)}{D_x}$$

$$\frac{1}{6} \frac{d^3 a(i)}{di^3} = -\frac{v^3 U(x+1, x+n)}{D_x}$$

etc.

The equation (32) may then be written ($S = S(x+1, x+n)$ etc.)

$$1+z-x = \frac{T - Uv\mathcal{A}i + Vv^2\mathcal{A}i^2 - \dots}{S - Tv\mathcal{A}i + Uv^2\mathcal{A}i^2 - \dots}$$

or

$$(32a) \quad 1+z-x = \frac{T}{S} \cdot \frac{1 - \frac{U}{T}v\mathcal{A}i + \frac{V}{T}v^2\mathcal{A}i^2 - \dots}{1 - \frac{T}{S}v\mathcal{A}i + \frac{U}{S}v^2\mathcal{A}i^2 - \dots}.$$

Replacing the numerator

$$(33) \quad \left\{ \begin{array}{l} \text{by} \\ 1 - \frac{U}{T}v\mathcal{A}i + \frac{V}{T}v^2\mathcal{A}i^2 - \dots \\ 1 - \frac{U}{T}v\mathcal{A}i + \frac{U^2}{T^2}v^2\mathcal{A}i^2 - \dots = \frac{1}{1 + \frac{U}{T}v\mathcal{A}i} \end{array} \right.$$

and the denominator

$$(33a) \left\{ \begin{array}{l} 1 - \frac{T}{S} v \mathcal{A} i + \frac{U}{S} v^2 \mathcal{A} i^2 - \dots \\ 1 - \frac{T}{S} v \mathcal{A} i + \frac{T^2}{S^2} v^2 \mathcal{A} i^2 - \dots = \frac{1}{1 + \frac{T}{S} v \mathcal{A} i} \end{array} \right. \text{ by}$$

we obtain

$$(34) \quad 1 + z - x = \frac{T}{S} \cdot \frac{1 + \frac{T}{S} v \mathcal{A} i}{1 + \frac{U}{T} v \mathcal{A} i}$$

or

$$(34a) \quad 1 + z - x = \frac{T}{S} \cdot \frac{1 + i + \frac{T}{S} \mathcal{A} i}{1 + i + \frac{U}{T} \mathcal{A} i}$$

For small values of $\mathcal{A} i$ we may write

$$(35) \quad 1 + z - x = \frac{T}{S}.$$

24. We give below, according to H^M , specimen values of a , as calculated by (29) by means of $(1 + z - x)$ determined from (34) or (35). The following notations have been used:

$$s(i, x) = -\frac{1}{v} \frac{d a(i)}{d i},$$

$$d(i, \mathcal{A}) = \frac{1 + i}{\mathcal{A} i} + 1 + z - x,$$

$$\mathcal{A} = \mathcal{A} i = i' - i,$$

whence

$$a(i') - a(i) = -\frac{s(i, x)}{d(i, \mathcal{A})}.$$

δ is the difference between the true value of a and the value as calculated by (29).

Table 11.

 HM ; $i = 0,04$; $a(i) = a_x$; $(1 + z - x)$ from 34.

x	20	40	60
$\log s(i, x)$	2,487060	2,281619	1,878677
$T : S$	13,9630	10,4717	6,7133
$U : T$	12,0483	8,9772	5,8592
Δi	$\log d(i, \Delta)$	$\log d(i, \Delta)$	$\log d(i, \Delta)$
-0,010	1,955821 n	1,971707 n	1,988314 n
-0,005	2,288190 n	2,295802 n	2,303876 n
0,005	2,346518	2,339587	2,331913
0,010	2,072593	2,059224	2,044465
0,020	1,822040	1,797464	1,769468
Δi	$a(i + \Delta i) \cdot 10^4 \bar{c}$	$a(i + \Delta i) \cdot 10^4 \bar{c}$	$a(i + \Delta i) \cdot 10^4 \bar{c}$
-0,010	23,0419 6	18,1760 2	11,2359 0
-0,005	21,2246 0	17,1026 0	10,8347 0
0,005	18,2617 0	15,2596 0	10,1068 0
0,010	17,0468 3	14,4659 1	9,7763 0
0,020	15,0198 21	13,0857 9	9,1731 1

Table 12.

 HM ; $i = 0,04$; $a(i) = a_x$; $\Delta i = 0,005$; $(1 + z - x)$ from (35).

x	$\log s(i, x)$	$\log d(i, \Delta)$	$\log (-d(i, -\Delta))$	$a(i + \Delta i) \cdot 10^4 \bar{c}$	$a(i - \Delta i) \cdot 10^4 \bar{c}$
10	2,55834	2,34928	2,28443	19,4582 10	22,9554 -15
20	2,48706	2,34627	2,28789	18,2609 8	21,2257 -11
30	2,40069	2,34298	2,29164	16,9888 5	19,4163 -7
40	2,28162	2,33939	2,29563	15,2593 3	17,1030 -4
50	2,11370	2,33564	2,29975	12,9361 2	14,1876 -2
60	1,87868	2,33185	2,30382	10,1068 0	10,8347 0
70	1,55649	2,32830	2,30758	7,1239 -1	7,4704 -1
80	1,14247	2,32523	2,31078	4,5386 1	4,6722 0
90	0,56573	2,32230	2,31378	2,6864 0	2,7218 -1

Table 13.

 H^M ; $i = 0,04$; $a(i) = a_{x, \overline{60-x}|}$; $(1+z-x)$ from (34).

x	20	30	40	50
$\log s(i, x)$	2,387387	2,237900	1,992413	1,496616
$T:S$	11,2413	8,9427	6,3710	3,5127
$U:T$	9,2854	7,3100	5,1891	2,9225
Δi	$\log d(i, \Delta)$	$\log d(i, \Delta)$	$\log d(i, \Delta)$	$\log d(i, \Delta)$
-0,010	1,968440 n	1,978675 n	1,989918 n	2,002199 n
-0,005	2,294179 n	2,299137 n	2,304630 n	2,310687 n
0,005	2,341123	2,336481	2,331238	2,325340
0,010	2,062339	2,053362	2,043126	2,031538
0,020	1,803458	1,786672	1,767176	1,744688
Δi	$a(i + \Delta i)$ $10^4 \bar{d}$	$a(i + \Delta i)$ $10^4 \bar{d}$	$a(i + \Delta i)$ $10^4 \bar{d}$	$a(i + \Delta i)$ $10^4 \bar{d}$
-0,010	20,9350 3	17,8350 1	13,7256 1	8,1291 0
-0,005	19,5505 0	16,8870 0	13,2071 0	7,9703 0
0,005	17,1987 0	15,2216 0	12,2615 0	7,6685 0
0,010	16,1974 1	14,4890 1	11,8300 1	7,5251 0
0,020	14,4747 16	13,1921 9	11,0401 3	7,2520 0

Table 14.

 H^M ; $i = 0,04$; $a(i) = a_{x, \overline{5}|}$; $\mathcal{A}i = 0,005$; $(1+z-x)$ from (35)

x	$\log s(i, x)$	$\log d(i, \Delta)$	$a(i + \Delta i)$ $10^5 \bar{d}$
20	0,94070	2,32218	4,53010 1
30	0,93900	2,32218	4,52868 0
40	0,93561	2,32216	4,49617 0
50	0,92742	2,32216	4,44320 0
60	0,90698	2,32214	4,31498 1
70	0,85790	2,32210	4,02804 0
80	0,73295	2,32197	3,40837 0

25. *Calculation of a_x (or N_x) for quinquennial ages and for a series of rates of interest.* From Table 14 it is seen that $a(i + \mathcal{A}i) = a_{x+5}^{i'}$ for $\mathcal{A}i = 0.005$ may be determined with great accuracy by the formula (29) and by the value of z derived from (35). If we, in this manner, calculate

$$a_{x+5}^{i'}, a_{x+10}^{i'}, a_{x+15}^{i'} \text{ etc.},$$

and if we calculate directly

$$D_x^{i'}, D_{x+5}^{i'}, D_{x+10}^{i'} \text{ etc.},$$

we find the whole-life annuities $a_x^{i'}$ (or $N_x^{i'}$) for the same ages by the formulae

$$(36) \quad \begin{cases} a_x^{i'} = a_{x+5}^{i'} + \frac{D_{x+5}^{i'}}{D_x^{i'}} a_{x+5}^{i'} \\ a_{x+5}^{i'} = a_{x+10}^{i'} + \frac{D_{x+10}^{i'}}{D_{x+5}^{i'}} a_{x+10}^{i'} \\ \text{etc.} \end{cases}$$

or

$$(37) \quad \begin{aligned} a_{x+5}^{i'} \cdot D_x^{i'} &= N_x^{i'} - N_{x+5}^{i'} \\ a_{x+10}^{i'} \cdot D_{x+5}^{i'} &= N_{x+5}^{i'} - N_{x+10}^{i'} \\ \text{etc.} \end{aligned}$$

From i' we may go further to $i'' = i' + \mathcal{A}$ in the following manner.

Going from i to i' we have

$$a(i') - a(i) = -\frac{s(i, x)}{d(i, \mathcal{A})}$$

using the same notations as in article 24.

Going from i' to i we have

$$a(i) - a(i') = -\frac{s(i', x)}{d(i', -\mathcal{A})}$$

whence

$$(38) \quad s(i', x) = -\frac{s(i, x)}{d(i, \mathcal{A})} d(i', -\mathcal{A}).$$

Now

$$d(i, \mathcal{A}) = \frac{1+i}{\mathcal{A}} + t(i),$$

where

$$t(i) = \frac{T}{S}$$

for $n=5$ is very nearly a linear function of i . If we have calculated $t(i)$ for two or three values of i , we know $t(i)$, and therewith $d(i, \mathcal{A})$, for all values of i .

We shall then have

$$\begin{aligned} a(i') - a(i) &= -\frac{s(i, x)}{d(i, \mathcal{A})}, \\ a(i'') - a(i') &= -\frac{s(i', x)}{d(i', \mathcal{A})}, \\ a(i''') - a(i'') &= -\frac{s(i'', x)}{d(i'', \mathcal{A})}, \\ &\text{etc.} \end{aligned}$$

where all the right-hand members are known.

It will be convenient to eliminate $s(i')$, $s(i'')$ etc., and we then get

$$\begin{aligned} a(i') - a(i) &= -\frac{s(i, x)}{d(i, \mathcal{A})} \\ (39) \quad a(i'') - a(i') &= \frac{s(i, x)}{d(i, \mathcal{A})} \cdot \frac{d(i', -\mathcal{A})}{d(i', \mathcal{A})} \\ a(i''') - a(i'') &= -\frac{s(i, x)}{d(i, \mathcal{A})} \cdot \frac{d(i', -\mathcal{A})}{d(i', \mathcal{A})} \cdot \frac{d(i'', -\mathcal{A})}{d(i'', \mathcal{A})} \\ &\text{etc.} \end{aligned}$$

26. We give below, according to H^M , an example of the application of (39).

Table 15 contains the values of $t(i) = \frac{T}{S}$ at $i = 0$, $i = 0.04$ and $i = 0.06$.

Table 15.

HM ; $t(i) = T(x+1, x+5) : S(x+1, x+5)$.

x	$i = 0$	$i = 0.04$	$i = 0.06$	$-10^4 \Delta t(i)$	
20	1.9966	1.9767	1.9668	199	99
25	1.9965	1.9766	1.9668	199	98
30	1.9959	1.9760	1.9661	199	99
35	1.9952	1.9753	1.9654	199	99
40	1.9945	1.9746	1.9647	199	99
45	1.9930	1.9730	1.9632	200	98
50	1.9910	1.9710	1.9611	200	99
55	1.9876	1.9675	1.9576	201	99
60	1.9819	1.9617	1.9517	202	100
65	1.9737	1.9534	1.9433	203	101
70	1.9593	1.9387	1.9285	206	102
75	1.9362	1.9152	1.9049	210	103
80	1.8999	1.8784	1.8678	215	106
85	1.8597	1.8378	1.8270	219	108
90	1.7546	1.7329	1.7224	217	105

Table 16.

HM ; $x = 20$; $\Delta i = 0.005$; i varies from 0 to 6 %;
 $\log s(0, x) = 0.991348$.

i	$\log d(i, \Delta)$	$\log (-d(i, -\Delta))$	$-\log \frac{-d(i, -\Delta)}{d(i, \Delta)}$
0.00	2.305344	2.296672	0.008672
0.02	2.313839	2.305380	0.008459
0.04	2.322171	2.313916	0.008255
0.06	2.330346	2.322288	0.008058

i	$-\log \frac{-d(i, -\Delta)}{d(i, \Delta)}$	$\log (-\Delta a(i))$	$-\Delta a(i)$	$a(i) = a_{20,5}^i$
0,000	0,008672	2,686004	0,048529	4,934496
0,005	0,008618	2,677386	0,047576	4,885967
0,010	0,008565	2,668821	0,046647	4,838391
0,015	0,008512	2,660309	0,045741	4,791744
0,020	0,008459	2,651850	0,044859	4,746003
0,025	0,008407	2,643443	0,043999	4,701144
0,030	0,008356	2,635087	0,043161	4,657145
0,035	0,008305	2,626782	0,042343	4,613984
0,040	0,008255	2,618527	0,041546	4,571641
0,045	0,008205	2,610322	0,040768	4,530095
0,050	0,008156	2,602166	0,040010	4,489327
0,055	0,008107	2,594059	0,039270	4,449317
0,060	0,008058			4,410047

The true value of $a(i) = a_{20,5}^i$ at $i = 0,04$: is 4,571643,
at $i = 0,06$: 4,410049.

26. Calculation of $s(i, x)$ or $\log s(i, x)$ for quinquennial ages and for a series of rates of interest. The formula (38) of the preceding article gives

$$s(i', x) = -s(i, x) \cdot \frac{d(i', -\mathcal{A})}{d(i, \mathcal{A})},$$

whence

$$(40) \quad \begin{cases} s(i'', x) = s(i, x) \frac{d(i', -\mathcal{A})}{d(i, \mathcal{A})} \cdot \frac{d(i'', -\mathcal{A})}{d(i', \mathcal{A})} \\ s(i''', x) = -s(i, x) \cdot \frac{d(i', -\mathcal{A})}{d(i, \mathcal{A})} \cdot \frac{d(i'', -\mathcal{A})}{d(i', \mathcal{A})} \cdot \frac{d(i''', -\mathcal{A})}{d(i'', \mathcal{A})} \\ \text{etc.} \end{cases}$$

As shown in the article (28) $\log s(i, x)$ may be applied in calculating annuities on two joint lives (a_{xy}) for quinquennial ages.

We give below (Table 17), according to H^M , an example of the application of (40). In order to avoid the cumulative effect of neglected digits, $t(i)$ has been calculated to 5, $\log d(i, \Delta)$ to 8 places of decimals. $\log s(i, x)$ is given to 7 places of decimals, the last figure, however, not always being to rely upon.

In the calculations of annuities on two joint lives a four-figure $\log s(i, x)$ will be sufficient. But if $\log s(i, x)$ shall be used as a basis for calculation of the financial elements S_x , a seven-figure $\log s(i, x)$ is necessary or desirable.

Table 17.

H^M ; $x = 20$; $\Delta i = 0,0025$; i varies from 3 to 6 %.

i	$\log d(i, \Delta)$	$\log (-d(i', -\Delta))$	$\log \frac{-d(i', -\Delta)}{d(i, \Delta)}$
0,03	2,61698106	2,61386255	-0,0031185
0,04	2,62115204	2,61807380	-0,0030782
0,05	2,62528338	2,62224458	-0,0030388
0,06	2,62937579	2,62637567	-0,0030001
i	$\log \frac{-d(i', -\Delta)}{d(i, \Delta)}$	$\log s(i, x)$	$\log (S_{21}^i - S_{20}^i - 5 N_{20}^i)$
0,0300	-0,0031185	0,9531160	5,6796504
0,0325	-0,0031084	0,9499975	5,6554752
0,0350	-0,0030983	0,9468891	5,6313610
0,0375	-0,0030882	0,9437908	5,6073076
0,0400	-0,0030782	0,9407026	5,5833147
0,0425	-0,0030683	0,9376244	5,5593820
0,0450	-0,0030584	0,9345561	5,5355092
0,0475	-0,0030486	0,9314977	5,5116960
0,0500	-0,0030388	0,9284491	5,4879420
0,0525	-0,0030291	0,9254103	5,4642471
0,0550	-0,0030194	0,9223812	5,4406109
0,0575	-0,0030097	0,9193618	5,4170332
0,0600	-0,0030001	0,9163521	5,3935137

27. *Transformation of $a_{\overline{n}|}$ into $a_{x\overline{n}|}$ for small values of n .* Knowing $a_{x\overline{n}|}$ we may calculate a_x (or N_x) for every n -th value of x , as shown in article 25. We shall now show another method of calculating $a_{x\overline{n}|}$ for small values of n (such as $n=5$), deriving it from $a_{\overline{n}|}$ (the annuity certain), by means of a transformation $\gamma=0$, applied on a »mortality table» $p=1$.

Putting

$$p=1, \quad \pi_x = p_x p = p_x$$

we get

$$(41) \quad \frac{\pi_x - p}{p} = p_x - 1 = -q_x.$$

The transformation $\gamma=0$ gives

$$(42) \quad \frac{p'_x - p}{p} = \frac{\beta}{\alpha + \beta x},$$

where $\beta = -1$, p'_x being $< p=1$. From (24) in article 20 we get

$$(43) \quad a(p') - a(p) = -\frac{1}{v} \frac{da(p)}{di} \cdot \frac{-1}{\alpha - x_0},$$

and we try to determine α so as to make $a(p') = a(\pi)$. If this condition is fulfilled for a value of α , corresponding to an age $z = x_0 + t$, we have

$$p'_z = p'_{x_0+t} = \pi_{x_0+t}$$

whence from (41) and (42)

$$-q_{x_0+t} = \frac{-1}{\alpha - (x_0 + t)}$$

or

$$\alpha = \frac{1}{q_{x_0+t}} + x_0 + t,$$

and substituting this value of α in (43) we find

$$a(p') - a(p) = \frac{1}{v} \frac{da(p)}{di} \cdot \frac{1}{\frac{1}{q_{x_0+t}} + t}$$

or

$$(44) \quad a_{x\bar{n}} - a_{\bar{n}} = \frac{1}{v} \frac{da_{\bar{n}}}{di} \cdot \frac{1}{\frac{1}{q_{x+t}} + t},$$

where x is written for x_0 .

The formula (44) is analogous to the formula (29) in article 22, t corresponding to $z - x_0$. For $n = 5$ t will be about = 1 (Vid. Table 15, where $t(i) = 1 + z - x_0$).

However, t can not be determined directly with sufficient accuracy, and we then proceed in the following manner.

We write

$$(45) \quad a_{x\bar{n}} - a_{\bar{n}} = \frac{\frac{1}{v} \frac{da_{\bar{n}}}{di}}{d(1:x)} = - \frac{s(i)}{d(1:x)},$$

where $d(1:x)$ may be called divisor for transformation of $a_{\bar{n}}$ into $a_{x\bar{n}}$. If the rate of interest is to be denoted, we write $d(i, 1:x)$. We calculate the divisors directly for two values of i , and (if $n = 5$) for every fifth value of x . For small values of n , such as $n = 5$, the divisors $d(i, 1:x)$, corresponding to a certain value of x , will be about a linear function of i , and moreover at most ages so slowly varying with i that it, for most practical purposes, may even be regarded as a constant. For instance (according to H^M) the divisors corresponding to $i = 0$ will give a_x correct to the third place of decimals at rates of interest up to 6 %, exceptions to be made only for the higher ages (Table 20).

We give below, according to H^M , some examples of the application of the formula (45).

Table 18 contains the values of $\log d(i, 1:x)$, for $n = 5$, as calculated by (45) at $i = 0$, $i = 0.04$ and $i = 0.06$. We

Table 18.

 H^M ; $\log d(i, 1 : x)$ for $n = 5$, as calculated by (45).

x	$i = 0$	$i = 0,04$	$i = 0,06$	$10^6 \Delta \log d(i, 1 : x)$	
10	2,38753	2,38601	2,38525	-152	-76
15	2,47292	2,47438	2,47510	+146	+72
20	2,18373	2,18392	2,18402	19	10
25	2,17330	2,17345	2,17353	15	8
30	2,10509	2,10523	2,10530	14	7
35	2,04438	2,04462	2,04474	24	12
40	1,98228	1,98240	1,98245	12	5
45	1,89373	1,89412	1,89431	39	19
50	1,78341	1,78371	1,78385	30	14
55	1,65772	1,65810	1,65829	38	19
60	1,50720	1,50760	1,50779	40	19
65	1,35214	1,35232	1,35241	18	9
70	1,19504	1,19526	1,19537	22	11
75	1,02051	1,02021	1,02007	-30	-14
80	0,87230	0,87159	0,87124	-71	-35
85	0,75479	0,75319	0,75240	-160	-79
90	0,64295	0,64105	0,64012	-190	-93

give $\log d$ to five places of decimals; but it should be noticed that this corresponds to 6 or 5 places of decimals in $a_{x\overline{5}}$, and almost as many places in a_x (4 places at very high ages).

Table 19 shows the calculation of a_x at $5\frac{1}{2}\%$, by means of $d(i, 1 : x)$, taken at the same rate of interest (interpolated from the values in Table 18).

Table 20 shows the calculation of a_x at 6% , by means of $d(i, 1 : x)$, taken at $i = 0$.

Table 21 gives the values of the logarithms of $s(i) = -\frac{1}{v} \frac{da_{\overline{n}}}{di}$ for $n = 5$ at different rates of interest.

Table 19.

H^M ; $i = 0,055$; a_x as calculated by means of $d(i, 1:x)$;
 $a_5 = 4,505150$.

x	$\log d(i, 1:x)$	$\log(a_5 - a_{x5})$	$a_5 - a_{x5}$	a_{x5}	D_x	$N_x - N_{x+5}$	N_x	a_x
90	0,64035	0,29052	1,9522	2,5530	11,793	30,108	31,279	2,6523
85	0,75260	0,17827	1,50755	2,99760	57,240	171,583	202,862	3,5441
80	0,87133	0,05954	1,14694	3,35821	192,200	645,448	848,310	4,4137
75	1,02010	1,91077	0,81428	3,69087	463,283	1709,92	2558,23	5,5220
70	1,19534	1,73553	0,54391	3,96124	898,517	3559,24	6117,47	6,8084
65	1,35239	1,57848	0,37886	4,12629	1518,485	6265,71	12383,18	8,1550
60	1,50774	1,42313	0,26493	4,24022	2369,829	10048,60	22431,78	9,4656
55	1,65824	1,27263	0,18734	4,31781	3499,623	15110,71	37542,49	10,7276
50	1,78382	1,14705	0,14030	4,36485	5001,114	21829,11	59371,60	11,8717
45	1,89426	1,03661	0,108795	4,396355	7002,977	30787,57	90159,17	12,8744
40	1,98244	2,94843	0,088804	4,416346	9655,337	42685,47	132844,64	13,7444
35	2,04471	2,88616	0,076941	4,428209	13245,827	58655,29	191499,93	14,4574
30	2,10528	2,82559	0,066926	4,438224	18030,874	80025,06	271524,99	15,0589
25	2,17351	2,75736	0,057195	4,447955	24403,731	108546,7	380071,7	15,5743
20	2,18400	2,74687	0,055830	4,449320	32978,409	146731,5	526803,2	15,9742
15	2,47492	2,45595	0,028572	4,476578	43997,776	196959,5	723762,7	16,4500
10	2,38544	2,54543	0,035110	4,470040	58543,058	261689,8	985452,5	16,8330

The values of $\log d(1:x)$ for $n=5$, as calculated by the formula

$$d(1:x) = \frac{1}{\frac{1}{q_{x+t}} + t}, \quad t = 1,$$

are given in Table 24.

The method applied in this article may also be used in calculating S_x for quinquennial ages. This will be shown in article 33.

Table 20.

H^M ; $i = 0,06$; a_x as calculated by means of $d(0, 1:x)$;
 $a_{\overline{5}} = 4,46511$.

x	$\log d(0)$	$\log(a_{\overline{5}} - a_x \overline{5})$	$a_{\overline{5}} - a_x \overline{5}$	$a_x \overline{5}$	Dx	$N_x - N_{x+5}$	N_x	a_x	$10^9 \%$
90	0,6430	0,2818	1,914	2,551	7,706	19,66	20,41	2,649	-13
85	0,7548	0,1700	1,479	2,986	38,297	114,35	134,76	3,519	-11
80	0,8723	0,0525	1,128	3,337	131,669	439,38	574,14	4,360	-6
75	1,0205	1,9043	0,8023	3,6628	324,968	1190,3	1764,4	5,429	-3
70	1,1950	1,7298	0,5368	3,9283	645,339	2535,1	4299,5	6,662	-1
65	1,3521	1,5727	0,3739	4,0912	1116,71	4568,6	8868,1	7,941	0
60	1,5072	1,4176	0,2616	4,2035	1784,48	7501,1	16369,2	9,173	0
55	1,6577	1,2671	0,1849	4,2802	2698,26	11549	27918	10,347	0
50	1,7834	1,1414	0,1385	4,3266	3948,18	17082	45000	11,398	0
45	1,8937	1,0311	0,1074	4,3577	5660,82	24668	69608	12,307	1
40	1,9823	2,9425	0,08760	4,37751	7999,83	35019	104687	13,0862	0,4
35	2,0444	2,8804	0,07593	4,38918	11225,6	49271	153958	13,7149	0,3
30	2,1051	2,8197	0,06603	4,39908	15646,4	68830	222788	14,2389	0,3
25	2,1733	2,7515	0,05643	4,40868	21683,1	95594	318382	14,6834	0,2
20	2,1837	2,7411	0,05509	4,41002	30002,8	132313	450695	15,0218	0,1
15	2,4729	2,4519	0,02831	4,43680	40985,4	181844	632539	15,4333	0,2
10	2,3875	2,5373	0,03445	4,43066	55839,5	247406	879945	15,7585	0,0

Table 21.

$$\log s(i) = \log \frac{1}{i} (a_{\overline{5}} - 5v^i).$$

i	$\log s(i)$	i	$\log s(i)$	i	$\log s(i)$
0,0300	0,96168	0,0400	0,94924	0,0500	0,93695
0,0325	0,95855	0,0425	0,94615	0,0525	0,93391
0,0350	0,95544	0,0450	0,94308	0,0550	0,93087
0,0375	0,95233	0,0475	0,94001	0,0575	0,92785
				0,0600	0,92483

28. *Calculation of a_{xy} (or N_{xy}) for quinquennial ages.* In article 27 we have shown a method of deriving $a_{x\bar{5}}$ from $a_{\bar{5}}$; we may, in the same manner, derive $a_{xy\bar{5}}$ from $a_{x\bar{5}}$, $x > y$, and knowing $a_{xy\bar{5}}$ we find a_{xy} (or N_{xy}) for quinquennial ages (in a similar manner as shown in article 25 for $a_{x\bar{5}}$ and a_x).

We put

$$r_x = p_x p_y$$

and get

$$\frac{r_x - p_x}{p_x} = p_y - 1 = -q_y.$$

The transformation $\gamma = 0$ gives

$$\frac{p'_x - p_x}{p_x} = \frac{\beta}{\alpha + \beta x},$$

where $\beta = -1$, p'_x being $< p_x$.

Proceeding as shown in article 27 we find

$$(45) \quad a_{xy\bar{n}} - a_{x\bar{n}} = \frac{1}{v} \frac{da_{x\bar{n}}}{di} \cdot \frac{1}{\frac{1}{q_{y+t}} + t},$$

which corresponds to the formula (44) for $a_{x\bar{n}} - a_{\bar{n}}$. As in the former case t is about $= 1$ for $n = 5$.

Proceeding as before we write

$$(46) \quad a_{xy\bar{n}} - a_{x\bar{n}} = \frac{\frac{1}{v} \frac{da_{x\bar{n}}}{di}}{d(i, x : xy)} = -\frac{s(i, x)}{d(i, x : xy)},$$

where $d(i, x : xy)$ may be called divisor for transformation of $a_{x\bar{n}}$ into $a_{xy\bar{n}}$.

The divisors

$$d(i, x : xy)$$

and

$$d(i, 1 : y)$$

have the same form

$$\frac{1}{q_{y+t}} + t,$$

where t for small values of n , such as $n=5$, can not differ very much in the two formulae (except at very high ages). Accordingly, $d(i, x:xy)$ will not differ very much from $d(i, 1:y)$, at most ages, and it may be expected that $d(i, x:xy)$ for $n=5$, both as a function of $(x-y)$, y being constant, and as a function of i , will be about constant, or so slowly varying that the divisor, corresponding to a certain difference of ages $(x-y)$, say $x-y=0$, or to a certain rate of interest, say $i=0$, will hold for a large series of differences of ages and rates of interest.

For most practical purposes we may put, for $n=5$,

$$(47) \quad d(i, x:xy) = d(i, 1:y) = d(0, 1:y);$$

and even

$$(48) \quad d = \frac{1}{q_{y+1}} + 1$$

will sometimes do good service.

We give below, according to H^M , some examples of the application of the formulae (46), (47) and (48).

Tables 22 and 23 contain the values of $\log d(i, x:xy)$ for $n=5$, as calculated by (46).

Table 23, for $d(i, x:xy)$, corresponds to Table 18, for $d(i, 1:y)$.

Table 24 contains the values of d and $\log d$ for $n=5$, as calculated by the formula (48).

Table 25 gives a_{xy} at 4 % for $x-y=20$, by means of $d(4\%, x:xy)$ for $x-y=0$.

Table 22.

 H^M ; $i = 0,04$; $\log d(i, x:xy)$ as calculated by (46), for $n = 5$.

$x - y$	$y = 10$	$y = 15$	$y = 20$	$y = 25$	$y = 30$	$y = 35$	$y = 40$	$y = 45$
0	2,3859	2,4745	2,1840	2,1735	2,1053	2,0447	1,9824	1,8943
10	2,3857	2,4746	2,1840	2,1735	2,1053	2,0447	1,9825	1,8944
20	2,3857	2,4747	2,1840	2,1735	2,1053	2,0448	1,9825	1,8946
30	2,3856	2,4749	2,1840	2,1735	2,1054	2,0449	1,9826	1,8954
40	2,3853	2,4753	2,1841	2,1737	2,1055	2,0454	1,9830	1,8969
50	2,3846	2,4764	2,1843	2,1740	2,1059	2,0464	1,9837	
$x - y$	$y = 50$	$y = 55$	$y = 60$	$y = 65$	$y = 70$	$y = 75$	$y = 80$	$y = 85$
0	1,7838	1,6583	1,5080	1,3526	1,1957	1,0192	0,8680	0,7418
5	1,7839	1,6585	1,5081	1,3527	1,1960	1,0187	0,8666	0,7336
10	1,7840	1,6586	1,5084	1,3529	1,1964	1,0181	0,8630	
15	1,7841	1,6589	1,5089	1,3533	1,1968	1,0166		
20	1,7843	1,6593	1,5096	1,3537	1,1980			
25	1,7846	1,6600	1,5105	1,3547		$x - y$	$y = 90$	
30	1,7852	1,6608	1,5126			0	0,6179	
35	1,7858	1,6628						
40	1,7873							

Table 23.

 H^M ; $\log d(i, x:xy)$, as calculated by (46), for $n = 5$.

$x - y = 0$	x	y	$i = 0$	$i = 0,04$	$i = 0,06$	$10^5 \Delta \log d(i, x:xy)$	
	10	10	2,38741	2,38588	2,38512	-153	-76
	15	15	2,47307	2,47453	2,47526	+146	+73
	20	20	2,18377	2,18396	2,18406	19	10
	25	25	2,17333	2,17348	2,17355	15	7
	30	30	2,10513	2,10526	2,10532	13	6
	35	35	2,04444	2,04468	2,04480	24	12
	40	40	1,98232	1,98243	1,98249	11	6

Table 23, continued.

$x - y = 0$	x	y	$i = 0$	$i = 0,04$	$i = 0,06$	$10^6 \Delta \log d(i, x : xy)$	
	45	45	1,89387	1,89426	1,89445	39	10
	50	50	1,78354	1,78384	1,78398	30	14
	55	55	1,65796	1,65834	1,65853	38	19
	60	60	1,50756	1,50797	1,50817	41	20
	65	65	1,35238	1,35257	1,35267	19	10
	70	70	1,19549	1,19571	1,19583	22	12
	75	75	1,01956	1,01925	1,01910	- 31	-15
	80	80	0,86875	0,86800	0,86763	- 75	-37
	85	85	0,74355	0,74181	0,74096	-174	-85
	90	90	0,61992	0,61792	0,61695	-200	-97
$x - y = 10$	20	10	2,38727	2,38574	2,38498	-153	-76
	30	20	2,18377	2,18397	2,18406	+ 20	+ 9
	40	30	2,10513	2,10527	2,10534	14	7
	50	40	1,98233	1,98245	1,98250	12	5
	60	50	1,78368	1,78397	1,78412	29	15
	70	60	1,50802	1,50843	1,50863	41	20
	80	70	1,19615	1,19638	1,19650	23	12
	90	80	0,86370	0,86296	0,86259	- 74	-37
$x - y = 20$	30	10	2,38723	2,38569	2,38492	-154	-77
	40	20	2,18379	2,18398	2,18408	+ 19	+10
	50	30	2,10516	2,10529	2,10536	13	7
	60	40	1,98238	1,98250	1,98255	12	5
	70	50	1,78401	1,78431	1,78445	30	14
	80	60	1,50921	1,50964	1,50985	43	21
	90	70	1,19776	1,19800	1,19812	24	12
$x - y = 30$	40	10	2,38712	2,38559	2,38482	-153	-77
	50	20	2,18381	2,18401	2,18411	+ 20	+10
	60	30	2,10522	2,10535	2,10543	13	8
	70	40	1,98252	1,98263	1,98269	11	6
	80	50	1,78487	1,78518	1,78534	31	16
	90	60	1,51214	1,51257	1,51279	43	22

Table 23, continued.

$x-y=40$	x	y	$i=0$	$i=0,04$	$i=0,06$	$10^6 \Delta \log d(i, x:xy)$	
	50	10	2,38684	2,38529	2,38453	-155	-76
	60	20	2,18390	2,18410	2,18420	+20	+10
	70	30	2,10537	2,10551	2,10559	14	8
	80	40	1,98285	1,98297	1,98302	12	5
	90	50	1,78697	1,78728	1,78744	31	16
$x-y=50$	60	10	2,38614	2,38459	2,38382	-155	-77
	70	20	2,18412	2,18433	2,18443	+21	+10
	80	30	2,10578	2,10593	2,10601	15	8
	90	40	1,98364	1,98375	1,98381	11	6
$x-y=60$	70	10	2,38442	2,38283	2,38204	-159	-79
	80	20	2,18473	2,18496	2,18506	+23	+10
	90	30	2,10678	2,10694	2,10702	16	8
$x-y=70$	80	10	2,37986	2,37821	2,37739	-165	-82
	90	20	2,18624	2,18651	2,18664	+27	+13
$x-y=80$	90	10	2,36881	2,36715	2,36631	-166	-84

Table 24.

H^M ; d and $\log d$, as calculated by (48), for $n=5$ ($t=1$).

x	d	$\log d$	x	d	$\log d$
10	251,6	2,4007	55	45,54	1,6584
15	309,0	2,4900	60	32,21	1,5080
20	149,7	2,1752	65	22,47	1,3516
25	150,6	2,1778	70	15,69	1,1956
30	127,3	2,1048	75	10,401	1,0171
35	110,8	2,0445	80	7,328	0,8650
40	96,36	1,9839	85	5,553	0,7445
45	78,28	1,8937	90	4,198	0,6230
50	60,99	1,7853			

Table 25.

HM ; $i = 0,04$; a_{xy} for $x - y = 20$, as calculated by means of $d(i, x : xy)$ for $x - y = 0$.

x	y	$\log s(i, x)$	$\frac{\log}{d(i, x : xy)}$	$\log (s : d)$	$\frac{a_{x\bar{5}} - a_{xy\bar{5}}}{a_{xy\bar{5}}}$	$a_{x\bar{5}}$	$a_{xy\bar{5}}$	$10^4 d$
90	70	0,4923	1,1957	$\bar{1},2966$	0,1980	2,5967	2,3987	10
85	65	0,6431	1,3526	$\bar{1},2905$	0,1952	3,0594	2,8642	4
80	60	0,7330	1,5080	$\bar{1},2250$	0,1679	3,4341	3,2662	7
75	55	0,8060	1,6583	$\bar{1},1477$	0,1405	3,7807	3,6402	3
70	50	0,8579	1,7838	$\bar{1},0741$	0,1186	4,0624	3,9438	1
65	45	0,8877	1,8943	$\bar{2},9934$	0,09849	4,23461	4,13612	0,6
60	40	0,9070	1,9824	$\bar{2},9246$	0,08407	4,35343	4,26936	0,3
55	35	0,9198	2,0447	$\bar{2},8751$	0,07501	4,43440	4,35939	0,2
50	30	0,9274	2,1053	$\bar{2},8221$	0,06639	4,48350	4,41711	-0,1
45	25	0,9324	2,1735	$\bar{2},7589$	0,05740	4,51636	4,45896	0,1
40	20	0,9356	2,1840	$\bar{2},7516$	0,05644	4,53725	4,48081	-0,1
35	15	0,9374	2,4745	$\bar{2},4629$	0,02903	4,54961	4,52058	0,1
30	10	0,9390	2,3859	$\bar{2},5531$	0,03574	4,56007	4,52433	-0,1
x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy\bar{5}}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^4 d$		
90	70	35,745	85,742	88,240	2,4686	10		
85	65	208,840	598,16	686,40	3,2867	6		
80	60	779,499	2546,0	3232,4	4,1468	8		
75	55	1976,30	7194,1	10426,5	5,2758	6		
70	50	3901,41	15386,4	25812,9	6,6163	4		
65	45	6576,03	27199,2	53012,1	8,0614	4		
60	40	10089,0	43073,6	96085,7	9,5238	3		
55	35	14543,0	63398,6	159484,3	10,9664	1		
50	30	20150,2	89005,6	248489,9	12,3319	1		
45	25	27200,5	121286	369776	13,5945	0		
40	20	36135,0	161914	531690	14,7140	1		
35	15	47057,9	212729	744419	15,8192	0		
30	10	60709,6	274670	1019089	16,7863	0		

Table 26.

H^M ; $i = 0,04$; a_{xy} for $x - y = 0$, as calculated by means of $d(i, 1 : y)$.

x	y	$\log s(i, x)$	$\log d(i, 1 : y)$	$\log(s : d)$	$\frac{a_{x\bar{5}} - a_{xy\bar{5}}}{a_{xy\bar{5}}}$	$a_{x\bar{5}}$	$a_{xy\bar{5}}$	$10^4 \delta$
90	90	0,4923	0,6411	1,8512	0,7099	2,5967	1,8868	-390
85	85	0,6431	0,7532	1,8899	0,7761	3,0594	2,2833	-207
80	80	0,7330	0,8716	1,8614	0,7268	3,4341	2,7073	-59
75	75	0,8060	1,0202	1,7858	0,6106	3,7807	3,1701	-15
70	70	0,8579	1,1953	1,6626	0,4598	4,0624	3,6026	4
65	65	0,8877	1,3523	1,5354	0,3431	4,2346	3,8915	2
60	60	0,9070	1,5076	1,3994	0,2508	4,3534	4,1026	2
55	55	0,9198	1,6581	1,2617	0,1827	4,4344	4,2517	1
50	50	0,9274	1,7837	1,1437	0,1392	4,4835	4,3443	0
45	45	0,9324	1,8941	1,0383	0,1092	4,5164	4,4072	0
40	40	0,9356	1,9824	2,9532	0,08978	4,53725	4,44747	0,0
35	35	0,9374	2,0446	2,8928	0,07812	4,54961	4,47149	0,1
30	30	0,9390	2,1052	2,8338	0,06821	4,56007	4,49186	0,2
25	25	0,9405	2,1735	2,7670	0,05848	4,57022	4,51174	0,0
20	20	0,9407	2,1839	2,7568	0,05712	4,57164	4,51452	0,1
15	15	0,9448	2,4744	2,4704	0,02954	4,60005	4,57051	0,1
10	10	0,9441	2,3860	2,5581	0,03615	4,59332	4,55717	0,1

x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy\bar{5}}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^4 \delta$
90	90	0,62475	1,1788	1,1838	1,8948	-391
85	85	10,483	23,936	25,120	2,3963	-231
80	80	84,185	227,91	253,03	3,0056	-87
75	75	348,386	1104,42	1357,45	3,8964	-36
70	70	933,390	3362,63	4720,08	5,0569	-9
65	65	1898,77	7389,06	12109,1	6,3773	-2
60	60	3294,04	13514,1	25623,2	7,7787	0
55	55	5116,58	21754,2	47377,4	9,2596	2
50	50	7442,39	32332,0	79709,4	10,7102	1
45	45	10394,1	45808,9	125518,3	12,0759	1
40	40	14102,5	62720,4	188238,7	13,3479	0
35	35	18865,3	84356,0	272594,7	14,4495	1

x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy} \bar{s}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^4 \bar{s}$
30	30	24898,9	111842	384437	15,4399	0
25	25	32486,4	146570	531007	16,3455	0
20	20	42256,3	190767	721774	17,0809	0
15	15	53571,7	244850	966624	18,0436	- 1
10	10	67556,4	307866	1274490	18,8656	0

Table 27.

H^M ; $i = 0,04$; a_{xy} for $x - y = 30$, as calculated by means of $d(i, 1 : y)$.

x	y	$\log s(i, x)$	$\log d(i, 1 : y)$	$\log (s : d)$	$\frac{a_{x\bar{s}} - a_{xy\bar{s}}}{a_{xy\bar{s}}}$	$a_{x\bar{s}}$	$a_{xy\bar{s}}$	$10^4 \bar{s}$
90	60	0,4923	1,5076	$\bar{2},9847$	0,0965	2,5967	2,5002	10
85	55	0,6431	1,6581	$\bar{2},9850$	0,0966	3,0594	2,9628	6
80	50	0,7330	1,7837	$\bar{2},9493$	0,0890	3,4341	3,3451	4
75	45	0,8060	1,8941	$\bar{2},9119$	0,0816	3,7807	3,6991	2
70	40	0,8579	1,9824	$\bar{2},8755$	0,0750	4,0624	3,9874	- 1
65	35	0,8877	2,0446	$\bar{2},8431$	0,0697	4,2346	4,1649	1
60	30	0,9070	2,1052	$\bar{2},8018$	0,0633	4,3534	4,2901	0
55	25	0,9198	2,1735	$\bar{2},7463$	0,0558	4,4344	4,3786	0
50	20	0,9274	2,1839	$\bar{2},7435$	0,0554	4,4835	4,4281	0
45	15	0,9324	2,4744	$\bar{2},4580$	0,0287	4,5164	4,4877	0
40	10	0,9356	2,3860	$\bar{2},5496$	0,0355	4,5372	4,5017	1

x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy} \bar{s}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^4 \bar{s}$
90	60	81,699	204,26	211,49	2,5886	11
85	55	417,09	1235,75	1447,2	3,4698	8
80	50	1425,52	4768,5	6215,7	4,3603	6
75	45	3427,08	12677,1	18893	5,5129	3
70	40	6534,02	26054	44947	6,8789	1
65	35	10778,8	44893	89840	8,3349	1
60	30	16310,0	69972	159812	9,7984	0
55	25	23218,9	101666	261478	11,2614	1
50	20	31937,6	141423	402901	12,6153	0
45	15	42497,3	190715	593616	13,9683	1
40	10	55588,1	250241	843857	15,1805	1

Table 28.

H^M ; $i=0,04$; a_{xy} for $x-y=0$, as calculated by means of d from (48).

x	y	$\log s(i, x)$	$\log d$	$\log (s : d)$	$\frac{ax\bar{5}}{axy\bar{5}} -$	$ax\bar{5}$	$axy\bar{5}$	$10^4 \delta$
90	90	0,4923	0,6230	$\bar{1},8693$	0,7401	2,5967	1,8566	-88
85	85	0,6431	0,7445	$\bar{1},8986$	0,7918	3,0594	2,2676	-50
80	80	0,7330	0,8650	$\bar{1},8680$	0,7379	3,4341	2,6962	52
75	75	0,8060	1,0171	$\bar{1},7889$	0,6151	3,7807	3,1656	30
70	70	0,8579	1,1956	$\bar{1},6623$	0,4595	4,0624	3,6029	1
65	65	0,8877	1,3516	$\bar{1},5361$	0,3437	4,2346	3,8909	8
60	60	0,9070	1,5080	$\bar{1},3990$	0,2506	4,3534	4,1028	0
55	55	0,9198	1,6584	$\bar{1},2614$	0,1826	4,4344	4,2518	0
50	50	0,9274	1,7853	$\bar{1},1421$	0,1387	4,4835	4,3448	-5
45	45	0,9324	1,8937	$\bar{1},0387$	0,1093	4,5164	4,4071	1
40	40	0,9356	1,9839	$\bar{2},9517$	0,0895	4,5372	4,4477	-2
35	35	0,9374	2,0445	$\bar{2},8929$	0,0781	4,5496	4,4715	0
30	30	0,9390	2,1048	$\bar{2},8342$	0,0683	4,5601	4,4918	1
25	25	0,9405	2,1778	$\bar{2},7627$	0,0579	4,5702	4,5123	-6
20	20	0,9407	2,1752	$\bar{2},7655$	0,0583	4,5716	4,5133	-12
15	15	0,9448	2,4900	$\bar{2},4548$	0,0285	4,6001	4,5716	11
10	10	0,9441	2,4007	$\bar{2},5434$	0,0349	4,5933	4,5584	-12
x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy\bar{5}}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^3 \delta$		
90	90	0,62475	1,160	1,165	1,865	-9		
85	85	10,483	23,771	24,936	2,379	-6		
80	80	84,185	226,98	251,92	2,992	5		
75	75	348,386	1102,85	1354,77	3,889	4		
70	70	933,390	3362,9	4717,7	5,054	2		
65	65	1898,77	7387,9	12105,6	6,375	2		
60	60	3294,04	13514,8	25620,4	7,778	1		
55	55	5116,58	21755	47375	9,259	1		
50	50	7442,39	32336	79711	10,710	0		
45	45	10394,1	45808	125519	12,076	0		
40	40	14102,5	62724	188243	13,348	0		
35	35	18865,3	84356	272599	14,450	0		

x	y	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy} \overline{5}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^3 \delta$
30	30	24898,9	111841	384440	15,440	0
25	25	32486,4	146588	531028	16,346	0
20	20	42256,3	190716	721744	17,080	1
15	15	53571,7	244908	966652	18,044	0
10	10	67556,4	307949	1274601	18,867	- 1

Table 26 gives a_{xy} at 4 % for $x - y = 0$, by means of $d(4\%, 1:y)$.

Table 27 gives a_{xy} at 4 % for $x - y = 30$, by means of $d(4\%, 1:y)$.

Table 28 gives a_{xy} at 4 % for $x - y = 0$, by means of d from (48), Table 24.

As to the calculation of a_{xy} by means of $d(i, 1:y)$, instead of $d(i, x:xy)$, the result, if necessary, can be improved, it being possible to derive $d(i, x:xy)$ from $d(i, 1:y)$ with good approximation. This will be shown in article 29.

29. *Relation between $d(i, x:xy)$ and $d(i, 1:y)$ for small values of n .* As shown in article 27 $\log d(i, 1:y)$, for $n = 5$, is nearly a linear function of i . We write

$$(49) \quad \log d(j, 1:y) = \log d(i, 1:y) + g(1:y)(j-i),$$

where $g(1:y)$ is practically independent of i .

We have from (45)

$$(50) \quad a_{xy\overline{5}}^i - a_{y\overline{5}}^i = -\frac{s(i, y)}{d(i, y:xy)} = -\frac{s(i, y)}{\frac{1}{q_{x+t}} + t};$$

$a_{xy\overline{5}}^i$ is equal to $a_{y\overline{5}}^{i'}$, taken at a rate of interest i' , which may be written

$$(51) \quad i' = i + (1+i)h(x),$$

where $h(x)$ may be easily found with a good approximation. We have from (29) and (35)

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$$(52) \quad a_{y\bar{5}}^{i'} - a_{y\bar{5}}^i = \frac{-s(i, y)}{\frac{1+i}{i'-i} + t(i)},$$

and comparing now (50) and (52) we find

$$\frac{1}{q_{x+t}} + t = \frac{1}{h(x)} + t(i),$$

where $t = t(i) - 1$ approximately. (Table 15). Hence

$$(53) \quad h(x) = \frac{q_{x+t}}{p_{x+t}}.$$

For

$$a_{x\bar{5}}^i = a_{\bar{5}}^{i'}$$

and

$$s(i, x) = s(i')$$

we find the same value of i' (approximately).

Comparing now

$$a_{xy\bar{5}}^i - a_{x\bar{5}}^i = -\frac{s(i, x)}{d(i, x : xy)}$$

with

$$a_{y\bar{5}}^{i'} - a_{\bar{5}}^{i'} = -\frac{s(i')}{d(i', 1 : y)}$$

we find

$$d(i, x : xy) = d(i', 1 : y)$$

whence by (49) and (51)

$$(54) \quad \log d(i, x : xy) = \log d(i, 1 : y) + g(1 : y) h(x) (1 + i).$$

We give below, according to H^M , some examples of the application of the formula (54).

Table 29 contains the values of $g(1:x)$ and $h(x)$.

$g(1:x)$ has been found from $\log d(i, 1:x)$ for $i=0,06$ and $i=0$ (Table 18).

In $h(x) = \frac{q_{x+t}}{p_{x+t}}$ t has been put $= 1$, except for $x = 90, 85$ and 80 , where t has been put $= 0,7, 0,8$ and $0,9$ (Table 15); the interpolation (by second differences for $x = 90$, by first differences for $x = 85$ and 80) has been made in $\log \frac{q_x}{p_x}$.

Table 30 gives some values of $\log d(i, x:xy)$, at $i = 0,04$, as calculated by (54).

Table 29.

H^M ; values of $g(1:x)$ and $h(x)$.

x	$g(1:x)$	$h(x)$	x	$g(1:x)$	$h(x)$
90	-0,047	0,433	45	0,010	0,013
85	-0,040	0,278	40	0,003	0,011
80	-0,018	0,186	35	0,006	0,009
75	-0,007	0,119	30	0,003	0,008
70	0,005	0,073	25	0,004	0,007
65	0,005	0,049	20	0,005	0,007
60	0,010	0,033	15	0,036	0,003
55	0,009	0,023	10	-0,038	0,004
50	0,007	0,017			

30. Calculation of a (or N) for 3, 4 etc. joint lives for quinquennial ages. Writing

$$a_{xyz\overline{5}|} - a_{xy\overline{5}|} = - \frac{s(i, xy)}{d(i, xy:xyz)}$$

$$a_{xyz\overline{5}|} - a_{xy\overline{5}|} = - \frac{s(i, xyz)}{d(i, xyz:xyz\overline{5}|)}$$

etc.,

Table 30.

 H^M ; $i = 0,04$; $\log d(i, x : xy)$, as calculated by (54).

	x	y	$\log d(i, 1 : y)$	$g(1 : y)h(x)(1 + i)$	$\log d(i, x : xy)$	10^{+5}
$x - y = 0$	90	90	0,6411	-0,0212	0,6199	-20
	85	85	0,7532	-0,0115	0,7417	1
	80	80	0,8716	-0,0035	0,8681	-1
	75	75	1,0202	-0,0008	1,0194	-2
	70	70	1,1953	0,0004	1,1957	0
	65	65	1,3523	0,0003	1,3526	0
	60	60	1,5076	0,0003	1,5079	1
	55	55	1,6581	0,0002	1,6583	0
	50	50	1,7837	0,0001	1,7838	0
$x - y = 20$	90	70	1,1953	0,0023	1,1976	4
	80	60	1,5076	0,0019	1,5095	1
	70	50	1,7837	0,0005	1,7842	1
	60	40	1,9824	0,0001	1,9825	0
	50	30	2,1052	0,0001	2,1053	0
$x - y = 60$	90	30	2,1052	0,0014	2,1066	3
	80	20	2,1839	0,0010	2,1849	1
	70	10	2,3860	-0,0029	2,3831	-3

For $x - y = 0$ the corresponding $a_{xy\bar{5}}$ and a_{xy} have been calculated in Table 33.

we have (approximately)

$$d(i, xy : xyz) = d(i', y : yz)$$

$$d(i, xyz : xyzu) = d(i', yz : yzu)$$

etc.,

where i' is equal to

$$i' = i + (1 + i)h(x)$$

from (51).

Further we have (approximately)

$$(55) \quad \begin{cases} \log d(j, x : xy) - \log d(i, x : xy) = g(1 : y)(j - i) \\ \log d(j, xy : xyz) - \log d(i, xy : xyz) = g(1 : z)(j - i) \\ \log d(j, xyz : xyzu) - \log d(i, xyz : xyzu) = g(1 : u)(j - i) \end{cases}$$

etc. (for a limited number of ages).

Hence

$$(55a) \quad \begin{cases} \log d(i, xy : xyz) = \log d(i', y : yz) = \\ \quad = \log d(i, y : yz) + g(1 : z)h(x)(1 + i) \\ \log d(i, xyz : xyzu) = \log d(i', yz : yzu) = \\ \quad = \log d(i, yz : yzu) + g(1 : u)h(x)(1 + i) \end{cases}$$

etc.

As to the determination of $s(i, xy)$, $s(i, xyz)$ etc. we may proceed in the following manner.

We have

$$a_{x\bar{5}} - a_{\bar{5}} = -\frac{s(i)}{d(i, 1 : x)};$$

writing now

$$(56) \quad a_{\bar{5}} - a_{x\bar{5}} = \frac{s(i, x)}{d(i, x : 1)}$$

where $d(i, x : 1)$ may be called divisor for transformation of $a_{x\bar{5}}$ into $a_{\bar{5}}$, we get

$$(57) \quad \frac{s(i, x)}{s(i)} = \frac{d(i, x : 1)}{d(i, 1 : x)}.$$

We find in this manner

$$(57a) \quad \begin{cases} \frac{s(i, xy)}{s(i, x)} = \frac{d(i, xy : x)}{d(i, x : xy)} \\ \frac{s(i, xyz)}{s(i, xy)} = \frac{d(i, xyz : xy)}{d(i, xy : xyz)} \\ \text{etc.} \end{cases}$$

Now $\log d(i, x:1)$, $\log d(i, xy:y)$ etc., for $n=5$, are nearly linear functions of i ; we write

$$(58) \quad \log d(j, x:1) = \log d(i, x:1) + g(x:1)(j-i),$$

where $g(x:1)$ is practically independent of i .

Further we have (approximately)

$$(59) \quad \begin{aligned} \log d(j, xy:x) - \log d(i, xy:x) &= g(y:1)(j-i) \\ \log d(j, xyz:xy) - \log d(i, xyz:xy) &= g(z:1)(j-i) \\ \log d(j, xyzu:xyz) - \log d(i, xyzu:xyz) &= g(u:1)(j-i) \end{aligned}$$

etc. (for a limited number of ages).

For

$$i' = i + (1+i)h(x)$$

we have (approximately)

$$(60) \quad \begin{aligned} s(i, x) &= s(i') \\ s(i, xy) &= s(i', y) \\ s(i, xyz) &= s(i', yz) \\ &\text{etc.,} \end{aligned}$$

and substituting these values in (57) and (57a), we find

$$\begin{aligned} \frac{s(i, xy)}{s(i, x)} &= \frac{s(i', y)}{s(i')} = \frac{d(i', y:1)}{d(i', 1:y)} \\ \frac{s(i, xyz)}{s(i, xy)} &= \frac{s(i', yz)}{s(i', y)} = \frac{d(i', yz:y)}{d(i', y:yz)} \\ &\text{etc.} \end{aligned}$$

Hence

$$\begin{aligned} \log \frac{s(i, xy)}{s(i, x)} &= \log d(i', y:1) - \log d(i', 1:y) \\ &= \log d(i, y:1) + g(y:1)h(x)(1+i) \\ &\quad - \log d(i, 1:y) - g(1:y)h(x)(1+i) = \\ &= \log \frac{s(i, y)}{s(i)} + [g(y:1) - g(1:y)]h(x)(1+i); \end{aligned}$$

writing now

$$(61) \quad g(y:1) - g(1:y) = g(y)$$

we find

$$(62) \quad \left\{ \begin{array}{l} \log s(i, xy) = \log s(i, x) + \log s(i, y) - \log s(i) + \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + g(y)h(x)(1+i), \\ \text{and in the same manner} \\ \log s(i, xyz) = \log s(i, xy) + \log s(i, yz) - \log s(i, y) + \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + g(z)h(x)(1+i) \\ \text{etc.} \end{array} \right.$$

Starting from $\frac{s(i, xy)}{s(i, y)}$ (instead of $\frac{s(i, xy)}{s(i, x)}$) we find

$$\log s(i, xy) = \log s(i, y) + \log s(i, x) - \log s(i) + \\ + g(x)h(y)(1+i),$$

and we shall then have (theoretically)

$$g(y)h(x) = g(x)h(y)$$

or

$$(63) \quad \frac{h(x)}{g(x)} = \frac{h(y)}{g(y)} = \text{constant}$$

for all ages, and accordingly for all mortality tables. For $n=5$ this constant is about 2.2.

However, the numbers h and g being only approximate, the equation (63), too, will be only approximate, and we then replace the number

$$g(y)h(x) \text{ or } g(x)h(y)$$

by the mean of them and write

$$(64) \quad \frac{1}{2}[g(x)h(y) + g(y)h(x)] = k(xy);$$

hence the formulae (62) take the form

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$$(65) \quad \left\{ \begin{array}{l} \log s(i, xy) = \log s(i, x) + \log s(i, y) - \log s(i) \\ \quad \quad \quad + k(xy)(1+i), \\ \log s(i, xyz) = \log s(i, xy) + \log s(i, yz) - \log s(i, y) \\ \quad \quad \quad + k(xz)(1+i), \\ \log s(i, xyzu) = \log s(i, xyz) + \log s(i, yzu) - \log s(i, yz) \\ \quad \quad \quad + k(xu)(1+i), \\ \text{etc.} \end{array} \right.$$

By substitution we get

$$\begin{aligned} \log s(i, xyz) &= \log s(i, xy) + \log s(i, z) - \log s(i) \\ &\quad + [k(xz) + k(yz)](1+i) = \end{aligned}$$

Table 31.

H^M ; $\log d(i, x:1)$ for $n=5$, as calculated by (56).

x	$i=0$	$i=0,04$	$i=0,06$	$10^5 \Delta \log d(i, x:1)$	
10	2,38232	2,38085	2,38013	-147	-72
15	2,46838	2,46992	2,47067	+154	+75
20	2,17508	2,17539	2,17555	31	16
25	2,16445	2,16472	2,16485	27	13
30	2,09472	2,09499	2,09513	27	14
35	2,03240	2,03280	2,03300	40	20
40	1,96847	1,96877	1,96891	30	14
45	1,87664	1,87726	1,87757	62	31
50	1,76129	1,76189	1,76218	60	29
55	1,62786	1,62866	1,62906	80	40
60	1,46432	1,46534	1,46584	102	50
65	1,28975	1,29084	1,29137	109	53
70	1,10230	1,10392	1,10472	162	80
75	0,87511	0,87700	0,87794	189	94
80	0,65257	0,65531	0,65665	274	134
85	0,44388	0,44709	0,44867	321	158
90	0,17766	0,18415	0,18731	649	316

$$= \log s(i, x) + \log s(i, y) + \log s(i, z) - 2 \log s(i) \\ + [k(xy) + k(xz) + k(yz)](1 + i)$$

etc.

However, it will generally be more convenient to use the formulae (65). Calculating, for instance, $a_{xy\overline{5}|}$ from $a_{xy\overline{5}|}$, and $a_{yzu\overline{5}|}$ from $a_{y\overline{5}|}$, we get $\log s(i, xy)$ and $\log s(i, yz)$, whence $\log s(i, xyz)$ etc.

The suppositions (55) and (59), for old ages (high values of q), hold only for a few lives, and the errors, in these cases, generally increase very rapidly, as the number of lives increases. For these ages we must then calculate the annuities directly, and comparing now the true values with the values obtained by the summary method, we find the point from which this method may be safely used.

Table 32.

HM ; $g(x:1)$, $g(1:x)$, $g(x)$ and $h(x)$.

x	$g(x:1)$	$g(1:x)$	$g(x)$	$h(x)$
90	0,161	-0,047	0,208	0,433
85	0,080	-0,040	0,120	0,278
80	0,068	-0,018	0,086	0,186
75	0,047	-0,007	0,054	0,119
70	0,040	0,005	0,035	0,073
65	0,027	0,005	0,022	0,049
60	0,025	0,010	0,015	0,033
55	0,020	0,009	0,011	0,023
50	0,015	0,007	0,008	0,017
45	0,016	0,010	0,006	0,013
40	0,008	0,003	0,005	0,011
35	0,010	0,006	0,004	0,009
30	0,007	0,003	0,004	0,008
25	0,007	0,004	0,003	0,007
20	0,008	0,005	0,003	0,007
15	0,038	0,036	0,002	0,003
10	-0,037	-0,038	0,001	0,004

31. *Application of the methods of the preceding article.* We give below, according to HM , some examples of the application of the above methods.

Table 31 gives the values of $\log d(i, x:1)$ as calculated by (56) for $n=5$, at $i=0$, $i=0,04$ and $i=0,06$.

Table 32 gives the values of $g(x:1)$ and $g(x)$. The values of $g(x:1)$ have been found from $\log d(i, x:1)$ for $i=0,06$ and $i=0$. We add the values of $g(1:x)$ and of $h(x)$ from Table 29.

Tables 33–36 give the joint-life annuities on 2, 3, 4 and 5 lives (of equal ages).

We have here used the same mortality table for all the lives. Of course, we may also use different mortality tables for the different lives.

Table 33.

HM ; $i=0,04$; $x=y$; a_{xy} as calculated by means of $\log d(i, x:xy)$ from (54).

x	$\log s(i, x)$	$\log d(i, x:xy)$ from (54)	$\log (s:d)$	$s:d$	$a_{x\overline{5} }$	$a_{xy\overline{5} }$	$10^4 \bar{c}$
90	0,4923	0,6199	1,8724	0,7454	2,5967	1,8513	—35
85	0,6431	0,7417	1,9014	0,7969	3,0594	2,2625	1
80	0,7330	0,8681	1,8649	0,7326	3,4341	2,7015	—1
75	0,8060	1,0194	1,7866	0,6117	3,7807	3,1690	—4
70	0,8579	1,1957	1,6622	0,4594	4,0624	3,6030	0
65	0,8877	1,3526	1,5351	0,3429	4,2346	3,8917	0
60	0,9070	1,5079	1,3991	0,2507	4,3534	4,1027	1
55	0,9198	1,6583	1,2615	0,1826	4,4344	4,2518	0
50	0,9274	1,7838	1,1436	0,1392	4,4835	4,3443	0
45	0,9324	1,8943	1,0381	0,1091	4,5164	4,4073	—1
40	0,9356	1,9824	2,9532	0,08978	4,53725	4,44747	0,0
35	0,9374	2,0447	2,8927	0,07811	4,54961	4,47150	0,0
30	0,9390	2,1053	2,8337	0,06819	4,56007	4,49188	0,0
25	0,9405	2,1735	2,7670	0,05846	4,57022	4,51174	0,0
20	0,9407	2,1840	2,7567	0,05711	4,57164	4,51453	0,0
15	0,9448	2,4745	2,4703	0,02953	4,60005	4,57052	0,0
10	0,9441	2,3859	2,5582	0,03616	4,59332	4,55716	0,0

x	$10^{-5} l_x D_y$	$10^{-5} l_x D_y a_{xy} \overline{5}$	$10^{-5} \Sigma l_x D_y$	a_{xy}	$10^4 \delta$
90	0,624748	1,1566	1,1616	1,8593	-36
85	10,48299	23,718	24,879	2,3733	-1
80	84,18507	227,43	252,31	2,9971	-2
75	348,3867	1104,0	1356,3	3,8931	-3
70	933,3901	3363,0	4719,3	5,0561	-1
65	1898,777	7389,5	12108,8	6,3772	-1
60	3294,039	13514	25623	7,7786	1
55	5116,576	21755	47378	9,2597	1
50	7442,390	32332	79710	10,7103	0
45	10394,09	45809	125519	12,0760	0
40	14102,54	62721	188240	13,3480	-1
35	18865,29	84356	272596	14,4496	0
30	24898,95	111843	384439	15,4400	-1
25	32486,43	146570	531009	16,3456	-1
20	42256,25	190767	721776	17,0809	0
15	53571,66	244850	966626	18,0436	-1
10	67556,42	307865	1274491	18,8656	0

Table 34.

H^M ; $i = 0,04$; $x = y = z$; a_{xyz} as calculated by the method shown in article 30.

x	$\log s(i, xy)$ from (65)	$\log d(i, xy:xyz)$ from (55a)	$\log(s:d)$	$s:d$	$a_{xy\overline{5}}$ from table 33	$a_{xyz\overline{5}}$	$10^4 \delta$
90	0,1291	0,5987	1,5304	0,3391	1,8513	1,5122	-51
85	0,3717	0,7302	1,6415	0,4380	2,2625	1,8245	12
80	0,5333	0,8646	1,6687	0,4663	2,7015	2,2352	-12
75	0,6696	1,0185	1,6511	0,4478	3,1690	2,7212	-10
70	0,7692	1,1960	1,5732	0,3743	3,6030	3,2287	-1
65	0,8274	1,3528	1,4746	0,2983	3,8917	3,5934	0
60	0,8652	1,5083	1,3569	0,2275	4,1027	3,8752	1
55	0,8906	1,6585	1,2321	0,1706	4,2518	4,0812	0
50	0,9058	1,7840	1,1218	0,1323	4,3443	4,2120	0

x	$\log s(i, xy)$ from (65)	$\log d(i, xy:xyz)$ from 55a)	$\log (s : d)$	$s : d$	$\frac{axy\bar{5}}{\text{from table 33}}$	$axy\bar{5}$	$10^4\bar{6}$
45	0,9156	1,8944	$\bar{1},0212$	0,1050	4,4073	4,3023	— 1
40	0,9220	1,9825	$\bar{2},9395$	0,08700	4,44747	4,36047	— 0,1
35	0,9256	2,0447	$\bar{2},8809$	0,07602	4,47150	4,39548	0,0
30	0,9288	2,1053	$\bar{2},8235$	0,06661	4,49188	4,42527	0,0
25	0,9318	2,1735	$\bar{2},7583$	0,05732	4,51174	4,45442	0,0
20	0,9322	2,1840	$\bar{2},7482$	0,05601	4,51453	4,45852	0,1
15	0,9403	2,4746	$\bar{2},4657$	0,02922	4,57052	4,54130	0,1
10	0,9389	2,3857	$\bar{2},5532$	0,03575	4,55716	4,52141	0,1
x	$10^{-10}l_xl_yD_x$	$10^{-10}l_xl_yD_xaxy\bar{5}$	$10^{-10}\Sigma l_xl_yD_x$	$axy\bar{5}$	$10^4\bar{6}$		
90	0,0091213	0,013793	0,013799	1,5128	—50		
85	0,56839	1,0370	1,0508	1,8487	12		
80	11,7270	26,212	27,263	2,3248	—11		
75	89,5040	243,56	270,82	3,0258	—11		
70	355,846	1148,9	1419,7	3,9896	— 2		
65	936,040	3363,6	4783,3	5,1101	— 1		
60	1939,07	7514,3	12297,6	6,3420	1		
55	3403,19	13889	26187	7,6948	0		
50	5412,55	22798	48985	9,0503	— 2		
45	8098,97	34844	83829	10,3506	— 2		
40	11604,1	50599	134428	11,5845	— 1		
35	16277,2	71546	205974	12,6541	0		
30	22375,4	99017	304991	13,6306	0		
25	30232,2	134667	439658	14,5427	0		
20	40660,2	181284	620942	15,2715	0		
15	52620,2	238964	859906	16,3417	0		
10	67556,4	305450	1165356	17,2501	0		

Table 35.

H^M ; $i = 0,04$; $x = y = z = u$; a_{xyzu} as calculated by the method shown in article 30.

x	$\log s(i, xyz)$ from (65)	$\log d(i, xyz : xyzu)$ from (55 a)	$\log (s : d)$	$s : d$	$a_{xyz} \bar{5}$ from table 34	$a_{xyzu} \bar{5}$	$10^6 \bar{d}$
90	1,8596	0,5775	1,2821	0,1914	1,5122	1,3208	45
85	0,1350	0,7187	1,4163	0,2608	1,8245	1,5637	40
80	0,3503	0,8612	1,4891	0,3084	2,2352	1,9268	-30
75	0,5400	1,0176	1,5224	0,3330	2,7212	2,3882	-21
70	0,6832	1,1964	1,4868	0,3068	3,2287	2,9219	-6
65	0,7681	1,3531	1,4150	0,2600	3,5934	3,3334	-2
60	0,8240	1,5086	1,3154	0,2067	3,8752	3,6685	1
55	0,8617	1,6588	1,2029	0,1595	4,0812	3,9217	-1
50	0,8842	1,7841	1,1001	0,1259	4,2120	4,0861	-1
45	0,8989	1,8945	1,0044	0,1010	4,3023	4,2013	-1
40	0,9085	1,9825	2,9260	0,08433	4,36047	4,27614	-0,2
35	0,9139	2,0448	2,8691	0,07398	4,39548	4,32150	0,0
30	0,9186	2,1053	2,8133	0,06506	4,42527	4,36021	0,1
25	0,9231	2,1735	2,7496	0,05618	4,45442	4,39824	0,1
20	0,9237	2,1840	2,7397	0,05492	4,45852	4,40360	0,2
15	0,9359	2,4747	2,4612	0,02892	4,54130	4,51238	0,2
10	0,9338	2,3855	2,5483	0,03534	4,52141	4,48607	0,1
x	$10^{-15} l_x l_y l_z Du$	$10^{-15} l_x l_y l_z Du$ $a_{xyzu} \bar{5}$	$10^{-15} \Sigma l_x l_y l_z Du$	a_{xyzu}	$10^6 \bar{d}$		
90	0,00013317	0,00017589	0,00017590	1,3209	45		
85	0,030818	0,048190	0,048366	1,5694	40		
80	1,63357	3,1476	3,1960	1,9565	-30		
75	22,9945	54,915	58,111	2,5272	-23		
70	135,663	396,39	454,50	3,3502	-10		
65	461,440	1538,2	1992,7	4,3184	-6		
60	1141,45	4187,4	6180,1	5,4143	-2		
55	2263,56	8877,0	15057,1	6,6520	-2		
50	3936,33	16084	31141	7,9112	-1		
45	6310,64	26513	57654	9,1360	-2		

x	$10^{-15} l_x l_y l_z Du$	$10^{-15} l_x l_y l_z Du$ $a_{xyzu} \overline{5}$	$10^{-15} \Sigma l_x l_y l_z Du$	a_{xyzu}	$10^4 \hat{\gamma}$
40	9548,34	40830	98484	10,3143	- 2
35	14044,1	60692	159176	11,3340	- 1
30	20107,7	87674	246850	12,2764	- 1
25	28134,4	123742	370592	13,1722	- 1
20	39124,5	172289	542881	13,8757	0
15	51685,7	233226	776107	15,0159	0
10	67556,4	303063	1079170	15,9744	- 1

Table 36.

HM ; $i = 0,04$; $x = y = z = u = v$; a_{xyzuv} as calculated by the method shown in article 30.

x	$\log s(i, xyzu)$ from (65)	$\log d(i, xyzu : xyzuv)$ from (55a)	$\log(s : d)$	$s : d$	$a_{xyzuv} \overline{5}$ from table 35	$a_{xyzuv} \overline{5}$	$10^4 \hat{\gamma}$
90	1,6838	0,5563	1,1275	0,1342	1,3208	1,1866	309
85	1,9330	0,7072	1,2258	0,1682	1,5637	1,3955	94
80	0,1839	0,8577	1,3262	0,2119	1,9268	1,7149	- 51
75	0,4173	1,0168	1,4005	0,2515	2,3882	2,1367	- 38
70	0,5998	1,1967	1,4031	0,2530	2,9219	2,6689	- 14
65	0,7100	1,3534	1,3566	0,2273	3,3334	3,1061	- 6
60	0,7833	1,5090	1,2743	0,1880	3,6685	3,4805	- 1
55	0,8330	1,6590	1,1740	0,1493	3,9217	3,7724	0
50	0,8629	1,7842	1,0787	0,1199	4,0861	3,9662	0
45	0,8823	1,8947	2,9876	0,0972	4,2013	4,1041	- 1
40	0,8951	1,9825	2,9126	0,08177	4,27614	4,19437	0,1
35	0,9022	2,0448	2,8574	0,07201	4,32150	4,24949	0,1
30	0,9085	2,1053	2,8032	0,06356	4,36021	4,29665	0,2
25	0,9144	2,1735	2,7409	0,05507	4,39824	4,34317	0,1
20	0,9152	2,1841	2,7311	0,05384	4,40360	4,34976	0,1
15	0,9314	2,4748	2,4566	0,02862	4,51238	4,48376	0,3
10	0,9286	2,3854	2,5437	0,03493	4,48607	4,45114	0,2

x	$10-20 l_x l_y l_z l_u D_v$	$10-20 l_x l_y l_z l_u D_v$ $a_{xyzuv} 5$	$10-20 \sum l_x l_y l_z l_u D_v$	a_{xyzuv}	$10^4 \delta$
90	0,0000019443	0,0000023071	0,0000023071	1,1866	309
85	0,00167095	0,0023318	0,0023341	1,3969	95
80	0,22756	0,39024	0,39257	1,7251	-50
75	5,90751	12,623	13,016	2,2033	-41
70	51,7200	138,04	151,06	2,9207	-20
65	227,476	706,56	857,62	3,7702	-11
60	671,927	2338,6	3196,2	4,7568	-4
55	1505,56	5679,6	8875,8	5,8953	-2
50	2862,74	11354	20230	7,0667	-2
45	4917,19	20181	40411	8,2183	-3
40	7856,76	32954	73365	9,3378	-1
35	12117,4	51493	124858	10,3040	-1
30	18069,8	77640	202498	11,2064	0
25	26182,1	113713	316211	12,0774	-1
20	37646,8	163755	479966	12,7492	0
15	50767,8	227631	707597	13,9379	0
10	67556,4	300703	1008300	14,9253	0

32. Calculation of a_x^{aa} (or N_x^{aa}) for quinquennial ages. Putting

$$p_x^{aa} = (1 - q_x^a)(1 - i_x)$$

we have from (46) and (48)

$$(66) \quad \frac{a_x^{aa}}{a_{x+1}^{aa}} - \frac{a_x^a}{a_{x+1}^a} = \frac{\frac{1}{v} \frac{d a_x^a}{d i}}{\frac{1}{i_{x+1}} + 1} = - \frac{s^a(i, x)}{d(x^a : x^{aa})},$$

and calculating D_x^{aa} for quinquennial ages we find a_x^{aa} (or N_x^{aa}) for the same ages.

We give below (Table 37) the values of a_x^{aa} , according to O^M , 4 %, and K^{I*} , where $q_x^a = q_x$ from O^M , and $i_x = \bar{i}_x$ from K^I . Hence $a_x^a = a_x$ etc.

* K^I = KARUP, Invaliditätsverhältnisse unter dem Nicht-Fahrpersonal deutscher Eisenbahnen 1877-89 (Reform des Rechnungswesens der Gothaer Lebensversicherungsbank a. G. Tabelle 19).

Table 37.

$OM; i = 0.04; K^I$. Calculation of a_x^{aa} for quinquennial ages.
 $N_{70}^{aa} = 1482.3$.

x	$\log s(i, x)$	$\log d(x : x^{aa})$	$\log (s : d)$	$a_{x\overline{5}} - a_{x\overline{5}}^{aa}$	$a_{x\overline{5}}$	$a_{x\overline{5}}^{aa}$	$10^4\delta$
65	0.8892	0.9116	$\overline{1},9776$	0,9497	4,2442	3,2945	107
60	0.9083	1,1556	$\overline{1},7527$	0,5658	4,3618	3,7960	23
55	0.9206	1,4661	$\overline{1},4545$	0,2847	4,4395	4,1548	- 18
50	0.9284	1,7527	$\overline{1},1757$	0,1498	4,4901	4,3403	- 6
45	0.9334	2,0310	$\overline{2},9024$	0,0799	4,5232	4,4433	- 5
40	0.9368	2,2739	$\overline{2},6629$	0,0460	4,5456	4,4996	- 2
35	0.9392	2,5540	$\overline{2},3852$	0,0243	4,5618	4,5375	0
30	0.9412	2,9106	$\overline{2},0306$	0,0107	4,5751	4,5644	- 3
25	0.9427	3,1488	$\overline{3},7939$	0,0062	4,5856	4,5794	- 1
20	0.9438	3,3668	$\overline{3},5770$	0,0038	4,5930	4,5892	0

x	D_x^{aa}	$D_x^{aa} \cdot a_{x\overline{5}}^{aa}$	N_x^{aa}	a_x^{aa}	$10^3\delta$
65	1671.3	5506.1	6988.4	4,181	11
60	3815.9	14485	21473	5,627	7
55	6484.6	26942	48415	7,466	3
50	9554.9	41471	89886	9,407	1
45	13095	58185	148071	11,307	1
40	1727 :	77717	225788	13,072	0
35	22244	100932	326720	14,688	0
30	28168	128570	455290	16,163	0
25	35321	161749	617039	17,469	0
20	44020	202017	819056	18,606	0

33. Calculation of S_x for quinquennial ages. We have

$$(Ia)_{x\overline{n}} = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x}$$

$$(Ia)_{\overline{n}} = \sum_0^{n-1} (n+1)v^n,$$

and proceeding as shown in article 27 we find

$$(67) \quad (Ia)_{x\overline{n}|} - (Ia)_{\overline{n}|} = \frac{\frac{1}{v} \frac{d(Ia)_{\overline{n}|}}{di}}{\frac{1}{qx+t} + t} = -\frac{\sigma(i)}{\delta(i, 1:x)},$$

where

$$\sigma(i) = -\frac{1}{v} \frac{d(Ia)_{\overline{n}|}}{di} = \sum_1^{n-1} n(n+1)v^n.$$

As in the former cases t is about $= 1$ for $n = 5$.

We give below, according to H^M , an example of the application of the formula (67).

Table 38 gives the values of $\log \delta(i, 1:x)$ for $n = 5$ as calculated by (67) at $i = 0$, $i = 0.04$ and $i = 0.06$.

Table 39 gives the values of $(Ia)_{\overline{n}|}$ and $\log \sigma(i)$ for $n = 5$ at different rates of interest.

Table 40 shows the calculation of S_x at $5\frac{1}{3}\%$, by means of $\delta(i, 1:x)$ taken at the same rate of interest (interpolated from the values in Table 38). The values of N_x have been taken from Table 19.

34. In preparing the subsidiary tables of the preceding articles, $d(i, 1:x)$, $d(i, x:1)$, $s(i, x)$ etc., we have to calculate, for one or two rates of interest, the financial elements for all ages. It will be convenient in this calculation to use the following elements (for $n = 5$):

$$\begin{aligned} l_x, \quad v l_{x+1}, \quad v^2 l_{x+2}, \quad v^3 l_{x+3}, \quad v^4 l_{x+4}; \\ l_{x+5}, \quad v l_{x+6}, \quad v^2 l_{x+7}, \quad v^3 l_{x+8}, \quad v^4 l_{x+9}; \\ l_{x+10}, \quad v l_{x+11}, \quad v^2 l_{x+12}, \quad v^3 l_{x+13}, \quad v^4 l_{x+14}; \end{aligned}$$

etc.

These elements contain only four different multiplicands v, v^2, v^3, v^4 , and we then get an easy and rapid calculation, which moreover may be immediately verified: $\Sigma v l = v \Sigma l$ etc.

Table 38.

 $HM; \log \delta(i, 1 : x)$ for $n = 5$, as calculated by (67).

x	$i = 0$	$i = 0,04$	$i = 0,06$	$10^3 \Delta \log \delta(i, 1 : x)$	
10	2,39726	2,39606	2,39547	-120	-59
15	2,46378	2,46496	2,46555	+118	+59
20	2,18253	2,18266	2,18272	13	6
25	2,17237	2,17249	2,17256	12	7
30	2,10423	2,10434	2,10439	11	5
35	2,04286	2,04305	2,04315	19	10
40	1,98158	1,98167	1,98171	9	4
45	1,89131	1,89161	1,89176	30	15
50	1,78160	1,78183	1,78194	23	11
55	1,65534	1,65564	1,65580	30	16
60	1,50470	1,50502	1,50517	32	15
65	1,35098	1,35112	1,35119	14	7
70	1,19366	1,19383	1,19392	17	9
75	1,02238	1,02214	1,02202	-24	-12
80	0,87677	0,87620	0,87591	-57	-29
85	0,76502	0,76374	0,76310	-128	-64
90	0,65512	0,65359	0,65283	-153	-76

Table 39.

 $(Ia)_{\overline{n}}$ and $\log \sigma(i) = \log \sum_{l=1}^{n-1} n(n+1)v^n$ for $n = 5$.

i	$(Ia)_{\overline{5}}$	$\log \sigma(i)$	i	$(Ia)_{\overline{5}}$	$\log \sigma(i)$
0,0300	13,87254	1,56049	0,0450	13,35906	1,54027
0,0325	13,78476	1,55709	0,0475	13,27647	1,53693
0,0350	13,69788	1,55371	0,0500	13,19471	1,53361
0,0375	13,61188	1,55033	0,0525	13,11377	1,53029
0,0400	13,52675	1,54697	0,0550	13,03363	1,52699
0,0425	13,44248	1,54361	0,0575	12,95429	1,52369
			0,0600	12,87573	1,52041

Table 40.
 H^M ; $i = 0,055$; S_x as calculated by means of $\delta(i, 1 : x)$. N_x and D_x from Table 19.

x	$\log \delta(i, 1 : x)$	$\log \langle (I_a)_{\bar{a}} - (I_a)_{x\bar{a}} \rangle$	$(I_a)_{\bar{a}} - (I_a)_{x\bar{a}}$	$(I_a)_{x\bar{a}}$	$S_x - S_{x+5} - 5 N_{x+5}$	$S_x - S_{x+5}$	S_x
90	0,65302	0,87397	7,4811	5,5525	65,481	71,338	72,896
85	0,76326	0,76373	5,8040	7,2290	413,82	570,22	643,12
80	0,87598	0,65101	4,4772	8,5564	1044,54	2658,85	3301,97
75	1,02205	0,50494	3,1984	9,8352	4556,48	8798,03	12100,00
70	1,19380	0,33309	2,1532	10,8804	9776,22	22567,37	34667,37
65	1,35117	0,17582	1,49906	11,53457	17615,07	48102,42	82769,79
60	1,50513	0,02186	1,05162	11,98201	28395,31	90311,21	173081,0
55	1,65576	1,87123	0,74341	12,39022	43011,1	155170,0	328251,0
50	1,78191	1,74508	0,55601	12,47762	62402,0	250114,5	578365,5
45	1,89172	1,63527	0,43179	12,60184	88250,4	385108,4	963473,9
40	1,98170	1,54529	0,35099	12,68264	122582,0	573377,8	1536851,7
35	2,04312	1,48387	0,30470	12,72893	168605,2	832828,4	2369680,1
30	2,10438	1,42261	0,26461	12,76902	230236,6	1187736,3	3557416,4
25	2,17254	1,35445	0,22018	12,80745	312549,6	1670174,5	5227590,9
20	2,18271	1,34428	0,22094	12,81269	422542,1	2322900,6	7550491
15	2,46540	1,06159	0,11324	12,91839	568380,4	3202396	10752887
10	2,39562	1,13137	0,13532	12,89831	755106,5	4373920	15126807

On a formula for the transformation of mortality tables.

By THV. RICHARDT.

(Continued from page 71.)

35. *Calculation of annuities on 2, 3, 4 etc. joint lives for quinquennial ages.* As shown in article 31 the formulae (54) (55 a) and (62) or (65) give very good approximations, except for old ages, where a direct calculation may be necessary.

A direct calculation (according to O^H) shows that a slight modification of the above mentioned formulae may be convenient. So we find that $h(x)(1+i)$ of these formulae may be replaced by a new number which is independent of the rate of interest. Denoting the new number by $h(x)$ we find (for $n=5$)

$$h(x) = 2,44 g(x).$$

The former $h(x)$ was about $2,2 g(x)$ for $n=5$.

Further we write the above mentioned formulae in the following form, where ω denotes the »new» age:

$$(68) \begin{cases} \log d(i, x: x\omega) &= \log d(i, 1: \omega) &+ \varphi(x; 1: \omega) \\ \log d(i, xy: xy\omega) &= \log d(i, y: y\omega) &+ \varphi(x; y: y\omega) \\ \log d(i, xyz: xyz\omega) &= \log d(i, yz: yz\omega) &+ \varphi(x; yz: yz\omega) \end{cases}$$

etc., and

$$(69) \begin{cases} \log s(i, xy) = \log s(i, x) + \log s(i, y) - \log s(i) + \psi(xy) \\ \log s(i, xyz) = \log s(i, xz) + \log s(i, yz) - \log s(i, z) + \psi(xy; z) \\ \log s(i, xyz u) = \log s(i, xz u) + \log s(i, yz u) - \\ \quad - \log s(i, zu) + \psi(xy; zu) \\ \text{etc.} \end{cases}$$

We then have with good approximation

$$(70) \quad \begin{aligned} \varphi(x; 1:\omega) &= \varphi(x; y:y\omega) = \varphi(x; yz:yz\omega) = \dots = g(1:\omega) h(x) \\ \psi(xy) &= \psi(xy; z) = \psi(xy; zu) = \dots = g(x) h(y) = g(y) h(x), \end{aligned}$$

except for old ages. As to the rate of interest the effect generally is insignificant; however, for old ages it may be desirable to take it into account.

36. We give below specimen values according to OM .

Table 41 gives $a_{x\overline{5}}^i$ and $\log s(i, x)$ for $n=5$ at different rates of interest.

Table 42 gives $\log d(i, 1:x)$ at 3 % and 6 %; further $g(1:x)$, $g(x)$ and $h(x)$, where

$$(71) \quad \begin{aligned} (j-i)g(1:x) &= \log d(j, 1:x) - \log d(i, 1:x) \\ (j-i)g(x) &= \log s(j, x) - \log s(i, x) - \log s(j) + \log s(i). \\ h(x) &= 2,44 g(x); \end{aligned}$$

here we have put $j=0,06$, $i=0,03$. The formula for $g(x)$ is easily found from (57) and (61).

Tables 43 and 44 give $\varphi(x; 1:y)$, $\psi(xy)$ and $\varphi(x; y:yz)$ for some ages at 3 % and 6 %.

Table 45 gives φ and ψ at 3 % and 6 % for 2, 3, 4 and 5 lives of equal ages.

Table 43, 44 and 45 give the true values of φ and ψ ; comparing these values with the values calculated by the formulae (70) we see that the direct calculations may be confined to a few ages.

Table 41.

OM	Values of az					Values of $\log s(i, x)$ for $n = 5$					OM
	$i = 0,03$	$i = 0,04$	$i = 0,05$	$i = 0,06$		$i = 0,03$	$i = 0,04$	$i = 0,05$	$i = 0,06$	x	
95	2,07170	2,05377	2,03639	2,01953		0,27318	0,26379	0,25453	0,24540	95	
90	2,58064	2,55094	2,52223	2,49445		0,49279	0,48233	0,47202	0,46184	90	
85	3,09250	3,04994	3,00885	2,96916		0,64952	0,63834	0,62731	0,61643	85	
80	3,53731	3,48299	3,43060	3,38006		0,75572	0,74408	0,73259	0,72126	80	
75	3,88605	3,82222	3,76071	3,70140		0,82588	0,81395	0,80218	0,79057	75	
70	4,14145	4,07054	4,00222	3,93638		0,87171	0,85960	0,84765	0,83586	70	
65	4,32010	4,24418	4,17105	4,10058		0,90145	0,88923	0,87716	0,86526	65	
60	4,44111	4,36176	4,28534	4,21173		0,92063	0,90833	0,89620	0,88422	60	
55	4,52109	4,43947	4,36088	4,28517		0,93291	0,92057	0,90839	0,89637	55	
50	4,57321	4,49010	4,41009	4,33301		0,94076	0,92839	0,91618	0,90413	50	
45	4,60731	4,52323	4,44228	4,36431		0,94582	0,93343	0,92121	0,90914	45	
40	4,63029	4,54555	4,46398	4,38540		0,94920	0,93680	0,92456	0,91247	40	
35	4,64708	4,56186	4,47983	4,40081		0,95164	0,93923	0,92698	0,91489	35	
30	4,66059	4,57499	4,49259	4,41322		0,95359	0,94118	0,92892	0,91682	30	
25	4,67144	4,58553	4,50283	4,42318		0,95516	0,94274	0,93048	0,91837	25	
20	4,67906	4,59294	4,51003	4,43018		0,95626	0,94384	0,93157	0,91946	20	
15	4,68360	4,59735	4,51431	4,43434		0,95692	0,94449	0,93223	0,92012	15	
10	4,68594	4,59961	4,51652	4,43648		0,95726	0,94483	0,93257	0,92045	10	

Table 42.

OM	$\log d(i, 1 : x)$		$10^4 g(1 : x)$	$h(x)$	$10^4 g(x)$
x	$i = 0,03$	$i = 0,06$			
95	0,53919	0,53645	-913	0,737	3022
90	0,63198	0,63022	-588	0,480	1968
85	0,75093	0,74991	-340	0,305	1252
80	0,88988	0,88938	-166	0,194	797
75	1,04205	1,04190	- 50	0,125	513
70	1,20152	1,20160	+ 24	0,0814	334
65	1,36290	1,36310	69	0,0537	220
60	1,52078	1,52106	92	0,0361	148
55	1,66941	1,66971	98	0,0251	103
50	1,80366	1,80394	94	0,0181	74
45	1,92114	1,92139	85	0,0135	55,5
40	2,02312	2,02335	77	0,0106	43,4
35	2,11645	2,11667	75	0,0085	34,8
30	2,20956	2,20980	80	0,0068	28,0
25	2,30213	2,30237	79	0,0055	22,5
20	2,38146	2,38165	61	0,0045	18,5
15	2,43665	2,43676	36	0,0039	16,1
10	2,46806	2,46811	16	0,0036	14,8

Table 43.

OM	$x-y$	$10^4 \varphi(x; 1 : y)$				$10^4 \phi(xy)$			
		$x=95$	$x=90$	$x=85$	$x=80$	$x=95$	$x=90$	$x=85$	$x=80$
$i = 0,03$	0	-648	-277	-103	-32	1928	879	370	153
	5	-422	-161	- 50	-10	1309	573	238	99
	10	-246	- 79	- 15	+ 5	861	371	154	64
	15	-120	- 24	+ 8	13	561	241	101	43
	20	- 36	+ 12	21	18	366	158	67	29
	25	+ 19	33	28	19	240	104	45	20
	30	51	44	30	18	160	71	31	14
	35	68	47	29	16	108	49	23	11
	40	73	45	26	15	75	35	17	8
	45	70	41	23	15	54	27	13	7
	50	63	37	23	16	41	21	11	5
	55	57	36	24	15	32	17	9	4
	60	56	38	24	12	26	13	7	4
	65	59	38	19	7	21	11	6	3
	70	58	29	11	3	17	9	5	3
$i = 0,06$	0	-647	-278	-104	-32	1910	877	371	154
	5	-422	-162	- 51	-10	1301	573	239	99
	10	-246	- 79	- 15	+ 5	857	371	154	64

Table 44.
Values of $10^4 \varphi(x; y; yz)$.

OM	$x-z$	$x-y=0$			$x-y=5$			$x-y=10$		$x-y=5$		$x-y=20$	
		$x=95$ $y=95$	$x=90$ $y=90$	$x=85$ $y=85$	$x=95$ $y=90$	$x=90$ $y=85$	$x=85$ $y=80$	$x=95$ $y=85$	$x=90$ $y=80$	$x=95$ $y=80$	$x=90$ $y=75$	$x=95$ $y=75$	$x=90$ $y=75$
$i=0,03$	0	-487	-258	-104	—	—	—	—	—	—	—	—	—
	5	-323	-151	-51	-373	-158	-51	—	—	—	—	—	—
	10	-189	-74	-15	-218	-78	-15	-233	-79	—	—	—	—
	15	-92	-22	8	-107	-23	8	-114	-23	-117	-24	—	—
	20	-27	12	22	-31	12	22	-33	12	-35	12	-35	—
	25	16	33	29	18	34	29	19	34	19	34	19	—
	30	42	43	31	48	45	31	50	45	51	45	51	—
	35	56	46	30	63	48	30	67	48	68	48	68	—
	40	59	44	27	67	45	27	71	46	72	46	73	—
	45	56	40	24	64	41	24	68	41	69	41	70	—
	50	50	36	24	58	37	24	61	38	62	37	63	—
$i=0,06$	0	-477	-254	-104	—	—	—	—	—	—	—	—	—
	5	-316	-149	-51	-367	-157	-51	—	—	—	—	—	—
	10	-185	-73	-15	-215	-77	-15	-231	-79	—	—	—	—

Table 45.
Values of φ and ψ for $x = y = z = u = v$.

OM	x	$10^4 \varphi(x; 1 : y)$	$10^4 \varphi(x; y : yz)$	$10^4 \varphi(x; yz : yzu)$	$10^4 \varphi(x; yzu : yzuu)$	$10^4 \psi(xy)$	$10^4 \psi(xy; z)$	$10^4 \psi(xyz; u)$
$i = 0,03$	95	-648	-487	-264	-132	1928	1253	641
	90	-277	-258	-195	-130	879	748	592
	85	-103	-104	-95	-80	370	360	317
	80	-32	-33	-33	-31	153	156	152
	75	-6	-6	-6	-6	64	65	66
$i = 0,06$	95	-647	-477	-256	-128	1910	1221	621
	90	-278	-254	-190	-127	877	735	518
	85	-104	-104	-94	-78	371	357	312
	80	-32	-33	-33	-31	154	156	150
	75	-6	-6	-6	-6	64	66	66