

THE COUPLED CIRCUIT  
BY THE METHOD OF GENERALIZED ANGULAR  
VELOCITIES\*

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ABSTRACT OF PAPER

In an oscillating-current circuit there is no impressed electromotive force and the sinusoids which are involved are damped.

In the alternating-current circuit, a certain function, called the impedance, may be used for the purpose of generalizing Ohm's law to apply to such circuits.

In order further to generalize Ohm's law so that it may be applied to oscillating-current circuits, an initial voltage must be used instead of an impressed voltage. The function which may be used to change this voltage amplitude into a current amplitude may be called the "*threshold impedance*."

The alternating-current involves an angular velocity. In the oscillating-current circuit this angular velocity may be generalized to include the decrement of the circuit, and it then becomes a complex quantity. From this complex generalized angular velocity may be formed by analogy a generalized impedance. This generalized impedance is always zero for free oscillations. This law enables us to determine the generalized angular velocities, and hence the frequencies and decrements, present in the free oscillation.

The threshold impedance is derived by a single differentiation from the generalized impedance. The use of the threshold impedance furnishes a second law to be used in the determination of the amplitudes of oscillation.

These two laws completely solve the oscillating-current circuit. They are of importance only when there are several generalized angular velocities simultaneously present.

The inductively coupled circuit furnishes an example of the

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\*Received by the Editor November 29, 1916.

utility of the method. In order to render this solution of greater practical value, a short approximate method is given in an appendix, for solving a fourth degree algebraic equation which appears.

A list of symbols will be found at the end of the paper.

## INTRODUCTORY

An oscillating-current circuit is one which oscillates in the absence of impressed electromotive force. The quantities involved are thus damped sinusoids. When there is only a single frequency of oscillation, and a single decrement to correspond, such a circuit may be readily solved by the use of differential equations. When, however, there are several frequencies and decrements simultaneously present, such a solution soon becomes cumbersome, particularly as regards the determination of the constants of integration in accordance with initial conditions.

Similar difficulties were experienced in the solution of alternating-current networks. The simple series circuit was readily solved by differential equations; but complicated networks presented difficulties.

A practical method was obtained for the alternating-current case by the introduction of the concept of impedance. This impedance was simply the function of the constants of the circuit which divided into the amplitude of voltage would give the amplitude of current. The solution of a network, then, required simply a knowledge of the rules for forming the impedance of the network from the several impedances of the branches.

If we attempt to generalize this law in a similar manner so that we may apply it to the oscillating-current circuit, we are confronted with the fact that here we have no impressed electromotive force. We must then use some other voltage; and for this purpose the initial voltage present in the circuit offers itself. This initial voltage may be due to an initial charge present in a condenser; or it may be due to an initial current thru a resistance.

We seek, then, a function of the constants of an oscillating network such that it may be divided into an initial voltage to give an initial amplitude of current oscillation.

## ANGULAR VELOCITIES

In forming the impedance of an alternating-current circuit we encounter the "*angular velocity*" of the circuit. This is the

time rate of change of the argument in the impressed voltage expression:

$$e = E \cos (\omega t) \quad \text{volts.}$$

It receives the name "angular velocity" because of the usual representation of such a quantity by means of a revolving plane vector.

If we make use also of the symbolic operation  $j$ , such a vector may be represented by the expression:

$$e = E \varepsilon^{j \omega t} \quad \text{volts.}$$

In an oscillating-current circuit we have present also logarithmic decrements.

In the expression:

$$A \varepsilon^{n t} \quad \angle$$

where  $n = -\alpha + j \omega$  hyp./sec.  $\angle$

we have both the usual angular velocity and the decrement present, for the expression may be rewritten in the form:

$$A \varepsilon^{-\alpha t} \varepsilon^{j \omega t} \quad \angle$$

when it is seen to consist of the alternating-current term, multiplied by a damping factor.

It will be convenient to call the complex quantity  $n$ , which includes both the angular velocity and the decrement, the "generalized angular velocity" of the circuit.

Using this generalized angular velocity, we may form generalized impedances, admittances, etc., by analogy with the alternating-current case.

### FIRST LAW OF OSCILLATING-CURRENT CIRCUITS

If an alternating current of angular velocity  $\omega$  passes thru a branch of impedance  $Z$ , the voltage across the branch is given by the product of the current and impedance.

It may readily be shown that this is also true if the current is oscillatory with a generalized angular velocity  $n$ <sup>1</sup>.

Hence it follows, since the impressed voltage in an oscillating-current circuit is zero, that *the generalized impedance of the entire circuit must be zero*.

This fact, which appears in Rayleigh's Theory of Sound, in Heaviside's "Electrical Papers," and in Helmholtz's works, may be taken as the first law of oscillating-current circuits.

The application of the law enables us to determine the un-

<sup>1</sup> "The Impedances, Angular Velocities, and Frequencies of Oscillating-Current Circuits"—A. E. Kennelly, "Proc. I. R. E.," 1915.

known generalized angular velocities of any given oscillating circuit, for the equation obtained on equating the generalized impedance to zero, may be solved for  $n$ . There may, of course, be several values of  $n$ , the generalized angular velocity, which appear as roots.

Upon separating these values of  $n$  into their real and imaginary portions, the decrements and angular velocities may be respectively found. In this way we may determine the damping factors and frequencies present in the free oscillation of any network.

The application of this law to practical circuits has been investigated by Eccles, Campbell and Kennelly.<sup>2</sup>

## SECOND LAW OF OSCILLATING-CURRENT CIRCUITS

When the circuit oscillates at a single frequency, the amplitude of oscillation may be found by inspection. When several frequencies are simultaneously present, there is needed a law which will determine the amplitude of the various terms of the current expression.

If the equation from the first law:

$$z=0 \qquad \text{ohms } \angle$$

yields as roots

$$n_1, n_2, \quad \cdot \quad \cdot \quad \cdot \qquad \text{hyp./sec. } \angle$$

these are the generalized angular velocities of free oscillation; and it follows that the current in the circuit will be of the form:

$$i = A_1 \epsilon^{n_1 t} + A_2 \epsilon^{n_2 t} + \quad \cdot \quad \cdot \quad \cdot \qquad \text{amperes.}$$

It is our problem to determine the  $A$ 's. If  $E$  is the initial voltage of the circuit, we seek a function which will divide into  $E$  to give  $A$ .

Such a function will be found in the expression:

$$n \frac{dz}{dn} \qquad \text{ohms } \angle$$

and this expression may appropriately be called the "*threshold impedance*" of the circuit. This fact is here given without proof; as a formal proof is necessarily too long for a paper of this character.

<sup>2</sup>Eccles, "Electrician," 1915; "Phys. Soc. Proc.," 24, 1912.  
Campbell, "Proc. A. I. E. E.," 1911.  
Kennelly, "Proc. I. R. E.," 1915.

Heaviside<sup>3</sup> gives without proof the following formula for the current in a network when a voltage  $E$  is suddenly applied:

$$i = \frac{E}{z(0)} + \sum_{r=1}^m \frac{E}{n_r \left( \frac{dz}{dn} \right)_{n_r}}$$

$$= \frac{E}{z(0)} + \frac{E}{n_1 \left( \frac{dz}{dn} \right)_{n=n_1}} \varepsilon^{n_1 t} + \frac{E}{n_2 \left( \frac{dz}{dn} \right)_{n=n_2}} \varepsilon^{n_2 t} + \dots$$

$$+ \frac{E}{n_m \left( \frac{dz}{dn} \right)_{n=n_m}} \varepsilon^{n_m t}$$

where  $n_1, n_2 \dots n_m$  are the roots of  $z(n) = 0$ .

$z$  is the generalized impedance of the circuit, a function of  $n$ .  $z(0)$  is the value of the generalized impedance obtained on inserting  $n = 0$ .

$\left( \frac{dz}{dn} \right)_{n_r}$  is the value of  $\frac{dz}{dn}$  obtained on inserting  $n_r$  for  $n$ .

Wagner<sup>4</sup> proves this formula by the use of the function theory. A summary of Wagner's proof is given in Appendix B of this paper.

In circuits in which the charged element is a condenser,  $z(0)$  is  $\infty$ ; so that the first term disappears. The second term may also be considered the current on discharge for such a case, since the charge and discharge currents are equal and opposite. If the charged element is not a condenser, the first term may not disappear, and the full formula should be used.

Heaviside's formula applies to any system, physical or electrical, of any number of degrees of freedom, in which the relation between the magnitudes involved may be expressed by linear differential or algebraic equations. The oscillating-current circuit is a special case to which the formula applies. For this case the proof as given by Wagner is valid without qualification.

The use of this formula is particularly advantageous in finding oscillating-current solutions; because of the fact that  $z$  may be formed by rules already familiar from the alternating-current circuit, without referring to the differential equations of the circuit.

<sup>3</sup>Heaviside, "Elec. Papers," Vol. 2, page 373; "Electromagnetic Theory," Vol. II.

<sup>4</sup>Karl Willy Wagner, "Archiv für Elektrotechnik," 1916, IV Band.

We may then state as a second law of oscillating-current circuits:

*The initial amplitude of current oscillation equals the initial voltage of the circuit divided by the threshold impedance.*

There will be a value of  $n$ , a value of  $n \frac{dz}{dn}$ , and hence a term in the current expression, corresponding to each root of the equation:  $z=0$ .

In forming the threshold impedance, it is necessary to form the generalized impedance  $z$  of the circuit considering the initially charged branch of the circuit as the main branch.

### SOLUTION OF CIRCUITS

If a network contains an initial store of energy in one branch, corresponding to an initial voltage  $E$ , the current of free oscillation in the circuit may be found by the following steps:

1. Form the generalized impedance  $z$  of the circuit, considering the initially charged branch as the main branch.

2. Equate to zero, and solve for  $n$ . Call the roots of the equation  $n_1, n_2, \dots$

3. Form the threshold impedance  $n \frac{dz}{dn}$

4. Write the current expression in the form;

$$i = \left( \frac{E}{n \frac{dz}{dn}} \varepsilon^{n t} \right)_{n=n_1} + \left( \frac{E}{n \frac{dz}{dn}} \varepsilon^{n t} \right)_{n=n_2} + \dots \quad \text{amperes}$$

or:

$$i = \sum \frac{E}{n \frac{dz}{dn}} \varepsilon^{n t} \quad \text{amperes.}$$

In this expression the generalized angular velocities, and the amplitudes are in general complex quantities.

Upon reducing by the use of the identity:

$$\varepsilon^{j \omega t} = \cos \omega t + j \sin \omega t$$

the imaginary portions of the expression will cancel out, leaving a real expression for  $i$ .

If there are several stores of energy initially present, they may be considered separately and the results added.

The current or voltage in a distant portion of the network may be found by combining the generalized impedances of the elements of the circuit, in the manner of simple resistances.

A case of suddenly applied electromotive force may be considered as the inverse of discharge from the final state attained.

## ILLUSTRATION

As a simple example to show the method, consider a condenser of capacitance  $C$ , initially charged to voltage  $E$ , and discharging thru resistance  $R$  (Figure 1)

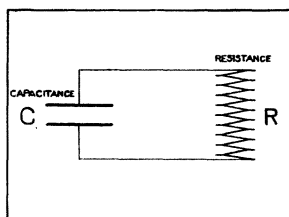


FIGURE 1

Here the generalized impedance is:

$$z = R + \frac{1}{Cn} \quad \text{ohms.}$$

Equating to zero we obtain:

$$n = -\frac{1}{RC} \quad \text{hyp./sec.}$$

The threshold impedance is:

$$n \frac{dz}{dn} = -\frac{1}{Cn} \quad \text{ohms.}$$

Hence the current in the circuit is:

$$i = \frac{E}{n \frac{dz}{dn}} \epsilon^{nt} = \left( \frac{E}{-\frac{1}{Cn}} \epsilon^{nt} \right)_{n = -\frac{1}{RC}} = \frac{E}{R} \epsilon^{-\frac{t}{RC}} \quad \text{amperes.}$$

This result may be checked by inspection.

## THE COUPLED CIRCUIT

The method is very useful for the solution of the circuits which occur in radio work.

It will be illustrated on the simple inductively coupled circuit.

In the circuit of Figure 2,  $R_1$ ,  $L_1$ ,  $C_1$ , are the primary, and  $R_2$ ,  $L_2$ ,  $C_2$  the secondary constants.  $M$  is the coefficient of mutual induction. The primary condenser is considered as discharging from an initial voltage  $E$ .

Form the generalized impedance of the circuit, considering the primary as the main branch.

For an alternating-current of angular velocity  $\omega$ , the impedance of such a circuit is well known to be:

$$z_1 - \frac{(M \omega)^2}{z_2} \quad \text{ohms}$$

where  $z_1$  is the impedance of the primary and  $z_2$  of the secondary alone.<sup>5</sup>

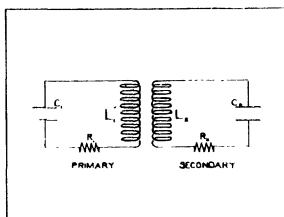


FIGURE 2

Hence, by analogy, we have as our generalized impedance:

$$z = R_1 + L_1 n + \frac{1}{C_1 n} - \frac{M^2 n^2}{R_2 + L_2 n + \frac{1}{C_2 n}} \quad \text{ohms. } \angle$$

Equate to zero, and clear of fractions and we obtain:

$$C_1 C_2 (L_1 L_2 - M^2) n^4 + C_1 C_2 (R_1 L_2 + R_2 L_1) n^3 + (C_1 L_1 + C_2 L_2 + C_1 C_2 R_1 R_2) n^2 + (C_1 R_1 + C_2 R_2) n + 1 = 0.$$

Solve this fourth degree equation for  $n$ ; and we obtain as roots the four values of the free generalized angular velocity:

$$n_1, n_2, n_3, n_4 \quad \text{hyp./sec. } \angle$$

Since these four roots are in general all complex, the solution of this equation is often laborious. It may, of course, be solved to any desired degree of accuracy by straightforward algebraic methods. If it is wished to avoid this labor, the approximate method given in appendix A may be used. This method gives results sufficiently accurate for most engineering purposes. The exact method may, however, be used if desired.

The threshold impedance may be found from  $z$  by a simple differentiation, and becomes on simplifying:

<sup>5</sup>"Impedance of Mutually Inductive Circuits," A. E. Kennelly, "The Electrician," London, Vol. XXXI, 1893, page 699.



$$n \frac{dz}{dn} = L_1 n - \frac{1}{C_1 n} - M^2 \frac{L_2 n^3 + 2 R_2 n^2 + \frac{3n}{C_2}}{\left(R_2 + L_2 n + \frac{1}{C_2 n}\right)^2} \quad \text{ohms. } \angle$$

Into this expression we may insert the four values of  $n$  found above.

The primary current is then given by the expression:

$$i_1 = \sum \frac{E}{n \frac{dz}{dn}} \epsilon^{nt} \quad \text{amperes } \angle$$

where the summation extends over the roots of  $n$  found from  $z=0$ .

The voltage induced in the secondary is found by multiplying  $i_1$  by  $-M n$ , and is:

$$e_2 = \sum -M n \frac{E}{n \frac{dz}{dn}} \epsilon^{nt} \quad \text{volts. } \angle$$

The secondary current is then found by dividing this voltage by the generalized secondary impedance:

$$i_2 = \sum \frac{-M n}{R_2 + L_2 n + \frac{1}{C_2 n}} \frac{E}{n \frac{dz}{dn}} \epsilon^{nt} \quad \text{amperes. } \angle$$

#### NUMERICAL EXAMPLE

This solution was applied to a test circuit at the Massachusetts Institute of Technology, and the results were checked by oscillograms and by comparison with the usual approximate methods of solution.

The constants of the circuit were:

$$\begin{aligned} R_1 &= 1.937 \text{ ohms} \\ R_2 &= 2.531 \text{ ohms} \\ L_1 &= 7.52 \times 10^{-3} \text{ henries} \\ L_2 &= 7.63 \times 10^{-3} \text{ henries} \\ C_1 &= 13.51 \text{ microfarads} \\ C_2 &= 24.62 \text{ microfarads} \\ M &= 3.475 \times 10^{-3} \text{ henries} \\ E &= 7.2 \text{ volts, initial} \end{aligned}$$

Inserting these values in the equation  $z=0$ , and reducing we obtain:

$$n^4 + 7.45 \times 10^2 n^3 + 1.930 \times 10^7 n^2 + 5.88 \times 10^9 n + 6.635 \times 10^{13} = 0$$

On solving this algebraic equation by the method of the appendix, there was obtained:

$$\left. \begin{array}{l} -249.2 \pm j 3827 \\ -123.3 \pm j 2129 \end{array} \right\} \text{hyp./sec. } \angle$$

for the four free generalized angular velocities.

These four values of  $n$ , and the values of the constants inserted in the expression of  $n \frac{dz}{dn}$ , give four values for the threshold impedance. Dividing each of these values into the values of  $E$  gave the four amplitudes:

$$\left. \begin{array}{l} -0.00396 \mp j 0.1460 \\ 0.00396 \mp j 0.0228 \end{array} \right\} \text{amperes. } \angle$$

Thus the primary current can be written:

$$\begin{aligned} i_1 = & (-0.00396 - j 0.1460) \varepsilon^{(-249.2 + j 3827)t} \\ & + (-0.00396 + j 0.1460) \varepsilon^{(-249.2 - j 3827)t} \\ & + (0.00396 - j 0.0228) \varepsilon^{(-123.3 + j 2129)t} \\ & + (0.00396 + j 0.0228) \varepsilon^{(-123.3 - j 2129)t} \text{ amperes } \angle \end{aligned}$$

Reducing the exponential terms to their trigonometric forms, and combining, this expression becomes:

$$\begin{aligned} i_1 = & \varepsilon^{-249.2t} (-0.00792 \cos 3827t + 0.292 \sin 3827t) \\ & + \varepsilon^{-123.3t} (0.00792 \cos 2129t + 0.0456 \sin 2129t) \text{ amperes.} \end{aligned}$$

or better:

$$\begin{aligned} i_1 = & 0.292 \varepsilon^{-249.2t} \sin (3827t - 0.0272) \\ & + 0.046 \varepsilon^{-123.3t} \sin (2129t + 0.1719) \text{ amperes.} \end{aligned}$$

Here we have the amplitudes, phase relations, and damping factors for the two terms of the primary current.

To obtain the secondary current amplitudes, we have to multiply the primary amplitudes by the ratio

$$- \frac{M n}{R_2 + L_2 n + \frac{1}{C_2 n}} \text{ numeric. } \angle$$

Inserting the values of the constants and of the roots for this ratio takes the four values:

$$\begin{aligned} & -0.732 \sqrt[3]{32.6'} \\ & -0.732 \sqrt[3]{32.6'} \\ & 2.528 \sqrt[6]{14.3'} \\ & 2.528 \sqrt[6]{14.3'} \text{ numeric. } \angle \end{aligned}$$

Multiplying the four primary amplitudes by these respective ratios, gives the four secondary amplitudes:

$$\begin{aligned} & -0.00370 + j0.1069 \\ & -0.00370 - j0.1069 \\ & +0.00370 - j0.0585 \\ & +0.00370 + j0.0585 \qquad \text{amperes. } \angle \end{aligned}$$

The secondary current expression may now be written in the same manner as was the primary current expression. It reduces to the form:

$$\begin{aligned} i_2 = & \varepsilon^{-249.2t} (-0.00740 \cos 3827 t - 0.2138 \sin 3827 t) \\ & + \varepsilon^{-123.3t} (+0.00740 \cos 2129 t + 0.1170 \sin 2129 t) \qquad \text{amperes} \end{aligned}$$

or:

$$\begin{aligned} i_2 = & -0.214 \varepsilon^{-249.2t} \sin (3827 t + 0.0346) \\ & + 0.117 \varepsilon^{-123.3t} \sin (2129 t + 0.0632) \qquad \text{amperes.} \end{aligned}$$

**SUMMARY:** The generalized impedance  $z$  of an oscillating circuit may be formed from the generalized angular velocity of oscillation,  $n$ , by analogy with the alternating-current circuit.

Equating this generalized impedance to zero, and solving for  $n$ , gives the free generalized angular velocities of oscillation. The real and imaginary portions of these free generalized angular velocities are used to find respectively the damping factors and frequencies of oscillation of the circuit.

From  $z$  may be found the threshold impedance of the circuit  $n \frac{dz}{dn}$ .

Dividing the initial voltage of the circuit by this threshold impedance gives the initial amplitudes of current oscillation.

The use of these two rules determines the complete expression for the oscillating current in any oscillating-current network.

The method applies to the simple inductively coupled circuit. A complete exact solution for the case of primary condenser discharge may be readily obtained. The method is of particular service in numerical problems.

An approximate method of solving the biquadratic obtained when coupled circuits are considered is given in an appendix to the paper.

## APPENDIX A.

Approximate method for the solution of the fourth degree algebraic equation occurring in the coupled circuit problem.

This equation:

$$C_1 C_2 (L_1 L_2 - M^2) n^4 + C_1 C_2 (R_1 L_2 + R_2 L_1) n^3 + (C_1 L_1 + C_2 L_2 + C_1 C_2 R_1 R_2) n^2 + (C_1 R_1 + C_2 R_2) n + 1 = 0$$

numeric  $\angle$

may be written in the form:

$$n^4 + \alpha n^3 + \beta n^2 + \gamma n + \delta = 0 \quad (\text{hyp./sec.})^4 \angle$$

where

$$\alpha = \frac{L_1 R_2 + L_2 R_1}{L_1 L_2 - M^2}$$

$$\beta = \frac{L_1 C_1 + L_2 C_2 + R_1 R_2 C_1 C_2}{C_1 C_2 (L_1 L_2 - M^2)}$$

$$\gamma = \frac{R_1 C_1 + R_2 C_2}{C_1 C_2 (L_1 L_2 - M^2)}$$

$$\delta = \frac{1}{C_1 C_2 (L_1 L_2 - M^2)}$$

The approximate method depends upon the fact that the absolute values of the roots of this equation are not greatly different from the absolute values of the roots of the equation found for  $R_1 = R_2 = 0$ .

The equation for the circuit without resistance will be:

$$n^4 + \lambda n^2 + \delta = 0 \quad (\text{hyp./sec.})^4 \angle$$

where

$$\lambda = \frac{L_1 C_1 + L_2 C_2}{C_1 C_2 (L_1 L_2 - M^2)}.$$

This equation is readily solved; and will yield as roots a pair of imaginary values:

$$j x_1 \text{ and } j x_2 \quad \text{hyp./sec.} \angle$$

Now if the desired roots of our complete equation are:

$$\begin{aligned} -a \pm j b \\ -c \pm j d \end{aligned} \quad \text{hyp./sec.} \angle$$

we may express these also in polar form as:

$$\begin{aligned} y_1 \angle \theta_1, y_1 < \theta_1 \\ y_2 \angle \theta_2, y_2 < \theta_2 \end{aligned} \quad \text{hyp./sec.} \angle$$

and by examining the relations between the roots and coefficient of our algebraic equation, write:

$$\begin{aligned}(1) \quad a+c &= \frac{\alpha}{2} \\(2) \quad y_1^2+y_2^2+4ac &= \beta \\(3) \quad 2ay_2^2+2cy_1^2 &= \gamma \\(4) \quad y_1^2y_2^2 &= \delta = x_1^2x_2^2\end{aligned}$$

From (4), since  $x_1$  and  $y_1$ ,  $x_2$  and  $y_2$  are nearly equal, we may write as a first approximation:

$$\begin{aligned}y_1 &= x_1(1-q) \\y_2 &= x_2(1+q)\end{aligned}\quad \text{hyp./sec. } \angle$$

where  $q$  is a small quantity.

Also from (1):

$$\frac{\alpha}{4} - a = c - \frac{\alpha}{4} = p \quad \text{hyp./sec.}$$

Substitute in (2) and (3)

$$\begin{aligned}x_1^2(1-q)^2+x_2^2(1+q)^2+4\left(\frac{\alpha}{4}-p\right)\left(\frac{\alpha}{4}+p\right) &= \beta \\ \left(\frac{\alpha}{4}-p\right)x_2^2(1+q)^2+\left(\frac{\alpha}{4}+p\right)x_1^2(1-q)^2 &= \gamma\end{aligned}\quad \text{(hyp./sec.)}^3$$

Expand, and neglect the square of  $q$  in comparison with unity:

$$\begin{aligned}(x_1^2+x_2^2)-2q(x_1^2-x_2^2)+\frac{\alpha^2}{4}-4p^2-\beta &= 0 \quad \text{(hyp./sec.)}^2 \\ \frac{\alpha}{4}(x_1^2+x_2^2)+p(x_1^2-x_2^2)-2pq(x_1^2+x_2^2)-q\frac{\gamma}{2}(x_1^2-x_2^2) &= \frac{\gamma}{2} \\ &\quad \text{(hyp./sec.)}^3\end{aligned}$$

Use as abbreviations:

$$\begin{aligned}x_1^2+x_2^2 &= s \\ x_1^2-x_2^2 &= t\end{aligned}\quad \text{(hyp./sec.)}^2$$

and the equations become:

$$\begin{aligned}s-2qt+\frac{\alpha^2}{4}-4p^2-\beta &= 0 \quad \text{(hyp./sec.)}^2 \\ \frac{\alpha s}{4}-2qp s+pt-\frac{\alpha qt}{2}-\frac{\gamma}{2} &= 0 \quad \text{(hyp./sec.)}\end{aligned}$$

These equations may be solved simultaneously for  $p$  and  $q$ , giving:

$$q = \frac{s t^2 + \alpha \gamma s - \gamma^2 - \alpha^2 \delta - \beta t^2}{2 t (2 s^2 + t^2 + \alpha \gamma - 2 s \beta)} \quad \text{numeric}$$

and:

$$p = \frac{2\gamma - \alpha s + \alpha q t}{4t - 8qs} \quad \text{hyp./sec.}$$

Since  $s$  and  $\beta$  are nearly equal, it is better to write a further abbreviation:

$$u = \beta - s = \frac{R_1 R_2}{L_1 L_2 - M^2}$$

and hence obtain:

$$q = \frac{\alpha \gamma s - t^2 u - \gamma^2 - \alpha^2 \delta}{2t(t^2 + \alpha \gamma - 2su)} \quad \text{numeric.}$$

We may now give the rule by which to obtain the free generalized angular velocities of the coupled circuit with constants  $R_1 L_1 C_1 R_2 L_2 C_2 M$ .

1. Solve the circuit without resistance. Call the absolute magnitude of the angular velocities obtained:  $x_1$  and  $x_2$ .

2. Form the function:

$$q = \frac{\alpha \gamma s - t^2 u - \gamma^2 - \alpha^2 \delta}{2t(t^2 + \alpha \gamma - 2su)}$$

where  $\alpha, \beta, \gamma, \delta$  are as given above,

$$s = x_1^2 + x_2^2$$

$$t = x_1^2 - x_2^2$$

and

$$u = \beta - s = \frac{R_1 R_2}{L_1 L_2 - M^2}$$

3. Write:

$$y_1 = x_1(1 - q) \quad y_2 = x_2(1 + q)$$

and  $y_1, y_2$  are the absolute values of the generalized angular velocities desired.

4. Form the function

$$p = \frac{2\gamma - \alpha s + \alpha q t}{4t - 8qs}$$

5. Write:

$$a = \frac{\alpha}{4} + p$$

$$c = \frac{\alpha}{4} - p$$

and  $a, c$  are the decrements desired, so that the generalized angular velocities are:

$$-a \pm j\sqrt{y_1^2 - a^2}$$

$$\text{and} \quad -c \pm j\sqrt{y_2^2 - c^2}.$$

As an illustration of the method to show the degree of approximation, a circuit with constants:

$$C_1 = 10^{-9} \text{ farads}$$

$$C_2 = 10^{-10} \text{ farads}$$

$$R_1 = 1000 \text{ ohms}$$

$$R_2 = 2000 \text{ ohms}$$

$$L_1 = 0.025 \text{ henries}$$

$$L_2 = 0.040 \text{ henries}$$

$$M = 0.020 \text{ henries}$$

where the resistances are purposely assumed large, was solved by the exact and the approximate methods, and gave results agreeing to at least five significant figures.

$q$ , in this case, was 0.000870, so that the assumption that the square of  $q$  could be neglected was evidently justified.

## APPENDIX B.

### SUMMARY OF WAGNER'S PROOF OF HEAVISIDE'S FORMULA

Wagner's proof is general, and applies to physical as well as electrical systems. This abstract of the proof will treat electrical oscillating systems only.

The constant voltage, which is suddenly applied to the network, is 0 when  $t < 0$ , and  $E$  when  $t > 0$ . Such a function may be represented by the Fourier integral:

$$f(t) = \frac{E}{2\pi j} \int_{-\infty}^{\infty} \frac{\epsilon^{nt}}{n} dn \quad (1)$$

where  $n$  is the complex variable of integration.<sup>6</sup>

This expression may be transformed to one with a closed path of integration as follows: About  $O$ , Figure 3, describe a circle of radius  $R$ . Examination will show that as  $R$  becomes infinite the integral vanishes along  $BCA$  for negative values of  $t$ , and along  $BDA$  for positive values of  $t$ . For negative values of  $t$  we may hence replace the open path of integration by the closed path of  $AOBCA$ , and for positive values of  $t$  by the path  $AOBDA$ .

Since the integrand is everywhere regular, except at the origin, it follows that the first of these integrals will be zero,

<sup>6</sup> Malcolm, "Transients in Submarine Cables," "The Electrician," May 10, 1912.

while the second will have the value  $2\pi j$  times the residual of the integrand at the origin. This residual is unity. Hence the expression:

$$f(t) = \frac{E}{2\pi j} \int_{A O B C A}^{\varepsilon^{nt}} \frac{dn}{n} \quad f(t) = \frac{E}{2\pi j} \int_{A O B D A}^{\varepsilon^{nt}} \frac{dn}{n} \quad (2)$$

has the value 0 when  $t < 0$ , and the value  $E$  when  $t > 0$ ; and hence faithfully represents our function.

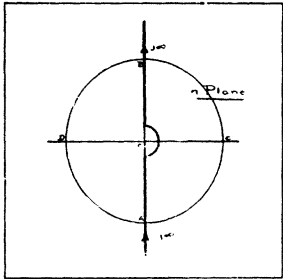


FIGURE 3

The voltage applied to a circuit, and the current in the circuit are always connected by a linear differential equation. This may be written symbolically:

$$e = F(D) i \text{ or } i = \frac{e}{F(D)}$$

where  $D$  represents the differential operator  $\frac{d}{dt}$ .

If  $e$  follows an exponential law of variation with the time, such as:

$$\varepsilon^{kt}$$

it is well known that the current will then be of the form:

$$i = \frac{\varepsilon^{kt}}{F(D)} = \frac{\varepsilon^{kt}}{F(k)}$$

Now in our Fourier integrals above we have expressed the impressed voltage as the sum of terms of the form:

$$\frac{dn}{n} \varepsilon^{nt}$$

and these terms follow the exponential law of time variation.



Hence corresponding to the voltage increment there will be a current increment:

$$\frac{d n}{n Z(n)} \varepsilon^{n t} \quad (3)$$

where  $Z(D)$  is the function of the differential operation from the equation of the circuit:

$$e = Z(D) i$$

The function  $Z$  is thus the generalized impedance of the circuit. Since the relations are linear, the effects of separate increments of the potential add simply to give the total effect.

Thus we obtain for the current in the circuit when  $t > 0$ , the expression:

$$i = \frac{E}{2 \pi j} \int_{A O B D A} \frac{\varepsilon^{n t}}{n Z(n)} d n \quad (4)$$

The other integral, which gives the current when  $t < 0$ , must be zero. Hence the function  $\frac{1}{Z(n)}$  can have no poles, or  $Z(n)$  can have no roots, which lie in the positive half of the real plane. This readily follows from physical considerations. This fact, and the limitations on  $Z(n)$  when more general systems are under consideration, cannot be entered into here.

The value of the expression (4) for the current may now be determined by the evaluation of the integral.

The integrand has poles at  $n = 0$ , and at the roots of  $Z(n) = 0$ . Suppose these roots to be:

$$n_1, n_2, \cdot \cdot \cdot n_m.$$

Then the value of the integral is  $2 \pi j$  times the sum of the residuals of the integrand at 0,  $n_1, n_2, \cdot \cdot \cdot n_m$ . This follows from the fact that, since the integrand is everywhere regular except at these points, the path  $A O B D A$ , Figure 4, may be deformed into a path consisting of a small circle about each pole, Figure 5. The value of the line integral for one circuit positively about a single pole is  $2 \pi j$  times the residual of the function of that pole.

If  $N_r$  is the residual of the function:

$$\frac{\varepsilon^{n t}}{n Z(n)} \quad (5)$$

at the pole  $n_r$ , and  $N_o$  at the origin, then the current is given by:

$$i = E N_o + E \sum_{r=1}^m N_r \quad (6)$$

To determine  $N_r$  the function (5) must be developed for the region about  $n_r$ , into a Laurent series in

$$(n - n_r)$$

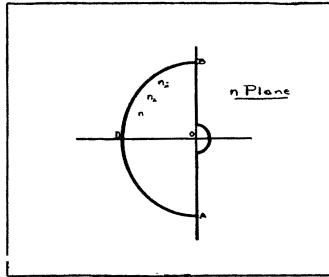


FIGURE 4

For abbreviation put  $(n - n_r) = \xi$

We have the following: 
$$\left\{ \begin{array}{l} \varepsilon^{n t} = \varepsilon^{n_r t} \varepsilon^{\xi t} = \varepsilon^{n_r t} \left( 1 + \xi t + \frac{\xi^2 t^2}{2!} + \dots \right) \\ n = n_2 + \xi \\ Z = Z(n_r) + \xi \left( \frac{dZ}{dn} \right)_{n_r} + \frac{1}{2} \xi^2 \left( \frac{d^2 Z}{dn^2} \right)_{n_r} + \dots \end{array} \right. \quad (7)$$

If the expansion of (5) is

$$\frac{\varepsilon^{n t}}{n Z} = \sum_{u=-\infty}^{u=\infty} A_u \xi^u \quad (8)$$

the coefficient  $A_{-1}$  is the residual  $N_r$  which we seek.

From (7) this may be seen to be:

$$N_r = \frac{\varepsilon^{n_r t}}{n_r \left( \frac{dZ}{dn} \right)_{n_r}} \quad (9)$$

To obtain the residual at the origin we use the expansion in the vicinity of the origin:

$$\frac{\varepsilon^{n t}}{n Z(n)} = \frac{\varepsilon^{n t}}{n \left( Z(0) + n \left( \frac{dZ}{dn} \right)_o + \dots \right)}$$

from which 
$$N_o = \frac{1}{Z(0)} \tag{10}$$

Using (9) and (10) in (6) we obtain finally:

$$i = \frac{E}{Z(0)} + \sum_{r=1}^m \frac{E}{n_r \left( \frac{dZ}{dn} \right)_{n_r}} \varepsilon^{n_r t} \tag{11}$$

which is the Heaviside formula.

In this expression the first term on the right hand side is the steady state term, and the remaining terms give the transient. In cases where the current is finally zero, the steady state term disappears.

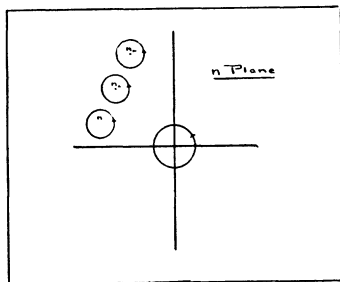


FIGURE 5

From the above derivation certain limitations as to the character of the roots of  $Z$  may be noted.

The roots must be negative in real part. Since a positive real part would mean physically, a circuit oscillating with a continuously increasing amplitude, this is of interest only in showing that Heaviside's formula is limited in application to such systems where this occurs; e. g., in the unstable arc.

The roots of  $Z$  must be distinct from each other and from zero. The case of multiple roots requires further treatment. This treatment will be found in Wagner's paper.

Singularities in  $Z$  do not appear in the treatment of the usual networks

## LIST OF SYMBOLS USED

$e$	Instantaneous electromotive force. Volts.
$i$	Instantaneous current. Amperes.
$E$	Maximum or initial value of voltage. Volts.
$I$	Maximum or initial value of current. Amps.
$\omega$	Angular velocity, $2\pi \times$ frequency. Radians per second.
$n$	Generalized angular velocity. Hyperbolic radians per second. $\angle$ .
$\alpha$	Logarithmic decrement. Hyps. per second.
$j$	The pure imaginary $\sqrt{-1}$ .
$\varepsilon$	Base of Napierian system of logarithms. 2.718 . . . .
$A$	A constant amplitude. $\angle$ .
$z$	Impedance. Ohms.
$Z$	Generalized impedance. Ohms. $\angle$ .
$R$	Resistance. Ohms.
$L$	Inductance. Henrys.
$C$	Capacitance. Farads.
$M$	Mutual inductance. Henrys.
$\alpha, \beta, \gamma, \delta, \lambda$	Coefficients of algebraic equation.
$x_1, x_2, y_1, y_2$	Absolute values of generalized angular velocities.
$a, b, c, d$	Rectangular components of generalized angular velocity.
$\theta_1, \theta_2$	Polar angles of generalized angular velocity.
$p, q$	Correcting factors.
$s, t, u$	Constants.
$\angle$	Sign of a complex quantity or equation.