THE SECOND THEOREM OF CONSISTENCY FOR SUMMABLE SERIES

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1. My object in writing this paper is to give a full proof of a theorem enunciated without proof in the tract "The general theory of Dirichlet's series", recently published by Dr. Marcel Riesz and myself*. The theorem is as follows:

If (i) *the series* Σc_n *is summable* (λ, κ), *to sum* C;

(ii) μ *is a logarithmico-exponential function of* λ *such that*

$$\mu = O(\lambda^{\Delta}),$$

where Δ *is a constant; then the series* Σc_n *is summable* (μ, κ) *to sum* C.

2. I begin by recalling Riesz's definition† of summability (λ, κ), *i.e.* summability by means of type λ and order κ. Suppose that (λ_n) is an ascending sequence of positive numbers whose limit is infinity; and let

$$C_\lambda(\tau) = c_1 + c_2 + \dots + c_n$$

if

$$\lambda_n < \tau \leqslant \lambda_{n+1}.$$

Further let

$$C_\lambda^\kappa(\omega) = C_\lambda(\omega) \tag{2.1}$$

if $\kappa = 0$, and

$$C_\lambda^\kappa(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\kappa c_n = \kappa \int_0^\omega C_\lambda(\tau)(\omega - \tau)^{\kappa - 1} \, d\tau \tag{2.2}$$

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* *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 18, p. 33 (Theorem 19). I refer to this tract as "H. and R."

† H. and R., p. 21.

if $\kappa > 0$. Then
$$\omega^{-\kappa} C_\lambda^\kappa(\omega)$$
is called the *typical mean of type* λ *and order* κ formed from the series Σc_n, and the series is said to be *summable* (λ, κ), to sum C, if this "typical mean" tends to the limit C when $\omega \to \infty$.

When $\lambda_n = n$, the means are said to be *arithmetic*. Arithmetic means are equivalent to Cesàro's means, or to the generalisations of Cesàro's means considered by Knopp and Chapman: a series is summable (n, κ) if and only if it is summable (C, κ).*

The *first theorem of consistency*† asserts that, if a series is summable (λ, κ), then it is summable (λ, κ'), to the same sum, for any value of κ' greater than κ. In particular a convergent series is summable by typical means of any positive order, since summability $(\lambda, 0)$ is equivalent to convergence. The general idea expressed by the first theorem of consistency is that, so long as the type remains the same, *the efficacy of a method of summation increases with the order.*

The *second theorem of consistency* lies somewhat deeper. The general idea which it expresses is that, when the order of a method of summation remains the same, *its efficacy increases as the type decreases*, that is to say as the rate of increase of the function λ_n which defines the type decreases. If
$$\lambda_n - \lambda_{n-1} > K\lambda_n,$$
that is to say if the rate of increase of λ_n is as great as that of an exponential e^{An}, then the efficacy of the method is *nil*: it will sum convergent series and no others‡. If λ_n runs through the functions of the logarithmico-exponential scale, such as
$$e^n,\ n,\ \log n,\ \log\log n,\ \ldots,$$
then we obtain a succession of systems of methods of gradually increasing efficacy.

The theorem suggested by this general idea is that *if a series is summable* (λ, κ) *then it is summable* (μ, κ), μ *being any function of* n *whose rate of increase is less than that of* λ. The actual theorem stated in § 1 is in one way less general and in another more general than this. In the first place, in order to ensure the truth of the theorem, we must suppose that the relation between the rates of increase of μ and λ is characterised by a certain regularity; and the most convenient way of

* Riesz, "Sur une méthode de sommation équivalente à la méthode des moyennes arithmétiques", *Comptes Rendus*, 12 June 1911.

† H. and R., p. 29 (Theorem 16).

‡ H. and R., p. 46 (Theorem 36).

ensuring this is to suppose that μ is a logarithmico-exponential function of λ, a phrase which we will define more precisely in a moment. But, when this limitation is made, we are able to assert rather more than our general principle suggests. The efficacy of the method increases, or at any rate does not decrease, as the rate of increase of the type decreases; a series summable (λ, κ) is certainly summable (μ, κ) if μ increases more slowly than λ. But the converse implication will also be true, and the two methods completely equivalent, if the difference between the rates of increase of λ and μ is not too pronounced, if in fact either function increases with a rapidity comparable to that of a power of the other. If, for example, both λ and μ are powers of n, then any series summable (λ, κ) will be summable (μ, κ), and conversely.

3. Proofs of certain special cases of this theorem have already been published. The most important case is that in which

(3.1) $$\mu = \log \lambda.$$

This case of the theorem was enunciated in 1909 by Riesz*; and his proof was published for the first time in our tract†. Another case is that in which

(3.2) $$\mu = P(\lambda),$$

where P is a polynomial. This case has been treated by Berwald‡, when $\lambda = n$ and κ is an integer. A third case§ is that in which $\kappa = 1$: the theorem then amounts to little more than a restatement in different language of a theorem of Cesàro.

I had conjectured the truth of the general theorem some years ago, when engaged, in collaboration with Mr. Chapman, on a paper dealing with the general theory of summability‖. At that time I had a proof not of the theorem itself, but of its analogue for integrals, and only in the two cases in which (i) κ is an integer or (ii) $0 < \kappa < 1$. As soon as I became familiar with Riesz's methods it became clear to me that my proof applied to series as well; but I was still unable to overcome the algebraical difficulties presented by the proof of the theorem in its most general form.

* "Sur la sommation des séries de Dirichlet", *Comptes Rendus*, 5 July 1909.

† H. and R., p. 30 (Theorem 17).

‡ "Solution nouvelle d'un problème de Fourier", *Arkiv för Matematik*, Vol. 9, 1913, No. 14.

§ Hardy, "On certain oscillating series", *Quarterly Journal*, Vol. 38, 1907, pp. 269–288.

‖ Hardy and Chapman, "A general view of the theory of summable series", *Quarterly Journal*, Vol. 42, 1911, pp. 181–216.

It was only when Riesz, in the course of the preparation of our tract, discovered an important simplification of his method of treatment of the case in which $\mu = \log \lambda$, that I was able to find a completely general proof.

*Definitions and lemmas from the Infinitärcalcul.**

4. A *logarithmico-exponential function*, or, shortly, *an L-function*, is a real one-valued function which can be defined by an explicit formula involving, each only a finite number of times, the ordinary algebraical symbols

$$+, \quad -, \quad \times, \quad \div, \quad \sqrt{},$$

and the symbols $$\log(\ldots), \quad e^{\cdots}$$

of the logarithmic and exponential functions.

The properties of L-functions which are required for the argument of this paper are as follows.

4.1. *Any L-function $\mu(\lambda)$ is continuous, of constant sign, and monotonic, from a certain value of λ onwards; and the same is true of any of its derivatives.*†

We may suppose, without real loss of generality, that $\mu(\lambda)$, and such of its derivatives as occur in the argument, satisfy these conditions for all values of λ in question.

4.2. *If $\mu \to \infty$, and a number Δ exists such that*

$$\mu = O(\lambda^{\Delta}), \tag{4.21}$$

then $$\mu^{(r)} = O\left(\frac{\mu}{\lambda^r}\right),$$

$\mu^{(r)}$ denoting the r-th derivative of $\mu(\lambda)$.‡

4.3. *If μ satisfies the conditions of* 4.2, *and ν lies between two posi-*

* See Hardy, "Orders of infinity", *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 12, 1910, or "Properties of logarithmico-exponential functions", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1912, pp. 54–90. I refer to the first of these publications as "O. I.".

† O. I., p. 18. See also "Properties &c.", p. 40.

‡ O. I., pp. 38 *et seq.*

tive numbers g and G, then positive numbers h and H exist, such that

$$h \leqslant \frac{\mu(\lambda\nu)}{\mu(\lambda)} \leqslant H.^* \tag{4.31}$$

For $$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\log\mu(\lambda\nu)-\log\mu(\lambda)} = e\left\{\lambda(\nu-1)\frac{\mu'(\theta)}{\mu(\theta)}\right\},$$

where θ lies between λ and $\lambda\nu$. Hence

$$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\lambda O(1/\lambda)} = e^{O(1)}.$$

It is evident that the same result holds for *decreasing* functions which decrease less rapidly than $\lambda^{-\Delta}$ for some value of Δ.

Proof of the theorem when κ is an integer.

5.1. In proving the theorem we may suppose c_n to be *real*: if c_n is complex we can consider the real and imaginary parts of the series separately. We may also suppose, without real loss of generality, that $C = 0$. If C is not zero we begin by proving the theorem for the series

$$(c_1 - C) + c_2 + c_3 + \ldots,$$

and afterwards add to this series the convergent series

$$C + 0 + 0 + \ldots .$$

We are given that

$$\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\,d\sigma = o(\eta^\kappa),\dagger \tag{5.11}$$

and we wish to prove that

$$\int_{\mu_1}^{\zeta} C_\mu(\tau)(\zeta-\tau)^{\kappa-1}\,d\tau = o(\zeta^\kappa). \tag{5.12}$$

In (5.12) we put $$\tau = \mu(\sigma),$$

and observe that $$C_\mu\{\mu(\sigma)\} = C_\lambda(\sigma).$$

* More precise results of this character will be found in my paper "Oscillating Dirichlet's integrals", *Quarterly Journal*, Vol. 44, 1913, pp. 1-40 (see p. 23 *et seq.*).

† Since $C_\lambda(\sigma) = 0$ for $0 \leqslant \sigma \leqslant \lambda_1$, it is a matter of indifference whether the lower limit is 0 or λ_1.

We thus obtain

$$(5.13) \qquad \int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu' d\sigma = o(\zeta^\kappa),$$

where $\mu = \mu(\sigma), \quad \zeta = \mu(\eta).$

From this point onwards our argument depends on the nature of κ. I shall suppose first that κ is an integer.

5.2. If κ is an integer, we have

$$C_\lambda(\sigma) = \frac{1}{\kappa!}\left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma).*$$

The integral (5.13) is therefore a constant multiple of

$$(5.21) \qquad J = \int_{\lambda_1}^{\eta} \frac{d}{d\sigma}(\zeta-\mu)^\kappa \left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma)\, d\sigma.$$

We transform this integral by κ integrations by parts. Observing that $C_\lambda^\kappa(\sigma)$ and its first $\kappa-1$ derivatives vanish for $\sigma = \lambda_1$, and that $\zeta-\mu$ vanishes for $\sigma = \eta$, we obtain

$$(5.22) \qquad \begin{aligned} J &= (-1)^{\kappa-1} C_\lambda^\kappa(\eta)\left[\left(\frac{d}{d\sigma}\right)^\kappa (\zeta-\mu)^\kappa\right]_{\sigma=\eta} \\ &\quad + (-1)^\kappa \int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\zeta-\mu)^\kappa\, d\sigma \\ &= J_1 + J_2, \end{aligned}$$

say.

5.3. In the first place

$$(5.31) \qquad J_1 = -\kappa!\, C_\lambda^\kappa(\eta)(\zeta')^\kappa,$$

where ζ' is the value of μ' when $\sigma = \eta$, $\mu = \zeta$. Hence

$$J_1 = -\kappa!\,\frac{C_\lambda^\kappa(\eta)}{\eta^\kappa}\left(\frac{\eta\zeta'}{\zeta}\right)^\kappa \zeta^\kappa = o(1)\,O(1)\,\zeta^\kappa = o(\zeta^\kappa),$$

by 4.2.

* H. and R., p. 28.

On the other hand, it is easily verified that

(5.32) $$\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\zeta-\mu)^{\kappa} = \Sigma A\zeta^{\kappa-r}\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\ldots,$$

where the A's are constants, and

(5.331) $$0 < s+s_1+s_2+\ldots = r \leqslant \kappa,$$

(5.332) $$s_1+2s_2+3s_3+\ldots = \kappa+1.$$

Hence our integral reduces to a sum of constant multiples of integrals of the types

(5.34) $$\zeta^{\kappa-r}\int_{\lambda_1}^{\eta} C_{\lambda}^{\kappa}(\sigma)\,\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\ldots\,d\sigma.$$

Observing that $$C_{\lambda}(\sigma) = o(\sigma^{\kappa}),$$

and $$\mu^{(r)} = O\left(\frac{\mu}{\lambda^{r}}\right),$$

by 4.2, we see that (5.34) is of the form

(5.35) $$o\left(\zeta^{\kappa-r}\int_{\lambda_1}^{\eta}\sigma^{\kappa-s_1-2s_2-3s_3-\ldots+1}\,\mu^{s+s_1+s_2+\ldots-1}\mu'\,d\sigma\right)$$

$$= o\left(\zeta^{\kappa-r}\int_{\lambda_1}^{\eta}\mu^{r-1}\mu'\,d\sigma\right) = o(\zeta^{\kappa}).$$

This completes the proof of the theorem when κ is an integer.

Proof when $0 < \kappa < 1$.

6.1. We consider next the case in which $0 < \kappa < 1$. I shall suppose first that the increase of μ is greater than that of $\log\lambda$, so that $\lambda\mu'$ tends steadily to infinity with λ.

We observe first that, in virtue of the first theorem of consistency, Σc_n is summable $(\lambda, 1)$, to sum zero, so that

(6.11) $$C_{\lambda}^{1}(\sigma) = o(\sigma).$$

Now let A be any positive constant. Then we can, when η is large enough, determine a unique number η_1 such that

(6.12) $$\lambda_1 < \eta_1 < \eta, \quad \zeta-\zeta_1 = A\eta_1\zeta_1',$$

where

(6.121) $$\zeta_1 = \mu(\eta_1), \qquad \zeta_1' = \mu'(\eta_1).$$

For, as σ increases from λ_1 to η, $\zeta-\mu$ decreases steadily from $\zeta-\mu_1$ (which is large when η is large) to zero, whereas $\sigma\mu'$ increases steadily with σ.

Let us suppose that $0 < A < 1$, and that η_1 has been chosen so as to satisfy (6.12). Then we can determine a positive constant h such that

(6.13) $$h\eta < \eta_1 < \eta.$$

For $$\zeta-\zeta_1 = (\eta-\eta_1)\,\zeta_2',$$

where ζ_2' is the value of μ' when σ has a certain value η_2 between η_1 and η. Thus
$$A\eta_1\zeta_1' = \zeta-\zeta_1 = (\eta-\eta_1)\,\zeta_2' = \frac{\eta-\eta_1}{\eta_2}\,\eta_2\zeta_2' > \frac{\eta-\eta_1}{\eta}\,\eta_1\zeta_1',$$

since $\sigma\mu'$ increases with λ. Hence

$$A > \frac{\eta-\eta_1}{\eta}, \qquad \eta_1 > (1-A)\,\eta,$$

and we may take $$h = 1-A.$$

6.2. We now write

(6.21) $$J = \int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu'd\sigma = \int_{\lambda_1}^{\eta_1} + \int_{\eta_1}^{\eta} = J_1+J_2,$$

say. We begin by considering J_1. Integrating by parts, we obtain

(6.22) $$J_1 = C_\lambda^1(\eta_1)(\zeta-\zeta_1)^{\kappa-1}\zeta_1' - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)\frac{d}{d\sigma}\{(\zeta-\mu)^{\kappa-1}\mu'\}\,d\sigma$$
$$= J_{1,1}+J_{1,2},$$

say. In the first place

(6.23) $$J_{1,1} = \frac{C_\lambda^1(\eta_1)}{\eta_1}(A\eta_1\zeta_1')^{\kappa-1}\eta_1\zeta_1' = o(\eta_1\zeta_1')^\kappa = o(\zeta_1^\kappa) = o(\zeta^\kappa).$$

Secondly,

(6.24) $$J_{1,2} = (\kappa-1)\int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\zeta-\mu)^{\kappa-2}\mu'^2 d\sigma - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\zeta-\mu)^{\kappa-1}\mu''\,d\sigma$$
$$= J_{1,2,1}+J_{1,2,2},$$

say. Now

(6.25)
$$\frac{\sigma\mu'}{\zeta-\mu} < \frac{\sigma\mu'}{\zeta-\zeta_1} < \frac{\sigma\mu'}{A\eta_1\zeta_1'} < \frac{1}{A},$$

since $\lambda_1 < \sigma < \eta_1$ and $\sigma\mu'$ increases with σ. Hence

(6.26)
$$J_{1,2,1} = (\kappa-1)\int_{\lambda_1}^{\eta_1} o(\sigma)(\zeta-\mu)^{\kappa-1}\mu'\frac{\sigma\mu'}{\zeta-\mu}\,d\sigma$$
$$= o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= o(\zeta^\kappa).$$

Also

(6.27)
$$J_{1,2,2} = -\int_{\lambda_1}^{\eta_1} o(\sigma)(\zeta-\mu)^{\kappa-1}O\left(\frac{\mu'}{\sigma}\right)d\sigma$$
$$= o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= o(\zeta^\kappa).$$

From (6.22)–(6.27) it follows that

(6.28)
$$J_1 = o(\zeta^\kappa).$$

6.3. It remains to consider J_2. We have

(6.311)
$$J_2 = \int_{\eta_1}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= \zeta'\int_{\eta_2}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\,d\sigma$$

if μ' increases; and

(6.312)
$$J_2 = \zeta_1'\int_{\eta_1}^{\eta_2} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\,d\sigma$$

if μ' decreases, η_2 denoting in either case a number between η_1 and η. Now

$$\frac{d}{d\sigma}\frac{\zeta-\mu}{\eta-\sigma} = \frac{\zeta-\mu-(\eta-\sigma)\mu'}{(\eta-\sigma)^2} = \frac{\bar{\zeta}'-\mu'}{\eta-\sigma},$$

$\bar{\zeta}'$ being the value of μ' for a value $\bar{\eta}$ of its argument between σ and η.

This is positive if μ' increases and negative if μ' decreases. Hence

$$\frac{\zeta-\mu}{\eta-\sigma}$$

is monotonic, and varies in the same sense as μ'; so that

$$\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1}$$

is monotonic in the sense opposite to that of the variation of μ'. We have therefore

$$(6.321) \qquad J_2 = \zeta' \int_{\eta_2}^{\eta} C_\lambda(\sigma) \left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma$$

$$= \zeta' \left(\frac{\zeta-\zeta_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_2}^{\eta_3} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' increases; and

$$(6.322) \qquad J_2 = \zeta_1' \int_{\eta_1}^{\eta_2} C_\lambda(\sigma) \left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma$$

$$= \zeta_1' \left(\frac{\zeta-\zeta_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' decreases, η_3 being in either case another number between η_1 and η. And in either case

$$\frac{\zeta-\zeta_2}{\eta-\eta_2}$$

lies between ζ_1' and ζ'.

The order of μ lies between $\log\lambda$ and λ^{Δ}, and that of μ' between $1/\lambda$ and $\lambda^{\Delta-1}$, and *a fortiori* between $\lambda^{-\Delta}$ and λ^{Δ}.* Also

$$h\eta < \eta_1 < \eta.$$

It follows from 4.3 that the ratio ζ_1'/ζ' lies between fixed positive limits. We have therefore in any case

$$(6.33) \qquad J_2 = O(\zeta')^{\kappa} \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma.$$

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* Evidently we may suppose $\Delta > 1$.

But*

(6.34) $$\left| \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}d\sigma \right| < 2 \operatorname*{Max}_{\lambda_1 \leqslant \tau \leqslant \eta} C_\lambda^\kappa(\tau).$$

Hence

(6.35) $$J_2 = O(\zeta')^\kappa o(\eta^\kappa) = o(\eta\zeta')^\kappa = o(\zeta^\kappa).$$

From (6.21), (6.28), and (6.35) it follows that

(6.36) $$J = o(\zeta^\kappa).$$

6.4. We have thus proved the theorem when $0 < \kappa < 1$ and the order of μ lies between $\log\lambda$ and λ^Δ. Suppose next that the order of μ lies between $\log\log\lambda$ and $(\log\lambda)^\Delta$; and let

$$\nu = (\log\lambda)^a,$$

where $a > 1$. The series is summable (λ, κ), and therefore, by what precedes, summable (ν, κ). But

$$\mu(\lambda) = \mu(e^{\nu^{1/a}})$$

is an L-function of ν whose order lies between $(1/a)\log\nu$ and $\nu^{\Delta a}$. Hence the series is summable (μ, κ). The theorem is thus proved when $0 < \kappa < 1$ and the order of μ is greater than $\log\log\lambda$. Repeating the argument, we prove it whenever the order of μ is greater than any one of

$$\log\log\log\lambda, \quad \log\log\log\log\lambda, \quad \ldots.$$

Since any L-function which tends to infinity must increase more rapidly than some one of the repeated logarithmic functions†, the theorem is true without restriction on μ. The proof when $0 < \kappa < 1$ is thus complete.

Proof when κ is greater than 1 *and not integral.*

7.1. The proof of the theorem when κ is greater than 1, but not an integer, presents no fresh difficulty of principle. All that is necessary is to combine in an appropriate manner the arguments used in 5 and 6.

We suppose that

(7.11) $$k < \kappa < k+1,$$

* H. and R., p. 28 (Lemma 7).

† O. I., p. 20; "Properties &c.", pp. 63 *et seq.*

k being an integer, and (for the moment) that μ is of higher order than $\log\lambda$. Integrating the integral (5.13) k times by parts, we obtain a constant multiple of

$$(7.12) \qquad J = \int_{\lambda_1}^{\eta} C_\lambda^k(\sigma)\left(\frac{d}{d\sigma}\right)^{k+1}(\zeta-\mu)^\kappa\, d\sigma.$$

We write, as in 6.2,

$$(7.13) \qquad J = \int_{\lambda_1}^{\eta_1} + \int_{\eta_1}^{\eta} = J_1 + J_2.$$

7.2. In order to obtain an upper limit for J_1 we integrate once more by parts. We thus obtain

$$(7.21) \quad J_1 = \frac{C_\lambda^{k+1}(\eta_1)}{k+1}\left(\frac{d}{d\eta_1}\right)^{k+1}(\zeta-\zeta_1)^\kappa - \frac{1}{k+1}\int_{\lambda_1}^{\eta_1} C_\lambda^{k+1}(\sigma)\left(\frac{d}{d\sigma}\right)^{k+2}(\zeta-\mu)^\kappa\, d\sigma$$

$$= J_{1,1} + J_{1,2},$$

say. Now

$$(7.22) \qquad \left(\frac{d}{d\sigma}\right)^{k+1}(\zeta-\mu)^\kappa = \Sigma A\,(\zeta-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\ldots,$$

where

$$(7.231) \qquad s+s_1+s_2+\ldots = \kappa,$$

$$(7.232) \qquad s_1+2s_2+3s_3+\ldots = k+1.$$

Hence

$$(7.24) \qquad J_{1,1} = \frac{C^{k+1}(\eta_1)}{k+1}\,\Sigma A\,(\zeta-\zeta_1)^s(\zeta_1')^{s_1}(\zeta_1'')^{s_2}\ldots.$$

But $$C^{k+1}(\eta_1) = o(\eta_1^{k+1}),$$

since the series is summable (λ, κ), and *a fortiori* summable $(\lambda, k+1)$, to sum 0. Also

$$\zeta-\zeta_1 = O(\eta_1\zeta_1'),$$

by (6.12), and

$$\zeta_1^{(r)} = O\left(\frac{\zeta_1'}{\eta_1^{r-1}}\right),$$

by 4.2. Hence

$$(7.25) \qquad J_{1,1} = \Sigma o\left\{\eta_1^{k+1}(\eta_1\zeta_1')^s\,\eta_1^{-s_2-2s_3-\ldots}(\zeta_1')^{s_1+s_2+\ldots}\right\}$$

$$= \Sigma o\left\{\eta_1^{s+s_1+2s_2+\ldots-s_2-2s_3-\ldots}(\zeta_1')^{s+s_1+s_2+\ldots}\right\}$$

$$= \Sigma\, o(\eta_1\zeta_1')^{s+s_1+s_2+\ldots} = o(\eta_1\zeta_1')^\kappa$$

$$= o(\zeta_1^\kappa) = o(\zeta^\kappa).$$

7.3. The integral $J_{1,2}$ is a sum of constant multiples of integrals of the type

(7.31) $$\int_{\lambda_1}^{\eta_1} C_\lambda^{k+1}(\sigma)(\zeta-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\dots d\sigma,$$

where now

(7.321) $$s+s_1+s_2+\dots = \kappa,$$

(7.322) $$s_1+2s_2+3s_3+\dots = k+2.$$

The integral (7.31) is of the form

(7.33) $$\int_{\lambda_1}^{\eta_1} o(\sigma^{k+1})(\zeta-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\dots(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma.$$

Now $$\zeta-\mu \geqslant \zeta-\zeta_1 = A\eta_1\zeta_1' \geqslant A\sigma\mu',$$

for $\lambda_1 \leqslant \sigma \leqslant \eta_1$; and

$$s-\kappa+1 = 1-s_1-s_2-\dots \leqslant 0.$$

Hence

(7.34) $$(\zeta-\mu)^{s-\kappa+1} = O(\sigma\mu')^{s-\kappa+1};$$

and

(7.35) $$\sigma^{k+1}(\zeta-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\dots$$
$$= O\left\{\sigma^{k+1}(\sigma\mu')^{s-\kappa+1}\sigma^{-s_2-2s_3-\dots}(\mu')^{s_1+s_2+\dots-1}\right\}$$
$$= O(1),$$

since $$k+1+s-\kappa+1-s_2-2s_3-\dots = 0$$

and $$s-\kappa+1+s_1+s_2+\dots-1 = 0,$$

in virtue of (7.321) and (7.322). Thus the integral (7.33) is of the form

$$o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma = o(\zeta_1^\kappa) = o(\zeta^\kappa);$$

so that

(7.36) $$J_{1,2} = o(\zeta^\kappa).$$

From (7.25) and (7.36) it follows that

(7.37) $$J_1 = o(\zeta^\kappa).$$

7.4. It remains to consider

(7.41) $$J_2 = \int_{\eta_1}^{\eta} C^k(\sigma) \left(\frac{d}{d\sigma}\right)^{k+1} (\zeta-\mu)^\kappa \, d\sigma,$$

which is a sum of constant multiples of integrals of the type

(7.42) $$\int_{\eta_1}^{\eta} C^k(\sigma)(\zeta-\mu)^s (\mu')^{s_1} (\mu'')^{s_2} \dots d\sigma,$$

where

(7.431) $$s+s_1+s_2+\dots = \kappa,$$

(7.432) $$s_1+2s_2+3s_3+\dots = k+1.$$

The integral (7.42) may be written in one or other of the forms

(7.441) $$(\zeta')^{s_1} (\zeta'')^{s_2} \dots \int_{\eta_2}^{\eta} C^k(\sigma)(\zeta-\mu)^s \, d\sigma,$$

(7.442) $$(\zeta_1')^{s_1} (\zeta_1'')^{s_2} \dots \int_{\eta_1}^{\eta_2} C^k(\sigma)(\zeta-\mu)^s \, d\sigma.$$

Arguing as in 6.3, we replace each of these integrals by one of the form

(7.45) $$j = (\zeta_3')^{s_1} (\zeta_3'')^{s_2} \dots \left(\frac{\zeta-\zeta_4}{\eta-\eta_4}\right)^s \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma,$$

where $\eta_3, \eta_4, \dots$ are numbers between η_1 and η, and $\zeta_3, \zeta_4, \dots, \zeta_3', \zeta_4', \dots$ are the corresponding values of μ and μ'. We write (7.45) in the form

(7.46) $$j = j_1 j_2,$$

where j_1 and j_2 denote the external factor and the integral in (7.45) respectively.

7.5. It follows from arguments similar to those employed in 6.3 that

(7.51) $$j_1 = O\{(\zeta')^{s+s_1}(\zeta'')^{s_2} \dots\} = O\{\eta^{-s_2-2s_3-\dots}(\zeta')^{s+s_1+s_2+\dots}\}.$$

In order to obtain an upper limit for j_2, we observe that

(7.52) $$[s] = k-s_1-s_2-\dots = k',$$

say, and integrate $k'+1$ times by parts. We thus obtain

(7.53) $$j_2 = \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma = F(\eta_6)-F(\eta_5)+j_4 = j_3+j_4,$$

say, where

$$(7.54)\quad F(\sigma)=\frac{1}{k+1}C^{k+1}(\sigma)(\eta-\sigma)^{s}+\frac{s}{(k+1)(k+2)}C^{k+2}(\sigma)(\eta-\sigma)^{s-1}+\dots$$

$$+\frac{s(s-1)\dots(s-k'+1)}{(k+1)(k+2)\dots(k+k'+1)}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'},$$

$$(7.55)\quad j_4=\frac{s(s-1)\dots(s-k')}{(k+1)(k+2)\dots(k+k'+1)}\int_{\eta_5}^{\eta_6}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'-1}\,d\sigma.$$

Any term of $F(\sigma)$ is of the form

$$o(\eta^{k+r+s-r+1})=o(\eta^{k+s+1});$$

and so

$$(7.56)\qquad j_3=o(\eta^{k+s+1}).$$

On the other hand $s-k'$ lies between 0 and 1, and so*

$$(7.57)\quad \left|\int_{\eta_5}^{\eta_6}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'-1}\,d\sigma\right|$$

$$\leqslant 2\frac{\Gamma(k+k'+2)\,\Gamma(s-k')}{\Gamma(s+k+2)}\operatorname*{Max}_{\lambda_1\leqslant\tau\leqslant\eta}|C^{s+k+1}(\tau)|$$

$$=o(\eta^{k+s+1}).$$

Thus both j_3 and j_4 are of this form, and so, therefore, is j_2; and therefore

$$(7.58)\qquad j=j_1j_2=o\left\{\eta^{k+s+1-s_2-2s_3-\dots}(\zeta')^{s+s_1+s_2+\dots}\right\}$$

$$=o(\eta\zeta')^{s+s_1+s_2+\dots}$$

$$=o(\eta\zeta')^{\kappa}=o(\zeta^{\kappa}).$$

Hence

$$(7.59)\qquad J_2=o(\zeta^{\kappa}).$$

7.6. From (7.13), (7.37), and (7.59) it follows that

$$(7.61)\qquad J=o(\zeta^{\kappa}).$$

The proof of the theorem is thus completed, provided that the order of μ is greater than that of $\log\lambda$. In order to extend the result to cover all possible cases we have only to repeat the argument of 6.4.

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* H. and R., p. 29 (Lemma 8).

Conclusion.

8. It remains only to show that the theorem is the best possible theorem of its kind. In order to prove this it is necessary to show that *if μ is any L-function of λ which tends to infinity more rapidly than any power of λ, then we can determine a number κ and a series Σc_n which is summable (λ, κ) and not summable (μ, κ).*

We may take $\lambda = n$, and we may suppose that μ is of lower order than e^n, since methods of type as high as e^n will sum convergent series only*. Consider the series

$$(8.1) \qquad \Sigma(-1)^n \left(\frac{\mu_n}{\mu'_n}\right)^\kappa,$$

which may be written in the form

$$\Sigma(-1)^n n^\kappa \left(\frac{\mu_n}{n\mu'_n}\right)^\kappa.$$

Since μ_n is of order higher than any power of n, we have

$$\frac{\mu_n}{n\mu'_n} = o(1).\dagger$$

Hence

$$\left(\frac{\mu_n}{n\mu'_n}\right)^\kappa$$

is an L-function which tends to zero, and so also are all its derivatives. All of these derivatives, moreover, are ultimately of constant sign; and the same is true of all the successive differences of the function. Also the series

$$\Sigma(-1)^n n^\kappa$$

is *finite* (C, κ). It follows from known theorems‡ that the series (8.1) is summable (C, κ), *i.e.* (n, κ).

But the series (8.1) is *not* summable (μ, κ). For if it were, we should have

$$\left(\frac{\mu_n}{\mu'_n}\right)^\kappa = o\left(\frac{\mu_{n+1}}{\mu_{n+1}-\mu_n}\right)^\kappa \S,$$

* H. and R., p. 46 (Theorem 36).

† O.I., p. 38.

‡ The theorem required is a special case of Theorem 1*a* (p. 61) of Bohr's dissertation "Bidrag til de Dirichlet'ske Rækkers Theori." (Copenhagen, 1910).

§ H. and R., p. 36 (Theorem 21).

and this is untrue, since $$\mu_{n+1} \sim \mu_n$$

and $$\mu_{n+1} - \mu_n \sim \mu_n',$$

when μ_n is an L-function of n of order less than e^n*.

For example, the series

$$1-1+1-1+\dots$$

is summable $(1, \kappa)$ for any positive value of κ, but is not summable (e^n, κ) for any value of κ. The series

$$\sqrt{1}-\sqrt{2}+\sqrt{3}-\dots$$

is summable $(n, 1)$ but not summable $(e^{\sqrt{n}}, 1)$; and so on.

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* O.I., pp. 41 *et seq.*