DIVISION OF THE LEMNISCATE INTO SEVEN EQUAL PARTS

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IN this note $\wp u$ stands for the function defined by

$$\wp'^2 u = 4\wp^3 u - 4\wp u,$$

and ω is the real half period given by

$$\omega = \int_0^1 \frac{dt}{\sqrt{(1-t^4)}} = \int_1^\infty \frac{dt}{\sqrt{(4t^3-4t)}}.$$

Taking μ to be any complex integer $(m+ni)$ not divisible by 7 (real values of μ are, of course, included), the quantity x_μ defined by

$$x_\mu = \wp\,\frac{2\mu\omega}{7},$$

is one of 24 values, which are the roots of the equation $\psi_7 = 0$, where

$$\psi_7 = 7x^{24} - 308x^{22} - 2954x^{20} + 19852x^{18} - 35231x^{16} + 82264x^{14} - 111916x^{12} + 42168x^{10} + 15673x^8 - 14756x^6 + 1302x^4 - 196x^2 - 1.$$

This polynomial is irreducible in the rational field; but by a series of adjunctions it can be broken up into the product of eight cubics in the following manner.

Every coefficient of ψ_7, except the last, is divisible by 7, and we have identically

$$\tfrac{1}{7}\psi_7 = u^2 - 7v^2,$$

where

$$u = x^{12} - 22x^{10} - 229x^8 + 1308x^6 - 633x^4 - 614x^2 - 3,$$

$$v = 8x^{10} + 88x^8 - 496x^6 + 240x^4 + 232x^2 + \tfrac{8}{7}.$$

Hence, if we put

$$a^2 = 7,$$

ψ_7 has the factor f_{12}, given by

$$f_{12} = x^{12} - (22+8a)\,x^{10} - (229+88a)\,x^8 + (1308+496a)\,x^6 - (633+240a)\,x^4 - (614+232a)\,x^2 - \tfrac{1}{7}(21+8a).$$

Next put
$$\beta^2 = 2\alpha,$$
it is found that f_{12} has the factor f_6, given by
$$f_6 = x^6-(11+4\alpha+7\beta+3\alpha\beta)\,x^4+(63+24\alpha+26\beta+10\alpha\beta)\,x^2$$
$$-\tfrac{1}{7}(371+140\alpha+161\beta+61\alpha\beta).$$

Finally, let
$$\gamma^2 = \beta(3-\alpha),$$
then f_6 has the factor f_3 given by
$$f_3 = x^3-\frac{\gamma}{4}(16+6\alpha+3\beta+\alpha\beta)\,x^2$$
$$+(5+2\alpha+3\beta+\alpha\beta)\,x-\frac{\alpha\beta\gamma}{28}(19+7\alpha+8\beta+3\alpha\beta).$$

By giving to α, β, γ all their different values, we obtain from f_3 the eight conjugate cubic factors of ψ_7. In particular, if we take α, β, γ all real and positive, the roots of $f_3 = 0$ are all real and positive, and are accordingly the values of x_1, x_2, x_3.

From an algebraical point of view this solution is as simple as can be desired; it may, however, be put into another shape, which is of much theoretical interest. In the transformation theory for $n = 7$, Klein's principal modulus τ is connected with the absolute invariant J by the relations
$$J : (J-1) : 1 = (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3$$
$$: (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau.$$

For the lemniscate functions $J = 1$, and consequently τ satisfies the equation
$$\phi(\tau) = \tau^4+14\tau^3+63\tau^2+70\tau-7 = 0.$$

This may be written
$$(\tau^2+7\tau+21)^2-28(\tau+4)^2 = 0,$$
whence
$$\tau^2+(7\pm 2\sqrt{7})\tau+(21\pm 8\sqrt{7}) = 0,$$
and hence
$$4\tau = -14+4\epsilon_1\sqrt{7}+\epsilon_2(\epsilon_1\sqrt{7}-1)\sqrt{(2\epsilon_1\sqrt{7})},$$
where ϵ_1, ϵ_2 are independent square roots of unity.

Let τ be any one of the four values of τ; then if we put
$$\alpha = -\tfrac{1}{18}(\tau^3+10\tau^2+14\tau-49),$$
$$\beta = \tfrac{1}{18}(\tau^3+16\tau^2+68\tau+35),$$

we have in virtue of $\phi(\tau) = 0$,

$$a^2 = 7, \quad \beta^2 = 2a,$$

and moreover $$a\beta = \tfrac{1}{18}(5\tau^3 + 56\tau^2 + 196\tau + 91).$$

By substituting these expressions, f_6 assumes the form

$$\begin{aligned} f_6 = x^6 &- \tfrac{1}{3}(3\tau^3 + 40\tau^2 + 168\tau + 152)x^4 \\ &+ \tfrac{1}{9}(26\tau^3 + 368\tau^2 + 1696\tau + 2065)x^2 \\ &- \tfrac{1}{63}(163\tau^3 + 2296\tau^2 + 10472\tau + 12362). \end{aligned}$$

We have also, in terms of τ,

$$9\gamma^2 = -\tau^3 - 4\tau^2 + 4\tau + 7,$$

so if we put $$-\tau^3 - 4\tau^2 + 4\tau + 7 = \sigma^2,$$

we have a cubic factor in the form

$$\begin{aligned} f_3 = x^3 &+ \tfrac{1}{108}(\tau^3 + 22\tau^2 + 158\tau + 389)\sigma x^2 \\ &+ \tfrac{1}{3}(\tau^3 + 14\tau^2 + 62\tau + 64)x \\ &+ \tfrac{1}{252}(17\tau^3 + 238\tau^2 + 1078\tau + 1253)\sigma = 0. \end{aligned}$$

It may be noticed that we can put

$$8\tau = \beta^3 + 4\beta^2 - 2\beta - 28,$$

with $$\beta^4 = 28.$$

I am indebted to my colleague, Mr. W. E. H. Berwick, for checking all of my work, except the calculation of ψ_7, and especially for performing the actual resolution of ψ_7 into its factors; this is entirely his work, and I have only partly verified it. As to the value of ψ_7, Mr. T. G. Creak was kind enough to work it out according to my directions, and since his result agreed with one I had found myself by an entirely different process, it is practically certain that the value given is correct.

Reference should be made to a paper by Brioschi (*American Journal*, Vol. XIII, 1891, p. 381); use was made of this in the course of the investigation.