THE DIVISORS OF NUMBERS

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1. The excess of the number of divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$ is a quantity of arithmetical importance which was studied by Jacobi in the *Fundamenta Nova*, and later by Glaisher,* who denoted it by $E(n)$. The present paper is a study of other numerical quantities of a similar nature.

If $\sigma_s(n)$ denote the sum of the s-th powers of the divisors of n, we have the well known identities

$$\sum_1^\infty \frac{m^s q^m}{1-q^m} = \sum_1^\infty \sigma_s(n) q^n, \quad \sum_1^\infty \frac{q^m}{(1-q^m)^2} = \sum_1^\infty \sigma_1(n) q^n,$$

and thence the well known double identity

$$\sum_1^\infty \frac{mq^m}{1-q^m} = \sum_1^\infty \frac{q^m}{(1-q^m)^2} = \sum_1^\infty \sigma_1(n) q^n,$$

which means when interpreted that the number of the divisors of n is equal to the number of the conjugates of divisors of n.

In general a double identity can always be obtained, a fact which will be in constant evidence in what follows.

If d be a divisor of n and a an arbitrary quantity, it is easy to establish the relations

$$(1) \qquad \sum_1^\infty \frac{aq^m}{1-aq^m} = \sum_1^\infty \frac{a^m q^m}{1-q^m} = \sum_1^\infty (\Sigma a^d) q^n,$$

by the method of expansion in row and summation by column introduced by Lambert.†

* *Proc. London Math. Soc.*, Ser. 1, Vol. 15.

† See also *Combinatory Analysis*.

We immediately deduce the relations

(2) $$\left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{a^m q^m}{1-q^m} = \sum_1 (\Sigma d^s a^d) q^n,$$

(3) $$\left(q\frac{d}{dq}\right)^u \left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \left(q\frac{d}{dq}\right)^u \left(a\frac{d}{da}\right) \sum_1^\infty \frac{a^m q^m}{1-q^m}$$

$$= \sum_1^\infty (\Sigma n^u d^s a^d) q^n,$$

which will be dealt with later in the paper.

At present I proceed in another manner from the relation (1).

Putting $$a = e^{ix},$$

where $$i = \surd(-1),$$

we find that

$$\frac{e^{ix}q^m}{1-e^{ix}q^m} = \frac{q^m \cos x - q^{2m}}{1-2q^m\cos x + q^{2m}} + i\,\frac{q^m \sin x}{1-2q^m \cos x + q^{2m}},$$

and since

$$\int \frac{e^{ix}q^m}{1-e^{ix}q^m}\,dx = \tan^{-1}\frac{q^m \sin x}{1-q^m\cos x} + \tfrac{1}{2}i \log(1-2q^m\cos x+q^{2m}),$$

the relations

(4) $$\sum_1^\infty \frac{q^m\cos x - q^{2m}}{1-2q^m\cos x+q^{2m}} = \sum_1^\infty \frac{q^m \cos mx}{1-q^m} = \sum_1^\infty (\Sigma \cos dx)\, q^n,$$

(5) $$\Sigma \frac{q^m \sin x}{1-2q^m \cos x + q^{2m}} = \sum_1^\infty \frac{q^m \sin mx}{1-q^m} = \sum_1^\infty (\Sigma \sin dx)\, q^n,$$

lead by integration to the relations

(6) $$\sum_1^\infty \tan^{-1}\frac{q^m \sin x}{1-q^m\cos x} = \sum_1^\infty \frac{1}{m}\frac{q^m \sin mx}{1-q^m} = \sum_1^\infty \left(\Sigma \frac{1}{d}\sin dx\right) q^n,$$

(7) $$\sum_1^\infty \log\frac{1}{1-2q^m\cos x+q^{2m}} = 2\sum_1^\infty \frac{1}{m}\frac{q^m\cos mx}{1-q^m} = 2\sum_1^\infty \left(\Sigma\frac{1}{d}\cos dx\right) q^n.$$

The left-hand side of relation (7) may be written

$$\log \frac{1}{\prod_1^\infty (1-2q^m \cos x + q^{2m})},$$

and transforming by a well known formula in the *Fundamenta Nova*, we obtain

$$\log \frac{\sin \frac{1}{2}x \prod_1^\infty (1-q^m)}{\sin \frac{1}{2}x - q \sin \frac{3}{2}x + q^3 \sin \frac{5}{2}x - \dots} = 2 \sum_1^\infty \frac{1}{m} \frac{q^m \cos mx}{1-q^m}$$

$$= 2 \sum_1^\infty \left(\Sigma \frac{1}{d} \cos dx \right) q^n,$$

where the exponents of q in the denominator series are the triangular numbers.

Differentiating with regard to x and changing sign throughout, we obtain the formula

$$(8) \quad \frac{\cos \frac{1}{2}x - 3q \cos \frac{3}{2}x + 5q^3 \cos \frac{5}{2}x - 7q^6 \cos \frac{7}{2}x + \dots}{\sin \frac{1}{2}x - q \sin \frac{3}{2}x + q^3 \sin \frac{5}{2}x - q^6 \sin \frac{7}{2}x - \dots}$$

$$= \cot \tfrac{1}{2}x + 4 \sum_1^\infty \frac{q^m \sin mx}{1-q^m} = \cot \tfrac{1}{2}x + 4\Sigma(\Sigma \sin dx) q^n,$$

which is fundamental for this research.

If in $\sin mx$ we give x the special value π/p where p is an integer, and m the successive values 1, 2, 3, ..., the values of $\sin mx$ recur with a period $2p$. Thus, to take the simplest case possible, $p = 2$, $\sin mx$ has the values 1, 0, -1, 0, ... in a period of 4.

In other words, if m have the forms $4m+1$, $4m+3$, the values are 1, -1 respectively, while the value for the forms $4m$, $4m+2$ is zero.

Hence we see that $\qquad \Sigma \sin dx,$

where d denotes a divisor of n, represents the excess of the number of divisors which have the form $4m+1$ over the number of divisors which have the form $4m+3$. In fact

$$\Sigma (\Sigma \sin dx) q^n = \Sigma E(n) q^n$$

in Glaisher's notation, when x has the value $\frac{1}{2}\pi$.

Before proceeding to the consideration of the identity (8) which as has been seen is derived directly from the *Fundamenta Nova*, an important simplification can be made, because by simple trigonometry it can be thrown into the form

$$(9) \quad \frac{2q \sin x - (q+3q^3) \sin 2x + (2q^3+4q^6) \sin 3x - (3q^6+5q^{10}) \sin 4x + \dots}{1-(1+q)\cos x + (q+q^3)\cos 2x - (q^3+q^6)\cos 3x + (q^6+q^{10})\cos 4x - \dots}$$

$$= 2 \sum_1^\infty (\Sigma \sin dx) q^n.$$

In the fraction the general terms of numerator and denominator are respectively

$$(-)^{m+1}\left\{(m-1)\,q^{\frac{1}{2}m(m-1)}+(m+1)\,q^{\frac{1}{2}m(m+1)}\right\}\sin mx,$$

$$(-)^{m}\left\{q^{\frac{1}{2}m(m-1)}+q^{\frac{1}{2}m(m+1)}\right\}\cos mx.$$

Writing these for brevity

$$N_m \sin mx, \qquad D_m \cos mx,$$

we have

$$\frac{N_1 \sin x+N_2 \sin 2x+N_3 \sin 3x+\ldots}{1+D_1 \cos x+D_2 \cos 2x+D_3 \cos 3x+\ldots}=2\sum_1^\infty (\Sigma \sin dx)\,q^n.$$

The case $x=\frac{\pi}{2}$ gives

$$(10)\quad \frac{q-q^3-2q^6+2q^{10}+3q^{15}-3q^{21}-4q^{28}+4q^{36}+\ldots}{1-q-q^3+q^6+q^{10}-q^{15}-q^{21}+q^{28}+q^{36}-\ldots}=\sum_1^\infty E^{(2)}(n)\,q^n,$$

equivalent to Glaisher's formula (*loc. cit.*, p. 4).

I have in the above denoted by $E^{(2)}(n)$ the quantity for which Glaisher's symbol is $E(n)$, because it is convenient to denote by $E^{(p)}(n)$ the arithmetical quantity obtained by putting $x=\frac{\pi}{p}$ in the formula.

Before proceeding to the general case I work out a few of the elementary cases.

The case $x=\frac{\pi}{3}$.

$\sin mx$ has the series of values of period 6,

$$\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,$$

and $\cos mx$ the series of values

$$\tfrac{1}{2},\ -\tfrac{1}{2},\ -1,\ -\tfrac{1}{2},\ \tfrac{1}{2},\ 1,$$

and we observe that if $E^{(3)}(n)$ represents the excess of the number of divisors of n which have the forms $6m+1$, $6m+2$ over the number of divisors which have the forms $6m+4$, $6m+5$,

$$\Sigma\left(\Sigma \sin\frac{d\pi}{3}\right)q^n=\frac{\sqrt{3}}{2}\,\Sigma\, E^{(3)}(x)\,q^n.$$

Inserting the values of the sines and cosines and throwing out the factors 2 and $\frac{\sqrt{3}}{3}$ we reach the relation

$$(11)\quad \frac{q-3q^3+3q^6+q^{10}-6q^{15}+6q^{21}+q^{28}-9q^{36}+9q^{45}-\dots}{1-2q+q^3+q^6-2q^{10}+q^{15}+q^{21}-2q^{28}+q^{36}+\dots} = \sum_1^\infty E^{(3)}(n)q^n.$$

In both numerator and denominator of this fraction the signs *recur* in the order $+, -, +$.

In the numerator

when the q exponent is of form $\frac{1}{2}(3m+1)(3m+2)$ the coefficient is unity,

" " $\frac{1}{2}(3m+2)(3m+3)$ " $-3(m+1)$,

" " $\frac{1}{2}(3m+3)(3m+4)$ " $+3(m+1)$.

In the denominator the coefficients recur in the order $1, -2, 1$. Thence the recurring formula

$$(12)\quad E^{(3)}(n)-2E^{(3)}(n-1)+E^{(3)}(n-3)+E^{(3)}(n-6)-\dots = 0$$ (or the number above specified according as n is not or is a triangular number).

The case $n = \frac{\pi}{4}$.

We have here a period of eight in the values of the sines and cosines.

$\sin\frac{m\pi}{4}$ has the values $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0,$

$\cos\frac{m\pi}{4}$ " $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1.$

Whence we gather that if $E^{(4)}(n)$ represents the excess of the number of divisors of n which have the forms $8m+1$, $8m+3$ over the number of divisors which have the forms $8m+5$, $8m+7$; and if $E^{(44)}(n)$ represents the excess of the number of divisors of n which have the form $8m+2$ over the number of divisors which have the form $8m+6$,

$$\Sigma\left(\Sigma\frac{d\pi}{4}\right)q^n = \frac{1}{\sqrt{2}}E^{(4)}(n)\,q^n+\Sigma\,E^{(44)}(n)\,q^n.$$

We find

$$\frac{\sqrt{2}(N_2-N_6+N_{10}-N_{14}+\ldots)+N_1+N_3-N_5-N_7+\ldots}{\sqrt{2}(1-D_4+D_7-D_{12}+D_{15}-\ldots)+D_1-D_3-D_5+D_7+\ldots}$$
$$= \sqrt{2}\,\Sigma E^{(4)}(n)\,q^n+\Sigma E^{(44)}(n)\,q^n,$$

simplifying to $$\frac{\sqrt{2}\,a+b}{\sqrt{2}\,c+d} = \sqrt{2}\,A+B,$$

where
$$\left.\begin{array}{l} a = -q-3q^3+5q^{15}+7q^{21} \\ b = 2q+2q^3+4q^6-4q^{10}-6q^{15} \end{array}\right\} \quad A = \Sigma E^{(4)}(n)\,q^n,$$
$$\left.\begin{array}{l} c = 1-q^6-q^{10}-q^{21}-q^{28}+\ldots \\ d = -1-q+q^3+q^6+q^{10}+q^{15}-\ldots \end{array}\right\} \quad B = \Sigma E^{(44)}(n) q^n.$$

Thence the relations
$$dA+cB = a,$$
$$2cA+dB = b,$$

and we are able to express A, B separately in terms of a, b, c, d; but if we did so the laws of the q series involved would not be so clear, so instead of writing

$$A = \frac{ad-bc}{d^2-2c^2}, \quad B = \frac{bd-2ac}{d^2-2c^2},$$

I prefer to retain the above simultaneous relations which are at length

(13) $$(-1-q+q^3+q^6+q^{10}+q^{15}-\ldots)\,\Sigma E^{(4)}(n)\,q^n$$
$$+(1-q^6-q^{10}-q^{21}-q^{28}+\ldots)\,\Sigma E^{(44)}(n)\,q^n$$
$$= -q-3q^3+5q^{15}+7q^{21}-\ldots,$$

(14) $$2(1-q^6-q^{10}-q^{21}-q^{28}+\ldots)\,\Sigma E^{(4)}(n) q^n$$
$$+(-1-q+q^3+q^6+q^{10}+q^{15}-\ldots)\,\Sigma E^{(44)}(n)\,q^n,$$
$$= 2\,(q+q^3+2q^6-2q^{10}-3q^{15}-\ldots),$$

leading to the simultaneous formulæ

(15) $$-E^{(4)}(n)-E^{(4)}(n-1)+E^{(4)}(n-3)+E^{(4)}(n-6)+E^{(4)}(n-10)$$
$$+E^{(4)}(n-15)-\ldots+E^{(44)}(n)-E^{(44)}(n-6)-E^{(44)}(n-10)$$
$$-E^{(44)}(n-21)-E^{(44)}(n-28)+\ldots$$
$$= \text{coefficient of } q^n \text{ in } -q-3q^3+5q^{15}+7q^{21}-\ldots,$$

(16) $2E^{(4)}(n)-2E^{(4)}(n-6)-2E^{(4)}(n-10)-2E^{(4)}(n-21)-2E^{(4)}(n-28)+\ldots$

$$-E^{44}(n)-E^{44}(n-1)+E^{(44)}(n-3)+E^{(44)}(n-6)+E^{(44)}(n-10)+\ldots$$

$$= \text{coefficient of } q^n \text{ in } 2(q+q^3+2q^6-2q^{10}-3q^{15}+\ldots).$$

The actual values of the trigonometrical functions are not necessary for the investigation. This will now appear.

The case $x = \frac{\pi}{5}$.

The values of the sines and cosines in a period of ten are

$$\sin\frac{\pi}{5},\ \sin\frac{2\pi}{5},\ \sin\frac{2\pi}{5},\ \sin\frac{\pi}{5},\ 0,\ -\sin\frac{\pi}{5},\ -\sin\frac{2\pi}{5},\ -\sin\frac{2\pi}{5},$$

$$-\sin\frac{\pi}{5},\ 0;$$

$$\cos\frac{\pi}{5},\ \cos\frac{2\pi}{5},\ -\cos\frac{2\pi}{5},\ -\cos\frac{\pi}{5},\ -1,\ -\cos\frac{\pi}{5},\ -\cos\frac{2\pi}{5},$$

$$\cos\frac{2\pi}{5},\ \cos\frac{\pi}{5},\ 1.$$

Whence, if $E^{(5)}(n)$ represents the excess of the number of divisors of n which have the forms $10m+1$, $10m+4$ over the number of divisors which have the forms $10m+6$, $10m+9$; and if $E^{(55)}(n)$ represents the excess of the number of divisors of n which have the forms $10m+2$, $10m+3$ over the number of divisors which have the forms $10m+7$, $10m+8$,

$$\Sigma\left(\Sigma\frac{d\pi}{5}\right)q^n = \sin\frac{\pi}{5}\,\Sigma E^{(5)}(n)\,q^n+\sin\frac{2\pi}{5}\,\Sigma E^{(55)}(n)\,q^n.$$

The left-hand side of the identity becomes as regards the numerator

$$\sin\frac{\pi}{5}(N_1+N_4-N_6-N_9+\ldots)+\sin\frac{2\pi}{5}(N_2+N_3-N_7-N_8+\ldots),$$

where the N subscripts are of forms $5m\pm 1$, $5m\pm 2$, respectively.

This is

$$\sin\frac{\pi}{5}(2q-3q^6-5q^{10}+5q^{15}+7q^{21}-8q^{36}-10q^{45}+\ldots)$$
$$+\sin\frac{2\pi}{5}(-q-q^3+4q^6-6q^{21}-8q^{28}+9q^{36}+\ldots)$$
$$=a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5}\text{ suppose,}$$

and as regards the denominator

$$1+\cos\frac{\pi}{5}(D_1-D_4-D_6+D_9+\ldots)+\cos\frac{2\pi}{5}(D_2-D_3-D_7+D_8+\ldots)$$
$$-D_5+D_{10}-D_{15}+\ldots.$$

This is

$$1+q^{10}+q^{15}+q^{45}+q^{55}+q^{105}+q^{120}+\ldots$$
$$+\cos\frac{\pi}{5}(-1-q-q^6-q^{10}-q^{15}-q^{21}-q^{36}-q^{45}-\ldots)$$
$$+\cos\frac{2\pi}{5}(q+2q^3+q^6+q^{21}+2q^{28}+q^{36}+\ldots)$$
$$=b_0+b_1\cos\frac{\pi}{5}+b_2\cos\frac{2\pi}{5}\text{ suppose.}$$

So that writing the right-hand side of the identity

$$2A_1\sin\frac{\pi}{5}+2A_2\sin\frac{2\pi}{5},$$

where $A_1=\Sigma E^{(5)}(n)\,q^n,\quad A_2=\Sigma E^{(55)}(n)\,q^n,$

$$\frac{a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5}}{b_0+b_1\cos\frac{\pi}{5}+b_2\cos\frac{2\pi}{5}}=2A_1\sin\frac{\pi}{5}+2A_2\sin\frac{2\pi}{5},$$

whence

$$\{(2b_0-b_2)A_1+(b_1+b_2)A_2\}\sin\frac{\pi}{5}+\{(b_1+b_2)A_1+(2b_0+b_1)A_2\}\sin\frac{2\pi}{5}$$
$$=a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5},$$

a relation which is of the form

$$\alpha_1\sin\frac{\pi}{5}+\alpha_2\sin\frac{2\pi}{5}=0,$$

and which can only be satisfied when

$$\alpha_1=\alpha_2=0.$$

Hence

$$(17)\qquad \begin{cases}(2b_0-b_2)A_1+(b_1+b_2)A_2=a_1,\\ (b_1+b_2)A_1+(2b_0+b_1)A_2=a_2,\end{cases}$$

where

$$2b_0-b_2=2-q-2q^3-q^6+2q^{10}+2q^{15}-q^{21}-2q^{28}-q^{36}+2q^{45}+2q^{55}+\ldots,$$
$$b_1+b_2=-1+2q^3-q^{10}-q^{15}+2q^{28}-q^{45}+\ldots,$$
$$2b_0+b_1=1-q-q^6+q^{10}+q^{15}-q^{21}-q^{36}+q^{45}+\ldots.$$

In the first of these series when the q exponent is

$\frac{1}{2}5m(5m+1)$	the coefficient is	$+2$,
$\frac{1}{2}(5m+1)(5m+2)$	,,	-1,
$\frac{1}{2}(5m+2)(5m+3)$	,,	-2,
$\frac{1}{2}(5m+3)(5m+4)$	,,	-1,
$\frac{1}{2}(5m+4)(5m+5)$	,,	$+2$,

while in a_1 and a_2 the law is evident.

We are led to the simultaneous formulæ

$$(18)\quad 2E^{(5)}(n)-E^{(5)}(n-1)-2E^{(5)}(n-3)-E^{(5)}(n-6)+2E^{(5)}(n-10)+\ldots$$
$$-E^{(55)}(n)+2E^{(55)}(n-3)-E^{(55)}(n-10)-E^{(55)}(n-15)+\ldots$$
$$=\text{coefficient of } q^n \text{ in } 2q-3q^6-5q^{10}+5q^{15}+7q^{21}-\ldots,$$

(19) $-E^{(5)}(n)+2E^{(5)}(n-3)-E^{(5)}(n-10)-E^{(5)}(n-15)+\dots$

$$+E^{(55)}(n)-E^{(55)}(n-1)-E^{(55)}(n-6)+E^{(55)}(n-10)+E^{(55)}(n-15)-\dots$$

$$= \text{coefficient of } q^n \text{ in } -q-q^3+4q^6-6q^{21}-8q^{28}+9q^{36}+\dots .$$

From these formulæ we calculate in succession

$$E^{(5)}(1) = 1,\quad E^{(5)}(2) = 1,\quad E^{(5)}(3) = 1,\quad E^{(5)}(4) = 2,\quad E^{(5)}(5) = 1,$$

$$E^{(5)}(6) = 0,\quad \&c.;$$

$$E^{(55)}(1) = 0,\quad E^{(55)}(2) = 1,\quad E^{(55)}(3) = 1,\quad E^{(55)}(4) = 1,\quad E^{(55)}(5) = 0,$$

$$E^{(55)}(6) = 2,\quad \&c.,$$

as a verification.

For the moment I omit the case $x = \frac{\pi}{6}$ because generally for the case $x = \frac{\pi}{6p}$ there is an exception to the general rule, due to the fact that in the period of $12p$ values of $\sin\frac{m\pi}{6p}$ there are two which are positive rational and not zero, viz. when $m = p$ and when $m = 3p$.

The case $x = \frac{\pi}{7}$ is very important, because it points clearly to a general law.

The sines and cosines in a period of fourteen are

$$\sin\frac{\pi}{7},\ \sin\frac{2\pi}{7},\ \sin\frac{3\pi}{7},\ \sin\frac{3\pi}{7},\ \sin\frac{2\pi}{7},\ \sin\frac{\pi}{7},\ 0,\ -\sin\frac{\pi}{7},\ -\sin\frac{2\pi}{7},$$

$$-\sin\frac{3\pi}{7},\ -\sin\frac{3\pi}{7},\ -\sin\frac{2\pi}{7},\ -\sin\frac{\pi}{7},\ 0;$$

$$\cos\frac{\pi}{7},\ \cos\frac{2\pi}{7},\ \cos\frac{3\pi}{7},\ -\cos\frac{3\pi}{7},\ -\cos\frac{2\pi}{7},\ -\cos\frac{\pi}{7},\ -1,\ -\cos\frac{\pi}{7},$$

$$-\cos\frac{2\pi}{7},\ -\cos\frac{3\pi}{7},\ \cos\frac{3\pi}{7},\ \cos\frac{2\pi}{7},\ \cos\frac{\pi}{7},\ 1.$$

Whence, if $E^{(7)}(n)$ represents the excess of the number of divisors of n of the forms $14m+1$, $14m+6$ over the number of divisors of the form $14m+8$, $14m+13$; and $E^{(77)}(n)$ represents the excess of the number of divisors of n of the forms $14m+2$, $14m+5$ over the number of divisors of the forms $14m+9$, $14m+12$; and $E^{(777)}(n)$ represents the excess of the number of divisors of the forms $14m+3$, $14m+4$ over the number of divisors of the forms $14m+10$, $14m+11$,

$$\sum_1^\infty \left(\Sigma \sin \frac{d\pi}{7}\right) q^n$$

$$= \sin \frac{\pi}{7} \Sigma E^{(7)}(n) q^n + \sin \frac{2\pi}{7} \Sigma E^{(77)}(n) q^n + \sin \frac{3\pi}{7} \Sigma E^{(777)}(n) q^n$$

$$= A_1 \sin \frac{\pi}{7} + A_2 \sin \frac{2\pi}{7} + A_3 \sin \frac{3\pi}{7} \text{ suppose.}$$

The numerator of the left-hand side of the identity is

$$\sin \frac{\pi}{7} (N_1 - N_6 + N_8 - N_{13} + \ldots)$$

$$+\sin \frac{2\pi}{7} (-N_2 + N_5 - N_9 + N_{12} - \ldots)$$

$$+\sin \frac{3\pi}{7} (N_3 - N_4 + N_{10} - N_{11} + \ldots),$$

and the denominator

$$1 + D_7 + D_{14} + D_{21} + D_{28} + \ldots$$

$$+\cos \frac{\pi}{7} (-D_1 - D_6 - D_8 - D_{13} - \ldots)$$

$$+\cos \frac{2\pi}{7} (D_2 + D_5 + D_9 + D_{12} + \ldots)$$

$$+\cos \frac{3\pi}{7} (-D_3 - D_4 - D_{10} - D_{11} - \ldots),$$

where in the coefficients of

$\sin \frac{\pi}{7}$, $\cos \frac{\pi}{7}$ the subscripts are of the form $7m \pm 1$,

$\sin \frac{2\pi}{7}$, $\cos \frac{2\pi}{7}$,, ,, $7m \pm 2$,

$\sin \frac{3\pi}{7}$, $\cos \frac{3\pi}{7}$,, ,, $7m \pm 3$.

Writing the identity in the abbreviated form

$$\frac{a_1 \sin\frac{\pi}{7} + a_2 \sin\frac{2\pi}{7} + a_3 \sin\frac{3\pi}{7}}{b_0 + b_1 \cos\frac{\pi}{7} + b_2 \cos\frac{2\pi}{7} + b_3 \cos\frac{3\pi}{7}} = 2A_1 \sin\frac{\pi}{7} + 2A_2 \sin\frac{2\pi}{7} + 2A_3 \sin\frac{3\pi}{7},$$

we find without difficulty the relation

$$\{(2b_0-b_2)A_1 + (b_1-b_3)A_2 + (b_2+b_3)A_3 - a_1\} \sin\frac{\pi}{7}$$

$$+\{(b_1-b_3)A_1 + (2b_0+b_3)A_2 + (b_1+b_2)A_3 - a_2\} \sin\frac{2\pi}{7}$$

$$+\{(b_2+b_3)A_1 + (b_1+b_2)A_2 + (2b_0+b_1)A_3 - a_3\} \sin\frac{3\pi}{7} = 0.$$

This can be converted into a quartic equation in $\sin\frac{\pi}{7}$; but it is known that $\sin\frac{\pi}{7}$ satisfies a sextic equation, and it follows that the relation cannot exist unless the trigonometrical functions have zero coefficients.

Hence

$$(20)\qquad \left\{\begin{aligned} (2b_0-b_1)A_1 + (b_1-b_3)A_2 + (b_2+b_3)A_3 &= a_1, \\ (b_1-b_3)A_1 + (2b_0+b_3)A_2 + (b_1+b_2)A_3 &= a_2, \\ (b_2+b_3)A_1 + (b_1+b_2)A_2 + (2b_0+b_1)A_3 &= a_3, \end{aligned}\right.$$

three simultaneous linear equations in

$$A_1 = \sum_1^{\infty} E^{(7)}(n) q^n,$$

$$A_2 = \sum_1^{\infty} E^{(77)}(n) q^n,$$

$$A_3 = \sum_1^{\infty} E^{(777)}(n) q^n,$$

leading to three simultaneous recurrent formulæ in the arithmetical quantities

$$E^{(7)}(n), \quad E^{(77)}(n), \quad E^{(777)}(n).$$

We can proceed to the case

$$x = \frac{\pi}{2p+1},$$

and are led to a relation

$$\frac{a_1 \sin\frac{\pi}{2p+1} + a_2 \sin\frac{2\pi}{2p+1} + a_3 \sin\frac{3\pi}{2p+1} + \ldots + a_p \sin\frac{p\pi}{2p+1}}{b_0 + b_1 \sin\frac{\pi}{2p+1} + b_2 \sin\frac{2\pi}{2p+1} + b_3 \sin\frac{3\pi}{2p+1} + \ldots + b_p \sin\frac{p\pi}{2p+1}}$$

$$= 2A_1 \sin\frac{\pi}{2p+1} + 2A_2 \sin\frac{2\pi}{2p+1} + \ldots + 2A_p \sin\frac{p\pi}{2p+1}.$$

and thence to the relation, for p of unlimited magnitude,

(21)

$$\begin{aligned}
&\{(2b_0-b_2)A_1 + (b_1-b_3)A_2 + (b_2-b_4)A_3 + (b_3-b_5)A_4 + (b_4-b_6)A_5 + \ldots - a_1\} \sin\frac{\pi}{2p+1} \\
&+ \{(b_1-b_3)A_1 + (2b_0-b_4)A_2 + (b_1-b_5)A_3 + (b_2-b_6)A_4 + (b_3-b_7)A_5 + \ldots - a_2\} \sin\frac{2\pi}{2p+1} \\
&+ \{(b_2-b_4)A_1 + (b_1-b_5)A_2 + (2b_0-b_6)A_3 + (b_1-b_7)A_4 + (b_2-b_8)A_5 + \ldots - a_3\} \sin\frac{3\pi}{2p+1} \\
&+ \{(b_3-b_5)A_1 + (b_2-b_6)A_2 + (b_1-b_7)A_3 + (2b_0-b_8)A_4 + (b_1-b_9)A_5 + \ldots - a_4\} \sin\frac{4\pi}{2p+1} \\
&+ \{(b_4-b_6)A_1 + (b_3-b_7)A_2 + (b_2-b_8)A_3 + (b_1-b_9)A_4 + (2b_0-b_{10})A_5 + \ldots - a_5\} \sin\frac{5\pi}{2p+1} \\
&+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&= 0.
\end{aligned}$$

The symmetry of the A matrix and the law of the elements will be observed.

To write down the matrix for a given value of p we notice first that

$$\cos\frac{(p+k+1)\pi}{2p+1} = -\cos\frac{(p-k)\pi}{2p+1},$$

and thence that $\qquad b_{p+k+1} = -b_{p-k}.$

Making this substitution, when k is zero or any positive number not exceeding $p-1$, we obtain the A matrix appertaining to

$$x = \frac{\pi}{2p+1}.$$

Thus the A matrix for $x = \frac{\pi}{11}$ is (retaining only the b terms)

(22)
$$\begin{matrix} (2b_0-b_2) & (b_1-b_3) & (b_2-b_4) & (b_3-b_5) & (b_4+b_5) \\ (b_1-b_3) & (2b_0-b_4) & (b_1-b_5) & (b_2+b_5) & (b_3+b_4) \\ (b_2-b_4) & (b_1-b_5) & (2b_0+b_5) & (b_1+b_4) & (b_2+b_3) \\ (b_3-b_5) & (b_2+b_5) & (b_1+b_4) & (2b_0+b_3) & (b_1+b_2) \\ (b_4+b_5) & (b_3+b_4) & (b_2+b_3) & (b_1+b_2) & (2b_0+b_1) \end{matrix}$$

The exact form of the relation for any value of p, which may be written

$$a_1 \sin\frac{\pi}{2p+1} + a_2 \sin\frac{2\pi}{2p+1} + \ldots + a_p \sin\frac{p\pi}{2p+1} = 0,$$

is therefore manifest.

This relation when expressed in powers of $\sin\frac{\pi}{2p+1}$ is of degree $2p-2$, and it is known that $\sin\frac{\pi}{2p+1}$ satisfies an equation of degree $2p$.

Hence the relation is impossible unless

$$a_1 = a_2 = \ldots = a_p = 0.$$

Hence we have the p relations obtained by giving m the values $1, 2, 3, \ldots, p$ in the relation

$$(b_{m-1}-b_{m+1})A_1 + (b_{m-2}-b_{m+2})A_2 + \ldots + (b_1-b_{2m-1})A_{m-1}$$

$$+(2b_0-b_{2m})A_m$$

$$+(b_1-b_{2m+1})A_{m+1} + (b_2-b_{2m+2})A_{m+2} + \ldots + (b_{p-m}-b_{p+m})A_p = a_m,$$

in which the first row of terms is to contain $m-1$ terms, and we are to write $-b_{p-k}$ for b_{p+k+1} throughout.

The arithmetical q series

$$A_1, \; A_2, \; \ldots, \; A_p,$$

which appertain to this case, may be defined. If we write down the $4p+2$ recurring values of $\sin \frac{m\pi}{2p+1}$, obtained by giving m the values 1, 2, 3, ..., the particular value $\sin \frac{k\pi}{2p+1}$, where $k \leqslant p$, occurs in the k-th and $(2p-k+1)$-th places and $-\sin \frac{k\pi}{2p+1}$ in the $(2p+k+1)$-th and $(4p-k+2)$-th places.

Hence $$A_k = \Sigma E^{(2p+1)^k}(n) q^n$$

is interpreted by the statement that $E^{(2p+1)^k}(n)$ represents the excess of the number of divisors of n which have the forms

$$(4p+2)m+k, \qquad (4p+2)m+2p-k+1$$

over the number of divisors which have the forms

$$(4p+2)m+2p+k+1, \qquad (4p+2)m+4p-k+2.$$

The matrix has row and column symmetry. Looking to the case of $x = \frac{\pi}{11}$ it will be observed that, if every element of the matrix which involves a b with a subscript > 3 be deleted, the remaining elements consolidated in a rectilinear manner constitute the matrix for the case $x = \frac{\pi}{7}$. The deleted elements occur in four sinister diagonal lines. By deletion of the elements which involve a b with a subscript > 4 we would similarly obtain the matrix for the case $x = \frac{\pi}{9}$. Generally we can in this way proceed from the matrix for the case $x = \frac{\pi}{2p+1}$ to the matrices corresponding to the division of π into an uneven number of parts and conversely proceed to higher matrices by a law that is readily ascertained.

Before taking the general case $x = \frac{\pi}{2p}$, p not a multiple of 3, we will consider the case $x = \frac{\pi}{8}$.

We are led to the relation

$$\frac{\sin\frac{\pi}{8}(N_1+N_7-N_9-N_{15}+\ldots)+\sin\frac{2\pi}{8}(N_2+N_6-N_{10}-N_{14}+\ldots)+\sin\frac{3\pi}{8}(N_3+N_5-N_{11}-N_{13}+\ldots)+N_4-N_{12}+N_{20}-N_{28}+\ldots}{1+\cos\frac{\pi}{8}(D_1-D_7-D_9+D_{15}+\ldots)+\cos\frac{2\pi}{8}(D_2-D_6-D_{10}+D_{14}+\ldots)+\cos\frac{3\pi}{8}(D_3-D_5-D_{11}+D_{13}+\ldots)}$$

$$= 2A_1\sin\frac{\pi}{8}+2A_2\sin\frac{2\pi}{8}+2A_3\sin\frac{3\pi}{8}+2A_4,$$

and thence writing the left-hand side

$$\frac{a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4}{1+b_1\cos\frac{\pi}{8}+b_2\cos\frac{2\pi}{8}+b_3\cos\frac{3\pi}{8}}$$

to

$$\begin{aligned}&\{(2-b_2)A_1+(b_1-b_3)A_2+b_2A_3+2b_3A_4\}\sin\frac{\pi}{8}\\ &+\{(b_1-b_3)A_1+2A_2+(b_1+b_3)A_3+2b_2A_4\}\sin\frac{2\pi}{8}\\ &+\{b_2A_1+(b_1+b_3)A_2+(2+b_2)A_3+2b_1A_4\}\sin\frac{3\pi}{8}\\ &+b_3A_1+b_2A_2+b_1A_3+2A_4\\ &= a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4,\end{aligned}$$

which is of the form

$$a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4=0.$$

This may be arranged as a sextic equation in $\sin\frac{\pi}{8}$, and it is seen to be incompatible with the equation that $\sin\frac{\pi}{8}$ is known to satisfy, viz.

$$16y^6-24y^4+10y^2-1=0,$$

unless $$a_1=a_2=a_3=a_4=0.$$

Hence we obtain the relations

(23)
$$\begin{cases} (2-b_2)A_1+(b_1-b_3)A_2+ \quad b_2A_3+2b_3A_4 = a_1, \\ (b_1-b_3)A_1+ \quad 2A_2+(b_1+b_3)A_3+2b_2A_4 = a_2, \\ b_2A_1+(b_1+b_3)A_2+(2+b_2)A_3+2b_1A_4 = a_3, \\ b_3A_1+ \quad b_2A_2+ \quad b_1A_3+ \quad 2A_4 = a_4. \end{cases}$$

If now p be unrestricted in magnitude I find the relation

$$\{(2-b_2)A_1+(b_1-b_3)A_2+(b_2-b_4)A_3+(b_3-b_5)A_4+(b_4-b_6)A_5+\ldots\}\sin\frac{\pi}{2p}$$
$$+\{(b_1-b_3)A_1+(2-b_4)A_2+(b_1-b_5)A_3+(b_2-b_6)A_4+(b_3-b_7)A_5+\ldots\}\sin\frac{2\pi}{2p}$$
$$+\{(b_2-b_4)A_1+(b_1-b_5)A_2+(2-b_6)A_3+(b_1-b_7)A_4+(b_2-b_8)A_5+\ldots\}\sin\frac{3\pi}{2p}$$
$$+\{(b_3-b_5)A_1+(b_2-b_6)A_2+(b_1-b_7)A_3+(2-b_8)A_4+(b_1-b_9)A_5+\ldots\}\sin\frac{4\pi}{2p}$$
$$+\{(b_4-b_6)A_1+(b_3-b_7)A_2+(b_2-b_8)A_3+(b_1-b_9)A_4+(2-b_{10})A_5+\ldots\}\sin\frac{5\pi}{2p}$$
$$+\ldots$$
$$= a_1\sin\frac{\pi}{2p}+a_2\sin\frac{2\pi}{2p}+a_3\sin\frac{3\pi}{2p}+a_4\sin\frac{4\pi}{2p}+a_5\sin\frac{5\pi}{2p}+\ldots,$$

in which the law of formation of the matrix is manifest.

For a definite value of p we have herein to put

$$b_p = 0, \qquad b_s = -b_{2p-s},$$

but in the last row $b_s = 0$ if $s > p$.

We thus verify the cases $p = 2$, $p = 4$, while for $p = 10$ we find

$$\{(2-b_2)A_1+(b_1-b_3)A_2+(b_2-b_4)A_3+ \quad b_3A_4+2b_4A_5\}\sin\frac{\pi}{10}$$
$$\{(b_1-b_3)A_1+(2-b_4)A_2+ \quad b_1A_3+(b_2+b_4)A_4+2b_3A_5\}\sin\frac{2\pi}{10}$$
$$\{(b_2-b_4)A_1+ \quad b_1A_2+(2+b_4)A_3+(b_1+b_3)A_4+2b_2A_5\}\sin\frac{3\pi}{10}$$
$$\{ \quad b_3A_1+(b_2+b_4)A_2+(b_1+b_3)A_3+(2+b_2)A_4+2b_1A_5\}\sin\frac{4\pi}{10}$$
$$\{ \quad b_4A_1+ \quad b_3A_2+ \quad b_2A_3+ \quad b_1A_4+ \quad 2A_5\}$$
$$= a_1\sin\frac{\pi}{10}+a_2\sin\frac{2\pi}{10}+a_3\sin\frac{3\pi}{10}+a_4\sin\frac{4\pi}{10}+a_5.$$

We gather that generally we have a relation

$$a_1 \sin\frac{\pi}{2p}+a_2\sin\frac{2\pi}{2p}+\ldots+a_{p-1}\sin\frac{(p-1)\pi}{2p}+a_p=0,$$

and it is not difficult to prove that this necessitates the relations

$$a_1=a_2=a_3=\ldots=a_{p-1}=a_p=0,$$

giving the solutions of the problem of expressing the arithmetical functions $A_1, A_2, \ldots, A_n$ in terms of the q series

$$a_1, a_2, a_3, \ldots, a_p, \quad b_1, b_2, b_3, \ldots, b_{p-1},$$

$$A_k = \Sigma E^{(2p)^k}(n)q^m,$$

where $E^{(2p)^k}(n)$ represents the excess of the number of divisors of n which have the forms $4mp+k$, $4mp+2p-k$ over the number of divisors which have the forms $4mp+2p+k$, $4mp+4p-k$.

For the remaining cases in which $x=\frac{\pi}{n}$, where n is a multiple of 6, it will suffice to show the nature of the results by considering in detail the case $x=\frac{\pi}{6}$.

From the sine and cosine values

$$\frac{1}{2},\ \frac{\sqrt{3}}{2},\ 1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ -1,\ -\frac{\sqrt{3}}{2},\ -\frac{1}{2},\ 0,$$

$$\frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ -1,\ -\frac{\sqrt{3}}{2},\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ \frac{\sqrt{3}}{2},\ 1,$$

we derive a relation of the form

$$\frac{a+b\sqrt{3}}{c+d\sqrt{3}} = A+B\sqrt{3},$$

where

$$a = N_1+N_5-N_7-N_{11}+\ldots+2(N_3-N_9+\ldots),$$

$$b = N_2+N_4-N_8-N_{10}+\ldots,$$

$$c = D_2-D_4-D_8+D_{10}+\ldots+2(1-D_6+D_{12}-\ldots),$$

$$d = D_1-D_5-D_7+D_{11}+\ldots,$$

$$A = \Sigma\left(d_{12m+1+5\atop -7-11}(n)+2d_{12m+3\atop -9}(n)\right)q^n,$$

$$B = \Sigma\left(d_{12m+2+4\atop -8-10}(n)\right)q^n,$$

where $d_{12m+\alpha \atop -\gamma}(n)$ denotes the excess of the sum of the divisors of n which are of form $12m+\alpha$ over the sum of those of form $12m+\gamma$, and $d_{12m+\alpha+\beta \atop -\gamma-\delta}(n)$ denotes the excess of the sum of the divisors of n which are of forms $12m+\alpha$, $12m+\beta$ over the sum of those which have the forms $12m+\gamma$, $12m+\delta$.

We thence derive

$$A = \frac{ac-3bd}{c^2-3d^2}, \quad B = \frac{bc-ad}{c^2-3d^2},$$

also

$$Ac+3Bd = a, \quad Bc+Ad = b$$

relations which, upon development, yield two simultaneous recurring formulæ.

2. I consider a theory similar to that of § 1, which is concerned only with those divisors of n which possess uneven conjugates. We will use the symbol δ to denote such a divisor.

We have

$$\sum_1^\infty \frac{aq^{2m-1}}{1-aq^{2m-1}} = \sum_1^\infty \frac{a^m q^m}{1-q^{2m}} = \sum_1^\infty (\Sigma a^\delta)\, q^n$$

and

$$\int \frac{e^{ix} q^{2m-1}}{1-e^{ix} q^{2m-1}}\, dx = \tan^{-1} \frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x} + \tfrac{1}{2} i \log(1-2q^{2m-1}\cos x + q^{4m-2}),$$

but

$$\frac{e^{ix} q^{2m-1}}{1-e^{ix} q^{2m-1}} = \frac{q^{2m-1}\cos x - q^{4m-2}}{1-2q^{2m-1}\cos x + q^{4m-2}} + i\, \frac{q^{2m-1}\sin x}{1-2q^{2m-1}\cos x + q^{4m-2}},$$

so that

$$\int \frac{q^{2m-1}\cos x - q^{4m-2}}{1-2q^{2m-1}\cos x + q^{4m-2}}\, dx = \tan^{-1} \frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x},$$

$$\int \frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x + q^{4m-2}}\, dx = \tfrac{1}{2}\log(1-2q^{2m-1}\cos x + q^{4m-2}),$$

and

$$\sum_1^\infty \tfrac{1}{2}\log(1-2q^{2m-1}\cos x + q^{4m-2}) = \sum_1^\infty \left(\Sigma - \frac{1}{\delta}\cos \delta x\right) q^n$$

or

$$\log \frac{1}{\prod_1^\infty (1-2q^{2m-1}\cos x + q^{4m-2})} = 2\sum_1^\infty \left(\Sigma\, \frac{1}{\delta}\cos \delta x\right) q^n.$$

Transforming by a well known formula of the *Fundamenta Nova*,

$$\log \frac{\prod_1^\infty (1-q^{2m})}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\dots} = 2\sum_1^\infty \left(\Sigma \frac{1}{\delta}\cos \delta x\right) q^n.$$

Differentiation with regard to x gives

(24) $$\frac{q\sin x-2q^4\sin 2x+3q^9\cos 3x-4q^{16}\cos 4x+\dots}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+2q^{16}\cos 4x-\dots} = \sum_1^\infty (\Sigma \sin \delta x) q^n.$$

the fundamental formula which presents itself for examination.

The case $x = \frac{\pi}{2}$.

The formula becomes

$$\frac{q-3q^9+5q^{25}-7q^{49}+\dots}{1-2q^4+2q^{16}-2q^{36}+\dots} = \Sigma \mathfrak{E}^{(2)}(n) q^n,$$

where $\mathfrak{E}^{(2)}(n)$ represents the excess of the number of divisors of n which have the form $4m+1$ over the number of divisors which have the form $4m+3$.

Thence the formula

$$\mathfrak{E}^{(2)}(n)-2\mathfrak{E}^{(2)}(n-4)+2\mathfrak{E}^{(2)}(n-16)-2\mathfrak{E}(n-36)+\dots = 0$$

or $(-)^m(2m+1)$ according as n is not or is of the form $(2m+1)^2$.

If we compare the two formulæ

(25)

$$\begin{cases} \dfrac{2q\sin x-(q+3q^3)\sin 2x+(2q^3+4q^6)\sin 3x-\dots}{1-(1+q)\cos x+(q+q^3)\cos 2x-(q^3+q^6)\cos 3x+\dots} = 2\sum_1^\infty (\Sigma \sin dx) q^n. \\[2ex] \dfrac{q\sin x-2q^4\sin 2x+3q^9\sin 3x-\dots}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\dots} = \sum_1^\infty (\Sigma \sin \delta x) q^n, \end{cases}$$

it will appear that the theories of the divisors d and δ from the point of view of this paper are absolutely parallel.

The trigonometrical functions are similarly placed and (bringing the factor 2 on the right-hand side of the first formula over to the left) they only differ in the coefficients of the sines and cosines. We have merely in the formulæ of the previous section to take for

$$N_1,\ N_2,\ N_3,\ \dots$$

$$D_1,\ D_2,\ D_3,\ \dots,$$

the values $2q,\ -4q^4,\ 6q^9,\ \ldots,$

$$-2q,\ +2q^4,\ -2q^9,\ \ldots,$$

and all the formulæ will be suitably transformed to the divisor δ. We can therefore at once proceed to some further developments affecting the divisor δ.

The δ fundamental formula is, in the notation of the *Fundamenta Nova*,

$$\frac{K}{2\pi} Z\left(\frac{Kx}{\pi}\right) = \Sigma(\Sigma \sin \delta x)\, q^n,$$

and squaring each side we find, by p. 136 of that work,

(26) $$\left(\frac{K}{2\pi}\right)^2 \left\{ Z\left(\frac{Kx}{\pi}\right) \right\}^2$$

$$= \tfrac{1}{2}\Sigma \frac{q^{2m}}{(1-q^{2m})^2} - \tfrac{1}{2}\Sigma \frac{mq^m \cos mx}{1-q^{2m}} + \tfrac{1}{2}\Sigma \frac{q^m(1+q^{2m}) \cos mx}{(1-q^{2m})^2}$$

$$= \{\Sigma(\Sigma \sin \delta x)\, q^n\}^2,$$

a remarkable theorem concerning the divisor δ.

I notice the relations

$$\Sigma \frac{q^{2m}}{(1-q^{2m})^2} = \Sigma \frac{mq^{2m}}{1-q^{2m}},$$

$$\Sigma \frac{mq^m \cos mx}{1-q^{2m}} = \Sigma \frac{(q^{2m-1}+q^{6m-3}) \cos x - 2q^{4m-2}}{(1-2q^{2m-1} \cos x + q^{4m-2})^2}.$$

Differentiating the fundamental relation we get

$$-\tfrac{1}{2}\frac{Q_0 - 2Q_1 \cos x + 2Q_2 \cos 2x - 2Q_3 \cos 3x + \ldots}{(1-2q \cos x + 2q^4 \cos 2x - 2q^9 \cos 3x + \ldots)^2} = \Sigma(\Sigma \delta \cos \delta x) q^n,$$

where $Q_0 = 2^2q^{1^2+1^2} + 4^2q^{2^2+2^2} + 6^2q^{3^2+3^2} + \ldots,$

$$Q_1 = 1^2q + 3^2q^{1^2+2^2} + 5^2q^{2^2+3^2} + 7^2q^{3^2+4^2} + \ldots,$$

$$Q_2 = 2^2q^{2^2} + 4^2q^{1^2+3^2} + 6^2q^{2^2+4^2} + 8^2q^{3^2+5^2} + \ldots,$$

$$Q_3 = 3^2q^{3^2} + 5^2q^{1^2+4^2} + 7^2q^{2^2+5^2} + 9^2q^{3^2+6^2} + \ldots,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

It will be noticed that in the series Q, the q exponents involve every partition of all numbers into one or two squares,

$$Q_m = \sum_{s=0}^{\infty} (2s+m)^2 q^{s^2+(s+m)^2} \quad \text{if} \quad m > 0.$$

Moreover

$$(1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\dots)^2$$
$$= R_0-2R_1\cos x+2R_2\cos 2x-2R_3\cos 3x+2R_4\cos 4x-\dots,$$

where
$$R_0 = 1+2\sum_1^{\infty} q^{s^2+s^2},$$

$$R_1 = 2\sum_0^{\infty} q^{s^2+(s+1)^2},$$

$$R_2 = q^{1^2+1^2}+2\sum_1^{\infty} q^{s^2+(s+2)^2},$$

$$R_3 = 2\sum_{-1}^{\infty} q^{s^2+(s+3)^2},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$R_{2m} = q^{m^2+m^2}+2\sum_{-(m-1)}^{\infty} q^{s^2+(s+m)^2},$$

$$R_{2m+1} = 2\sum_{-1}^{\infty} q^{s^2+(s+2m+1)^2}.$$

We have therefore

$$-\tfrac{1}{2}\,\frac{Q_0-2Q_1\cos x+2Q_2\cos 2x-\dots}{R_0-2R_1\cos x+2R_2\cos 2x-\dots} = \sum_1^{\infty} (\Sigma\,\delta\cos\delta x)\,q^n,$$

R_{2m} enumerates the compositions of numbers into two squares, the numbers squared differing in magnitude by $2m$ and being drawn from the series

$$-(m-1),\ -(m-2),\ \dots,\ +\infty,$$

where it must be noted that

$$a^2+b^2, \quad b^2+a^2$$

are different compositions, except when $a = \pm b$.

Similarly R_{2m+1} enumerates the compositions of numbers into two squares, the numbers squared differing in magnitude by $2m+1$ and being drawn from the series

$$-m,\ -(m-1),\ \dots,\ +\infty.$$

Putting $x = \frac{1}{2}\pi$,

$$-\frac{1}{2}\,\frac{Q_0-2Q_2+2Q_8-2Q_6+\dots}{R_0-2R_2+2R_4-2R_6+\dots} = \sum_1^{\infty} (\Sigma\,\delta\cos\tfrac{1}{2}\delta\pi)\,q^n,$$

and the right-hand side denotes the excess of the sum of the divisors δ of n which have the form $4m+4$ over the sum of the divisors which have the form $4m+2$. Calling this $F_1(n)$, we find that

$$\begin{aligned}(27)\quad & F_1(n)-4F_1(n-4)+4F_1(n-8)+4F_1(n-16)-8F_1(n-20) \\ & \qquad +4F_1(n-32)-4F_1(n-36)+8F_1(n-40)+\dots \\ & = \text{coefficients of } q^n \text{ in } -\tfrac{1}{2}(Q_0-2Q_2+2Q_4-2Q_6+\dots).\end{aligned}$$

The q function $\qquad R_0-2R_2+2R_4-2R_6+\dots$

is equal to $\dfrac{2k'K}{\pi}$ when q^4 is written for q. Its elliptic function expression is therefore

$$\frac{1}{\pi}\,\frac{\sqrt{2}}{\pi}\sqrt{[\sqrt{k'}\,(1+k')]}\,K,$$

while Jacobi gives for it the expression

$$1-4\,\Sigma\,\psi(n)\,q^{(4m-1)^2 4n}+4\,\Sigma\,\psi(n)\,q^{2^{l+1}\,(4m-1)^2 4n},$$

in which n is an uneven number whose prime factors are all of the form $\equiv 1 \bmod 4$; $\psi(n)$ is the number of divisors of n, and l, m assume all integer values from 0 to $+\infty$.*

It is also expressed in the forms

$$1-4\,\frac{q^4}{1+q^4}+4\,\frac{q^{12}}{1+q^{12}}-4\,\frac{q^{20}}{1+q^{20}}+\dots,$$

$$1-4\,\frac{q^4}{1+q^8}+4\,\frac{q^8}{1+q^{16}}-4\,\frac{q^{12}}{1+q^{24}}+\dots,$$

$$\left\{\frac{(1-q^4)(1-q^8)(1-q^{12})\dots}{(1+q^4)(1+q^8)(1+q^{12})\dots}\right\}^2.$$

* *Fundamenta Nova*, p. 105.

3. We now take up the relation

$$\left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \sum_1^\infty (\Sigma d^s a^d) q^n = \sum_1^\infty \frac{m^s a^m q^m}{1-q^m}.$$

The left-hand side is

$$\sum_1^\infty \frac{\sum_1^s c_{s,t}(aq^m)^t}{(1-aq^m)^{s+1}},$$

where $\sum_1^\infty c_{s,t}(aq^m)^t$, for the successive values of s, is

$$aq^m,$$
$$aq^m + a^2q^{2m},$$
$$aq^m + 4a^2q^{2m} + a^3q^{3m},$$
$$aq^m + 11a^2q^{2m} + 11a^3q^{3m} + a^4q^{4m},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the numerical coefficients, in tabular form, are

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1
...	...	...	...		

the number in the s-th row and t-th column being $c_{s,t}$.

We have $$c_{s,t} = tc_{s-1,t} + (s-t+1)c_{s-1,t-1}$$

and $$c_{s,t} = c_{s,s-t+1},$$

$$\sum_1^s c_{s,t} = s!.$$

The coefficients enjoy the further property

$$n^s = c_{s,1}\binom{n}{s} + c_{s,2}\binom{n+1}{s} + c_{s,3}\binom{n+2}{s} + \ldots + c_{s,s}\binom{n+s-1}{s},$$

and we readily obtain the evaluations

$$c_{s,1} = 1^s,$$

$$c_{s,2} = 2^s - \binom{s+1}{1} 1^s,$$

$$c_{s,3} = 3^s - \binom{s+1}{1} 2^s + \binom{s+1}{2} 1^s,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$c_{s,t} = t^s - \binom{s+1}{1}(t-1)^s + \binom{s+1}{2}(t-2)^s - \dots (-)^{t+1}\binom{s+1}{t-1} 1^s.$$

Putting $a = 1$, gives

$$\sum_1^\infty \frac{\sum_1^s c_{st} q^{mt}}{(1-q^m)^{s+1}} = \sum_1^\infty \sigma_s(n)\, q^n = \sum_1^\infty \frac{m^s q^m}{1-q^m}, \tag{28}$$

and $a = -1$ yields the analogous result connected with the excess of the sum of the s-th powers of the even divisors over the sum of the s-th powers of the uneven divisors.

Putting $a = \sqrt{(-1)} = i$,

$$\Sigma(\Sigma d^s i^d)\, q^n = i\Sigma E_s(n) q^n + \Sigma F_s(n) q^n,$$

where $E_s(n)$ is the excess of the sum of the s-th powers of the $4m+1$ divisors over the sum of the s-th powers of the $4m+3$ divisors, and $F_s(n)$ is the excess of the sum of the s-th powers of the $4m+4$ divisors over the sum of the s-th. powers of the $4m+2$ divisors.

If we take $s = 2$,

$$\sum_1^\infty E_2(n) q^n = \frac{q}{1-q} - 3^2 \frac{q^3}{1-q^3} + 5^2 \frac{q^5}{1-q^5} - \dots = \tfrac{1}{4} - \tfrac{1}{4} k'^2 \left(\frac{2K}{\pi}\right)^3,$$

and for $s = 3$,

$$\sum_1^\infty F_3(n) q^n = -2^3 \frac{q^2}{1-q^2} + 4^3 \frac{q^4}{1-q^4} - 6^3 \frac{q^6}{1-q^6} + \dots = -\tfrac{1}{2} + \tfrac{1}{2} k'^2 \left(\frac{2K}{\pi}\right)^4,$$

and we derive

$$\frac{1 + 2\sum_1^\infty F_3(n)\, q^n}{1 - 4\sum_1^\infty E_2(n) q^n} = \frac{2K}{\pi} = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + \dots,$$

a well known series.

Assuming conventionally

$$E_2(0) = -\tfrac{1}{4}, \quad F_3(0) = \tfrac{1}{2},$$

$$\frac{\sum_0^\infty F_3(n)q^n}{-2\sum_0^\infty E_2(n)q^n} = 1+4q+4q^2+4q^4+8q^5+\ldots,$$

leading to the formula

$$(29) \quad 2E_2(n)+8E_2(n-1)+8E_2(n-2)+8E_2(n-4)+16E_2(n-5)+\ldots = -F_3(n),$$

verified for $n = 4$, because

$$E_2(4) = 1, \quad E_2(3) = 1^2-3^2 = -8, \quad E_2(2) = 1, \quad E_2(0) = -\tfrac{1}{4},$$

$$F_3(4) = 4^3-2^3 = 56,$$

$$2.1+8(-8)+8.1+8(-\tfrac{1}{4}) = -56.$$

As another illustration we derive from well known series for

$$\frac{2kK}{\pi} \text{ and } \frac{4kK^2}{\pi^2},$$

$$(30) \quad F_3(n)+2F_3(n-2)+F_3(n-4)+2F_3(n-6)+2F_3(n-8)+\ldots$$
$$= -2E_2(n)-8E_2(n-1)-12E_2(n-2)-16E_2(n-3)-26E_2(n-4)$$
$$-24E_2(n-5)-28E_2(n-6)-\ldots.$$

The general formulæ which arise from the substitution of i for α are

$$(31) \begin{cases} \sum_1^\infty \dfrac{\sum_1^{s+1} (-)^{t+1} b_{s+1,t}\, q^{(2t-1)m}}{(1+q^{2m})^{s+1}} = \sum_1^\infty E_s(n)q^n = \sum_1^\infty (-)^{m+1} \dfrac{(2m-1)^s q^{2m-1}}{1-q^{2m-1}}, \\ 2^s \sum_1^\infty \dfrac{\sum_1^s (-)^{t+1} c_{s,t}\, q^{2tm}}{(1+q^{2m})^{s+1}} = \sum F_s(n)q^n = 2^s\sum(-)^{m+1} \dfrac{m^s q^{2m}}{1-q^{2m}}, \end{cases}$$

where $c_{s,t}$ has the values given above.

As regards b the table of numbers is

1					
1	1				
1	6	1			
1	23	23	1		
1	76	230	76	1	
1	237	1682	1682	237	1
...	...	...	...	...	

the number in the s-th row and t-th column being $b_{s,t}$.

We have $$b_{s,t} = (2t-1)\,b_{s-1,t}+(2s-2t+1)\,b_{s-1,t-1}$$

and $$b_{s,t} = b_{s,s-t+1},$$

$$\sum_1^s b_{s,t} = 2^{s-1}(s-1)!\,.$$

The coefficients enjoy the further property

$$(2n+1)^{s-1} = b_{s,1}\binom{n}{s-1}+b_{s,2}\binom{n+1}{s-1}+b_{s,3}\binom{n+2}{s-1}+\ldots+b_{s,s}\binom{n+s-1}{s-1},$$

and we readily obtain the evaluations

$$b_{s,1} = 1^{s-1},$$

$$b_{s,2} = 3^{s-1}-\binom{s}{1}1^{s-1},$$

$$b_{s,3} = 5^{s-1}-\binom{s}{1}3^{s-1}+\binom{s}{2}1^{s-1},$$

... ...

$$b_{s,t} = (2t-1)^{s-1}-\binom{s}{1}(2t-3)^{s-1}+\binom{s}{2}(2t-5)^{s-1}-\ldots+(-)^{t+1}\binom{s}{t-1}1^{s-1}.$$

4. The relation
$$\Sigma\frac{m^s a^m q^m}{1-q^m} = \Sigma\,(\Sigma d^s a^d)\,q^n$$

enables us to study the divisors which are restricted in magnitude by upper and lower limits.

For put $a = q^u$, so that

$$\Sigma \frac{m^s q^{m(u+1)}}{1-q^m} = \Sigma (\Sigma d^s)\, q^{n+ut}.$$

The general term, on the left, under the sign of summation is

$$m^s q^{m(u+1+k)},$$

where m is a divisor of $m(u+1+k)$ which is less than

$$m\,\frac{u+1+k}{u}$$

for all integer values of u and k.

Hence
$$\Sigma \frac{m^s q^{m(u+1)}}{1-q^m} = \Sigma (\Sigma d_u^s)\, q^n,$$

where d_u is a divisor of n which is less than n/u.

Since also
$$\Sigma \frac{m^s q^{m(u+v+1)}}{1-q^m} = \Sigma (\Sigma d_{u+v}^s)\, q^n,$$

where d_{u+v} is a divisor of n which is less than $n/(u+v)$, we obtain by subtraction

$$\text{(32)} \qquad \sum_1^\infty m^s \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma (\Sigma d_{u,\,u+v}^s)\, q^n,$$

where $d_{u,\,u+v}$ is a divisor of n such that

$$\frac{n}{u} > d_{u,\,u+v} \geqslant \frac{n}{u+v}.$$

In this formula, u, v may be any real positive numerical quantities whatever.

The case $s = 0$ is of particular interest because then the relation becomes

$$\text{(33)} \qquad \Sigma \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma \nu_{u,\,u+v}(n)\, q^n,$$

where $\nu_{u,\,u+v}$ denotes the *number* of the divisors $d_{u,\,u+v}$ of n such that

$$\frac{n}{u} > d_{u,\,u+v} \geqslant \frac{n}{u+v}.$$

I proceed to its examination.

Put $v = 1$, and we find that

$$\frac{q^{u+1}}{1-q^{u+1}} = \Sigma \nu_{v,\,u+1}(n)\,q^n,$$

showing, what is otherwise obvious, that $\nu_{u,\,u+1}$ is zero or unity according as n is not or is a multiple of $u+1$.

Put $v = 2$ leading to

$$\frac{\frac{1-q^2}{1-q}\,q^{u+1}-2q^{2u+3}}{(1-q^{u+1})(1-q^{u+2})} = \Sigma \nu_{u,\,u+2}(n)\,q^n,$$

and for $v = 3$,

$$\frac{\frac{1-q^3}{1-q}\,q^{u+1}-2\,\frac{1-q^3}{1-q}\,q^{2u+3}+3q^{3u+6}}{(1-q^{u+1})(1-q^{u+2})(1-q^{u+3})} = \Sigma \nu_{u,\,u+3}(n)\,q^n,$$

and, finally, the general formula

(34)

$$\frac{\frac{1-q^v}{1-q}\,q^{u+1}-2\,\frac{(1-q^v)(1-q^{v-1})}{(1-q)(1-q^2)}\,q^{2u+3}+3\,\frac{(1-q^v)(1-q^{v-1})(1-q^{v-2})}{(1-q)(1-q^2)(1-q^3)}\,q^{3u+6}-\ldots+(-)^{v+1}\,v\,q^{vu+\binom{v+1}{2}}}{(1-q^{u+1})(1-q^{u+2})(1-q^{u+3})\ldots(1-q^{u+v})}$$

$$= \Sigma \nu_{u,\,u+v}(n)q^n = \sum_1^v \frac{q^{u+m}}{1-q^{u+m}},$$

which is a valuable result.

(i) Put $u = 0$, obtaining

$$\frac{\frac{1-q^v}{1-q}\,q-2\,\frac{(1-q^v)(1-q^{v-1})}{(1-q)(1-q^2)}\,q^3+3\,\frac{(1-q^v)(1-q^{v-1})(1-q^{v-2})}{(1-q)(1-q^2)(1-q^3)}\,q^6+\ldots+(-)^{v+1}\,v\,q^{\binom{v+1}{2}}}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^v)}$$

$$= \Sigma \nu_{0,\,v}(n)\,q^n,$$

where the divisor $d_{0,\,v}$ is such that

$$\infty > d_{0,\,v} \geqslant \frac{n}{v},$$

i.e. the divisors enumerated are not less than n/v, a single condition.

(ii) In the last formula put $v = \infty$, obtaining

$$\text{(35)} \qquad \frac{\dfrac{q}{1-q} - 2\dfrac{q^3}{(1-q)(1-q^2)} + 3\dfrac{q^6}{(1-q)(1-q^2)(1-q^3)} - 4\dfrac{q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots}{(1-q)(1-q^2)(1-q^3)(1-q^4)\dots}$$

$$= \sum_1^\infty \nu(n)\, q^n,$$

where $\nu(n)$ is the number of divisors of n without any restrictions because

$$\infty > d_{0,\infty} \geqslant 0.$$

It will be recalled that $\sum \nu(n) q^n$

has two other well known expressions, viz.:

Lambert's $$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \dots,$$

and Clausen's $$q\,\frac{1+q}{1-q} + q^4\,\frac{1+q^2}{1-q^2} + q^9\,\frac{1+q^3}{1-q^3} + q^{16}\,\frac{1+q^4}{1-q^4} + \dots.$$

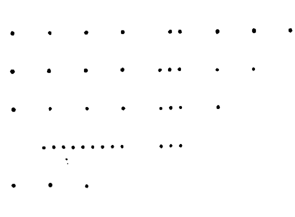
That obtained above, which connects the arithmetical entity $\nu(n)$ with the theory of partitions, is possibly known, though I do not remember meeting with it in other writings.

In the numerator of the left-hand side the general term is

$$(-)^{k+1}\,\frac{kq^{\binom{k+1}{2}}}{(1-q)(1-q^2)\dots(1-q^k)},$$

where $$\binom{k+1}{2} = 1+2+3+\dots+k.$$

If we denote this number by a lattice of nodes containing in the successive rows k, $k-1$, ..., 2, 1 nodes successively



and consider the partitions enumerated by

$$\frac{1}{(1-q)(1-q^2)\,..\,(1-q^k)},$$

we find that each of the latter involves k or fewer parts, of unrestricted magnitude, which may be graphically represented by k or fewer rows of nodes, the numbers of nodes in the rows being in descending order of magnitude. Adding to each of these the above graph of $\binom{k+1}{2}$, row to row, we obtain the graph of a partition in which there are no repetitions of parts.

Hence we gather that the numerator enumerates, with respect to the number n, the excess of $2j+1$ times the number of partitions into exactly $2j+1$ parts ($j = 0, 1, 2, \ldots$) without repetitions over $2k+2$ times the number of partitions into exactly $2k+2$ parts ($k = 0, 1, 2, \ldots$) without repetitions.

The numerator, to a few terms, is

$$q+q^2-q^3-q^4-3q^5-2q^7+q^8+2q^9+q^{10}+2q^{11}+4q^{12}+\ldots,$$

so that if p_n denote the number of partitions of n we are led to the relations

$$(36)\quad \begin{cases} \nu(n) = p_{n-1}+p_{n-2}-p_{n-3}-p_{n-4}-3p_{n-5}-2p_{n-7}+p_{n-8}+2p_{n-9} \\ \qquad\qquad +p_{n-10}+2p_{n-11}+4p_{n-12}+\ldots, \\ \nu(n)-\nu(n-1)-\nu(n-2)+\nu(n-5)+\nu(n-7)-\nu(n-12)-\ldots \\ \qquad\qquad = \text{coefficient of } q^n \text{ in the numerator,} \end{cases}$$

verified for $n = 10$ by

$$\nu(10) = p_9+p_8-p_7-p_6-3p_5-2p_3+p_2+2p_1+1,$$

$$4 \;= 30+22-15-11-21-6+2+2+1 = 57-53,$$

$$\nu(10)-\nu(9)-\nu(8)+\nu(5)+\nu(3) = 1,$$

$$4 \;-\; 3 \;-\; 4 \;+\; 2 \;+\; 2 \;= 1.$$

In considering the general case

$$\sum_1^\infty m^s \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma\Sigma d^s_{u,\,u+v}(n)q^n,$$

the left-hand side, as it stands, is a sum of an infinite number of terms and is readily converted into a sum of v terms. Thus, for $s = 1, 2, 3, \ldots$, we readily verify the expressions

$$\sum_1^v \frac{q^{u+m}}{(1-q^{u+m})^2} = \Sigma\Sigma d_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+q^{u+m})}{(1-q^{u+m})^3} = \Sigma\Sigma d^2_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+4q^{u+m}+q^{2u+2m})}{(1-q^{u+m})^4} = \Sigma\Sigma d^3_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+11q^{u+m}+11q^{2u+2m}+q^{3u+3m})}{(1-q^{u+m})^5} = \Sigma\Sigma d^4_{u,\,u+v}(n)q^n,$$

where the numerical coefficients in the series of numerators are for identification with the scheme of numbers met with in § 3.

In general we have

$$\sum_1^v \frac{\sum_1^s c_{s,t}\, q^{t(u+m)}}{(1-q^{u+m})^{s+1}} = \Sigma\Sigma d^s_{u,\,u+v}(n)q^n. \tag{37}$$

When $v = \infty$, so that $\frac{n}{u} > d_{u,\,s}$, the only condition,

$$\sum_1^\infty \frac{\sum_1^s c_{s,t}\, q^{t(u+m)}}{(1-q^{u+m})^{s+1}} = \Sigma\Sigma d^s_{u,\,\infty}(n)q^n.$$

When $v = 1$,

$$\frac{\sum_1^s c_{s,t}\, q^{t(u+1)}}{(1-q^{u+1})^{s+1}} = \Sigma\Sigma d^s_{u,\,u+1}(n)\, q^n.$$

The coefficient of $q^{n(u+1)}$ in the left-hand side is

$$c_{s,1}\binom{s+u-1}{n-1}+c_{s,2}\binom{s+u-2}{n-2}+\ldots+c_{s,n} = \Sigma d^s_{u,\,u+1}(nu+n).$$

Ex. gr., $s = 2,\ u = 3,\ n = 9$.

The divisors of 36 which are such that

$$\tfrac{36}{3} > d \geqslant \tfrac{36}{4}$$

are simply the one divisor 9 and

$$c_{2,1}\binom{10}{8}+c_{2,2}\binom{9}{7} = 45+36 = 9^2.$$

5. A similar theory exists for other classes of divisors.

Thus, for those whose conjugates are uneven, we start with

$$\Sigma \frac{a^m q^m}{1-q^{2m}} = \Sigma\Sigma a^\delta q^n,$$

and, differentiating s times with $a\dfrac{d}{da}$, we derive

$$\Sigma \frac{m^s a^m q^m}{1-q^{2m}} = \Sigma\Sigma \delta^s a^\delta q^n,$$

and putting $a = q^u$, $\quad \Sigma \dfrac{m^s q^{m(u+1)}}{1-q^{2m}} = \Sigma\Sigma \delta^s q^{n+\delta u}$,

and changing n so that

$$\Sigma \frac{m^s q^{m(u+1)}}{1-q^{2m}} = \Sigma\Sigma \delta_u^s q^n,$$

we find that
$$\frac{n}{u} > \delta_u.$$

Substituting $u+2v$ for u and subtracting

$$\Sigma \frac{m^s q^{m(u+1)}(1-q^{2mv})}{1-q^{2m}} = \Sigma\Sigma \delta^s_{u,\,u+2v}(n)\, q^n, \tag{38}$$

where
$$\frac{n}{u} > \delta_{u,\,u+2v} \geqslant \frac{n}{u+2v}.$$

Put $s = 0$, obtaining

$$\Sigma \frac{q^{m(u+1)}(1-q^{2mv})}{1-q^{2m}} = \Sigma\Sigma \delta^0_{u,\,u+2v}(n)\, q^n,$$

where now $\Sigma\delta^0_{u,\,u+2v}(n)$ is the number of divisors with uneven conjugates such that the above limits are satisfied.

For $v=1$, $$\frac{q^{u+1}}{1-q^{u+1}} = \Sigma\Sigma\delta^0_{u,\,u+2}(n)q^n,$$

a trivial result.

For $v=2$, $$\frac{\frac{1-q^4}{1-q^2}q^{u+1}-2q^{2u+4}}{(1-q^{u+1})(1-q^{u+3})} = \Sigma\Sigma\delta^0_{u,\,u+4}(n)q^n,$$

and, in general,

$$\frac{\frac{1-q^{2v}}{1-q^2}q^{u+1}-2\frac{(1-q^{2v})(1-q^{2v-2})}{(1-q^2)(1-q^4)}q^{2u+4}+3\frac{(1-q^{2v})(1-q^{2v-2})(1-q^{2v-4})}{(1-q^2)(1-q^4)(1-q^6)}q^{3u+9}-\ldots+(-)^{v+1}vq^{(u+v)v}}{(1-q^{u+1})(1-q^{u+3})\ldots(1-q^{u+2v-1})}$$

$$= \Sigma\Sigma\delta^0_{u,\,u+2v}(n)q^n;$$

put $u=0$, then

$$\text{(39)}\quad \frac{\frac{1-q^{2v}}{1-q^2}q-2\frac{(1-q^{2v})(1-q^{2v-2})}{(1-q^2)(1-q^4)}q^4+3\frac{(1-q^{2v})(1-q^{2v-2})(1-q^{2v-4})}{(1-q^2)(1-q^4)(1-q^6)}q^9-\ldots+(-)^{v+1}vq^{v^2}}{(1-q)(1-q^3)(1-q^5)\ldots(1-q^{2v-1})}$$

$$= \Sigma\Sigma\delta^0_{0,\,2v}(n)q^n,$$

where $$\infty > \delta_{0,\,2v} \geqslant \frac{n}{2v},$$

i.e. the divisors enumerated are not less than $n/2v$ and have uneven conjugates.

If we now put $v=\infty$,

$$\text{(40)}\quad \frac{\frac{q}{1-q^2}-2\frac{q^4}{(1-q^2)(1-q^4)}+3\frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}-4\frac{q^{16}}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}-\ldots}{(1-q)(1-q^3)(1-q^5)\ldots}$$

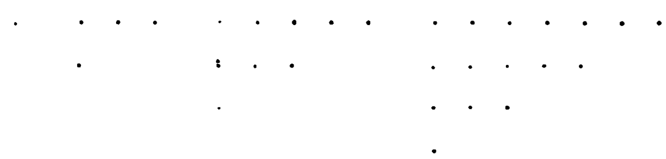
$$= \Sigma\Sigma\delta_{0,\,\infty}(n)q^n,$$

where $$\infty > \delta_{0,\,\infty} \geqslant 0,$$

and we have a new function which generates numbers which enumerate the whole of the divisors which are before us.

The numerator is interpretable.

Since q, q^4, q^9, q^{16}, ... are graphically representable as



and $(1-q^2, 1-q^4, \ldots, 1-q^{2m})^{-1}$ enumerates the partitions of numbers into n or fewer even parts, we can add on to the m-th of these graphs the graphs of such partitions, thus obtaining graphs of partitions into exactly m uneven parts, which involve neither *sequences nor repetitions of parts.*

Hence we gather that the numerator enumerates, with respect to the number n, the excess of $2j+1$ times the number of partitions into exactly $2j+1$ uneven parts which involve neither sequences nor repetitions over $2k+2$ times the number of partitions into exactly $2k+2$ uneven parts which involve neither sequences nor repetitions of parts, where

$$j = 0, 1, 2, \ldots, \qquad k = 0, 1, 2, \ldots .$$

The numerator, to a few terms, is

$$q+q^2+2q^3+q^5-q^6-4q^8-4q^{10}-6q^{12}+q^{13}+\ldots,$$

and the coefficient in this of q^n is

$$\delta(n)-\delta(n-1)-\delta(n-3)+\delta(n-4)-\delta(n-5)+\delta(n-6)-\delta(n-7)$$

$$+2\delta(n-8)-2\delta(n-9)+2\delta(n-10)-2\delta(n-11)+3\delta(n-12)-\ldots .$$

For $n = 12$, we verify that

$$\begin{aligned} &2-2-3+1-2+2-2 \\ &+2-4+2-2 \qquad = -6. \end{aligned}$$

Returning to the general case the left-hand side is expressible as the sum of v terms so that,

for $s=1$, $$\sum_1^v \frac{q^{u+2m-1}}{(1-q^{u+2m-1})^2} = \Sigma\Sigma\delta_{u,\,u+2v}(n)\,q^n,$$

,, $s=2$, $$\sum_1^v \frac{q^{u+2m-1}(1+q^{u+2m-1})}{(1-q^{u+2m-1})^3} = \Sigma\Sigma\delta^2_{u,\,u+2v}(n)q^n,$$

,, $s=3$, $$\sum_1^v \frac{q^{u+2m-1}(1+4q^{u+2m-1}+q^{2u+4m-2})}{(1-q^{u+2m-1})^4} = \Sigma\Sigma\delta^3_{u,\,u+2v}(n)q^n.$$

&c.,

the numerator numbers being those that have already appeared in § 3.

In general

(41) $$\sum_1^v \frac{\sum_1^s c_{s,\,t}\, q^{t\,(u+2m-1)}}{(1-q^{u+2m-1})^{s+1}} = \Sigma\Sigma\delta^s_{u,\,u+2v}(n)q^n.$$