ON A CERTAIN SERIES OF FOURIER

By W. H. YOUNG, Sc.D., F.R.S.

[Received May 13th, 1912.—Read June 13th, 1912.]

1. Apart from the practical importance which the detailed study of a particular series of Fourier may possess, such a study may be of theoretical interest in two ways—first, in the light it throws on the properties of the general Fourier series by showing what the possibilities are, and by suggesting theorems; secondly, in virtue of the fact that such a series, considered in conjunction with the general series, gives rise to definite convergence properties of series involving the coefficients of the Fourier series, whether these be numerical or themselves trigonometrical series connected with the general Fourier series. Of primary importance in this connexion are the two series, respectively sine and cosine series, whose coefficients are the reciprocals of the natural numbers in order. The sums of both these series are, of course, well known, and the fact that the sine series converges boundedly is classical; it is, moreover, a particular case of the theorem* that the Fourier series of functions of bounded variation converge boundedly. So little have these series, however, been studied that it was only quite recently† that the fact was enunciated that the partial summations of the sine series are all positive or all negative, according as x is positive or negative in the interval $(-\pi, \pi)$. The present paper is devoted to the cosine series, and more especially to the mode in which it diverges in the neighbourhood of the origin. Thus, it is shown that the chasm function, as well as the peak function, has the value of $+\infty$ at the origin, so that the divergence is what I have called ‡ uniform diver-

---

* W. H. Young, "On the Integration of Fourier Series," § 3, 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 352, 353.

† Dunham Jackson, "Ueber eine trigonometrische Summe," 1911, *Rend. di Palermo*, XXXII, pp. 257–262, where references to Fejér will be found.

‡ W. H. Young, "On Uniform and Non-Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," § 6, 1907, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 36.

gence. Moreover, the ratio of the sum-function to the n-th partial summation has the unique double limit unity as $n \to \infty$ and the point approaches the origin.

As an illustration of the application which may be made of these results to the general theory of Fourier series, a property of the coefficients of the cosine terms of these series is deduced which holds good for an extremely wide class of series. As is well known, the coefficients of the cosine terms and those of the sine terms behave in various respects differently from one another. The property here obtained has reference precisely to one of these points of difference. The series whose typical term is a sine-coefficient converges when each term is divided by the integer denoting its place in the series, but it is known * that this is not true for the cosine-coefficients in the general case. It is not difficult to see that, if the Fourier series has for its function one which remains summable when raised to some positive power greater than unity, the series in question converges; this follows, in fact, from the known sufficient condition for the convergence of the allied series of the Fourier series, coupled with a necessary condition that a function should be the integral of one which remains summable when raised to such a positive power. It is shown in the present paper that this series converges if the function be such that it remains summable when multiplied by the logarithm of its modulus, a condition which, of course, includes the condition for convergence above given as a very particular case. In the proof of this result the following inequality is required, and is, accordingly, proved:—

$$uv \leqslant u \log u + e^{v-1}, \quad (0 < u).$$

This is deducible by limiting process from the inequality

$$(p+1)\,uv \leqslant u^{1+p} + pv^{1+1/p},$$

which I have already † had occasion more than once to employ in the theory of Fourier series. Once the statement has been made, it can be proved still more easily by a direct process. In the general theory of classes of summable functions and elsewhere, this inequality, and others akin to it, are bound to play a prominent and increasingly important part.

It will be seen from perusal of the paper that the reasoning employed

---

* W. H. Young, "On the Nature of the Successions formed by the Coefficients of a Fourier Series," §§ 1 and 4, 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 344–348.

† W. H. Young, "On a Class of Parametric Integrals and their Application in the Theory of Fourier Series," § 7, 1911, *Proc. Roy. Soc.*, Ser. A, Vol. 85, p. 407.

is of a delicate nature as soon as the more obvious properties of the series, such as the distribution of the maxima and minima of the partial summations, have been disposed of. It is not impossible, therefore, that the corresponding discussion of the nature of the divergence of the most general Fourier series of the type here considered, that is of one whose function is continuous in the extended sense, periodic and monotone towards the origin, which is the sole infinity, may not require essentially more subtle considerations; they will necessarily be, however, at least in part, of a different character to those here employed. From this point of view, therefore, as well as from others, the publication of the present account seems desirable. A further remark may be added: the notion of extended continuity, in which infinite values are allowed, produces in the theory of integration no sensible simplification in the statement of results; a point at which there is continuity in the extended sense behaves for purposes of integration like a discontinuity. In the theory of series, on the other hand, I have more than once emphasized the inevitableness of the concept. Its importance is once more illustrated by the results here obtained.

2. Let

$$S_n(x) = \cos x + \tfrac{1}{2}\cos 2x + \ldots + \frac{1}{n}\cos nx, \quad (0 \leqslant x \leqslant \pi). \tag{1}$$

The greatest value of $S_n(x)$ is obviously at the origin, and since

$$\begin{aligned} S_n'(x) &= -\sin x - \sin 2x + \ldots - \sin nx \\ &= -\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\left(\cos\tfrac{1}{2}x - \cos\frac{2n+1}{2}x\right) \\ &= -\operatorname{cosec}\tfrac{1}{2}x \sin\frac{nx}{2}\sin\frac{(n+1)x}{2}, \end{aligned} \tag{2}$$

we see that the maxima of $S_n(x)$ are at the points 0, $2\pi/n$, $4\pi/n$, ..., and the minima at $2\pi/(n+1)$, $4\pi/(n+1)$, ...

Now taking $\lambda < k$,

$$\begin{aligned} &S_n[2k\pi/(n+1)] - S_n[2\lambda\pi/(n+1)] \\ &\qquad = \int_{2\lambda\pi/(n+1)}^{2k\pi/(n+1)} S_n'(x)\,dx \\ &\qquad = -\frac{2}{n+1}\int_{\lambda\pi}^{k\pi}\left(\sin x \cot\frac{x}{n+1} - \cos x\right)\sin x\,dx \\ &\qquad = -\frac{2}{n+1}\int_{\lambda\pi}^{k\pi}\sin^2 x\cot\frac{x}{n+1}\,dx, \end{aligned} \tag{3}$$

which is certainly negative, since $k \leqslant \frac{1}{2}(n+1)$, so that the cotangent in the integrand is positive.

Thus the minimal values of $S_n(x)$ decrease as we move to the right, so that the least value of $S_n(x)$ is at the point $[2k\pi/(n+1)]$, where k is the greatest integer in $\frac{1}{2}(n+1)$. Thus if n is odd, the least value is $S_n(\pi)$, while if n is even, it is $S_n\left(\pi-\frac{\pi}{n+1}\right)$.

In the former case we have

$$S_n(\pi) = -1+\tfrac{1}{2}-\ldots-\frac{1}{n} > -1.$$

In the latter case,

$$\begin{aligned} S_n\left(\pi-\frac{\pi}{n+1}\right) &\equiv S_n(\pi-p), \text{ say,} \\ &= -\cos p+\tfrac{1}{2}\cos 2p-\ldots+\frac{1}{n}\cos np \\ &= -\left(1-\frac{1}{n}\right)\cos p+\left(\tfrac{1}{2}-\frac{1}{n-1}\right)\cos 2p-\ldots \\ &\qquad +(-)^{\frac{1}{2}n}\left(\frac{2}{n}-\frac{1}{\frac{1}{2}n+1}\right)\cos \tfrac{1}{2}np. \end{aligned}$$

Since $$1-\frac{1}{n} > \tfrac{1}{2}-\frac{1}{n-1} > \frac{1}{3}-\frac{1}{n-2} > \ldots > \frac{2}{n}-\frac{1}{\frac{1}{2}n+1},$$

and $$\cos p \geqslant \cos 2p \geqslant \ldots \geqslant \cos \tfrac{1}{2}np,$$

we have $$S_n\left(\pi-\frac{\pi}{n+1}\right) > -\left(1-\frac{1}{n}\right)\cos p > -1.$$

Thus, in any case, $$S_n(x) > -1, \quad (0 \leqslant x \leqslant \pi). \tag{4}$$

3. If we take any interval $(0, e)$, we know that, for all values of n, such that

$$\frac{2\pi}{n+1} < e, \tag{5}$$

the least value of $S_n(x)$ in the interval is at the point $2\lambda\pi/(n+1)$, where

$$\frac{2\lambda\pi}{n+1} \leqslant e < \frac{2(\lambda+1)\pi}{n+1}. \tag{6}$$

Thus we have only to show that, as n increases indefinitely, the value of

$S_n(x)$ at this point is greater than a certain positive quantity, depending on e in such a way that, by choosing e small enough, the quantity is as large as we please, to prove that $S_n(x)$ diverges uniformly to $+\infty$ at the origin.

Now regarding k in (3) as giving the point already considered at which $S_n(x)$ assumes its least value in $(0, \pi)$, we know that the first term on the left of (3) is greater than -1; therefore it is only necessary for the proof of our statement to shew that the integral on the right, taken with sign $+$ instead of $-$, is as large as we please.

Now the integral may be written

$$\frac{2k}{n+1}\int_{\lambda\pi/k}^{\pi}(\sin kx)^2\cot\frac{kx}{n+1}\,dx \geqslant \frac{2k}{n+1}\int_{e_1}^{\pi}(\sin kx)^2\cot\frac{kx}{n+1}\,dx,$$

where $$e_1 = e\left(1+\frac{e}{2\pi-e}\right),$$

since $k \geqslant \frac{1}{2}n$, and therefore

$$\lambda\pi/k \leqslant 2\lambda\pi/n \leqslant \frac{2\lambda\pi}{n+1}\left(1+\frac{1}{n}\right) \leqslant e\left(1+\frac{1}{n}\right) \leqslant e\left(1+\frac{1}{-1+2\pi/e}\right),$$

using (6). Since $n \leqslant 2k \leqslant n+1$, our integral

$$\geqslant \frac{n}{n+1}\int_{e_1}^{\pi}\tfrac{1}{2}(1-\cos 2kx)\cot\frac{x}{2}\,dx.$$

But, by the theorem of Riemann-Lebesgue,* since $\cot\frac{1}{2}x$ is summable from e_1 to π,

$$\operatorname*{Lt}_{n\to\infty}\frac{n}{n+1}\int_{e_1}^{\pi}\cot\tfrac{1}{2}x\cos 2kx\,dx = 0.$$

But, by (5), $n\to\infty$, when $e\to 0$; therefore our integral

$$\geqslant\left(1-\frac{e}{2\pi}\right)\log\operatorname{cosec}\tfrac{1}{2}e+E,$$

where E has the unique limit zero, when e approaches zero.

This shows that our integral is as great as we please, which, as already pointed out, proves the statement that $S_n(x)$ diverges uniformly to $+\infty$ at the origin.

4. Write

$$\sigma_{m,n}(x) = S_{m+n}(x)-S_m(x) = \sum_{r=m+1}^{m+n} r^{-1}\cos rx, \quad (0 \leqslant x \leqslant \pi).$$

---

* See, for example, Hobson's *Theory of Functions of a Real Variable*, p. 674.

Differentiating,

$$\sigma'_{m,n}(x) = \sum_{r=m+1}^{m+n} (-\sin rx)$$

$$= \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x \left\{\cos(2m+1)\frac{x}{2} - \cos(2m+2n+1)\frac{x}{2}\right\}$$

$$= -\sin\frac{nx}{2}\sin\frac{(2m+n+1)x}{2}\Big/\sin\frac{x}{2}$$

$$= -\sin\tfrac{1}{2}p(2m+1)x \sin\tfrac{1}{2}(p+1)(2m+1)x/\sin\tfrac{1}{2}x, \qquad (7)$$

where we have written $n \doteq p(2m+1)$,

and we may, if we please, so choose our succession of n's, so that p is an odd integer. Thus $2m+n+1 = (2m+1)(p+1)$.

Since the origin is evidently a maximum of $\sigma_{m,n}(x)$, we see, from (7), that the maxima of $\sigma_{m,n}(x)$ are at the points $0, 2\pi/n, 4\pi/n, \ldots$, and the minima at the points $2\pi/(2m+n+1), 4\pi/(2m+n+1), \ldots$, the extreme right-hand minimum point being at $x = \pi$.

5. Let us write y, in (7), for $\frac{1}{2}(2m+1)x$. Then we have

$$\sigma_{m,n}(x) - \sigma_{m,n}(\pi) = \frac{2}{2m+1}\int_{(m+\frac{1}{2})x}^{(m+\frac{1}{2})\pi} \sin py \sin(p+1)y \operatorname{cosec}\frac{y}{2m+1}\,dy. \quad (8)$$

If, in the first place, the point $(m+\frac{1}{2})x$ lies at, or to the right of, the point $\frac{1}{2}\pi$, we use the Second Theorem of the Mean, and bring out the monotone decreasing factor cosec $y/(2m+1)$ of the integrand. We thus have

$$\sigma_{m,n}(x) - \sigma_{m,n}(\pi) = \frac{1}{2m+1}\operatorname{cosec}\tfrac{1}{2}x \int_{(m+\frac{1}{2})x}^{X} [\cos y - \cos(2p+1)y]\,dy$$

$$= \frac{1}{\pi}\,\frac{\pi}{x(m+\frac{1}{2})}\,\frac{\frac{1}{2}x}{\sin\frac{1}{2}x}\,4\theta,$$

where θ is numerically $\leqslant 1$.

Thus, since $\frac{1}{2}x/\sin\frac{1}{2}x \leqslant \frac{1}{2}\pi$,

$$|\sigma_{m,n}(x) - \sigma_{m,n}(\pi)| \leqslant 4 \quad (\tfrac{1}{2}\pi \leqslant x \leqslant \pi). \qquad (9)$$

In the second place, if $x < \frac{1}{2}\pi/(m+\frac{1}{2})$, and therefore $y < \frac{1}{2}\pi$, the function $\dfrac{\sin y}{\sin y/(2m+1)}$ is a monotone decreasing function of y up to $y=\frac{1}{2}\pi$. Thus dividing the interval of integration in (8) into two parts at the point $y=\frac{1}{2}\pi$, the second part gives us an integral which, as we have just seen, is numerically $\leqslant 4$, and the first part gives us an integral which may be written

$$\frac{2}{2m+1}\int_{(m+\frac{1}{2})x}^{\frac{1}{2}\pi}\frac{\sin y}{\sin y/(2m+1)}(\sin^2 py\cot y+\sin py\cos py)\,dy$$

$$=\frac{2}{2m+1}\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}$$

$$\times\left\{\int_{(m+\frac{1}{2})x}^{X}\sin^2 py\cot y\,dy+\frac{1}{2p}\left\{\sin^2 pX-\sin^2 p(m+\tfrac{1}{2})x\right\}\right\}\geqslant -\frac{1}{2p}.$$

Thus we have, finally, combining with (9),

$$\sigma_{m,n}(x)-\sigma_{m,n}(\pi)\geqslant -4-\frac{1}{2p}, \tag{10}$$

where $n=p(2m+1)$.

6. Hence $$S_m(x)-S_{m+n}(x)\leqslant 5+\frac{1}{2p},$$

since $\sigma_{m,n}(\pi)$ is numerically $\leqslant 1/(m+1)\leqslant 1$.

Now, let n increase without limit, $S_{m+n}(x)$ becomes in the limit $S(x)$, where, except at the origin,

$$S(x)=\tfrac{1}{2}\log\frac{1}{2(1-\cos x)}.$$

Combining this with a former result (§ 2), we have, since $p\to\infty$,

$$-1\leqslant S_m(x)\leqslant S(x)+5. \tag{11}$$

Hence $$-E\leqslant\int_E S_m(x)\,dx\leqslant 5E+\int_E S(x)\,dx,$$

which shows that $\int_E S_m(x)\,dx$ has the unique double limit zero, when $E\to 0$, $m\to\infty$.

It will be seen in § 9, that this last result, which is otherwise evident, is included in the more general statements of that article.

7. We shall now require the inequality referred to in the introduction. We know that, if $u \geqslant 0$ and $w \geqslant 0$,

$$(p+1)uw \leqslant u^{p+1}+pw^{1+1/p}.$$

Hence, subtracting u from both sides and dividing by p,

$$u\left(\frac{p+1}{p}w-\frac{1}{p}\right) \leqslant \frac{1}{p}\,[u(u^p-1)]+w^{1+1/p};$$

or, say,

$$uv \leqslant u\,\frac{u^p-1}{p}+\left(\frac{1+pv}{1+p}\right)^{1+1/p} \quad (v \geqslant -1/p), \qquad (12)$$

where

$$pv = (p+1)w-1;$$

and therefore

$$w = (1+pv)/(1+p).$$

Now

$$\operatorname*{Lt}_{p\to 0}\frac{1}{\log\left(\dfrac{1+pv}{1+p}\right)^{1+1/p}} = \operatorname*{Lt}_{p\to 0}\frac{p/(1+p)}{\log\dfrac{1+pv}{1+p}} = \frac{0}{0} = \operatorname*{Lt}_{p\to 0}\frac{1/(1+p)^2}{\dfrac{v}{1+pv}-\dfrac{1}{1+p}} = \frac{1}{v-1}.$$

Therefore, letting p approach zero, the inequality (12) gives us in the limit the following:—

$$uv \leqslant u\log u+e^{v-1} \quad (0 \leqslant u,\ -\infty < v). \qquad (13).$$

8. The preceding proof is of interest as showing how the inequality naturally suggested itself. The following proof is, however, more direct.

We have at once

$$v \leqslant e^{v-1};$$

and therefore

$$v-\log u \leqslant e^{v-1-\log u} \leqslant \frac{1}{u}\,e^{v-1},$$

whence

$$uv \leqslant u\log u+e^{v-1}.$$

9. We have already proved (§ 6), that

$$-1 \leqslant S_n(x) \leqslant S(x)+5. \qquad (11)$$

Hence

$$e^{-\frac{3}{2}} \leqslant e^{\frac{1}{2}S_n(x)-1} \leqslant e^{\frac{3}{2}}e^{\frac{1}{2}S(x)} \leqslant e^{\frac{3}{2}}\left(\frac{1}{2(1-\cos x)}\right)^{\frac{1}{2}} \leqslant (\tfrac{1}{2}e^3\operatorname{cosec}\tfrac{1}{2}x)^{\frac{1}{2}}.$$

This shews that $e^{\frac{1}{2}S_n(x)-1}$ is summable; moreover, if $u(x)$ is any positive function, such that $u \log u$ is summable, we have, by the inequality (13),

$$-\int_E \tfrac{1}{2}u(x)\,dx \leqslant \int_E u(x)\,\tfrac{1}{2}S_n(x)\,dx \leqslant \int_E u\log u\,dx + \int_E e^{\frac{1}{2}S_n(x)-1}\,dx$$

$$\leqslant \int_E u\log u\,dx + (\tfrac{1}{2}e^3)^{\frac{1}{2}}\int_E \frac{dx}{\sqrt{\sin\frac{1}{2}x}},$$

which shews that $\int_E u(x)\,S_n(x)\,dx$ has the unique double limit zero, as $n \to \infty$, $E \to 0$.

Again, if $f(x)$ is any function such that $|f(x)|\log|f(x)|$ is summable, and $u(x)$ denote either the function which is equal to $f(x)$ wherever positive and zero elsewhere, or the function which is equal to $|f(x)|$ wherever negative and zero elsewhere, $u(x)\log u(x)$ is evidently summable, and therefore $\int_E u(x)\,S_n(x)\,dx$ has the unique double limit zero, and therefore, since $f(x)$ is the difference of these two functions, $\int_E f(x)\,S_n(x)\,dx$ has the unique double limit zero.

10. Now the succession $f(x)\,S_n(x)$ converges, except at the origin, to the summable function $f(x)\,S(x)$. Therefore, Vitali's condition* having been just shown to be satisfied, the succession may be integrated term by term, and we have

$$\underset{n\to\infty}{\mathrm{Lt}}\int_0^x f(x)\,S_n(x)\,dx = \int_0^x f(x)\,S(x)\,dx.$$

Hence, since $S_n(x)$ is an even function, we see that, provided $|f(x)|\log|f(x)|$ is summable, not only from 0 to π, but also from $(-\pi)$ to 0, the same equation holds when we change the upper limit of integration from x to $-x$. Hence,

$$\underset{n\to\infty}{\mathrm{Lt}}\int_a^b f(x)\,S_n(x)\,dx = \int_a^b f(x)\,S(x)\,dx \quad (-\pi \leqslant a \leqslant b \leqslant \pi), \tag{14}$$

provided only $|f(x)|\log|f(x)|$ is summable from $-\pi$ to π.

---

* See my paper on "Semi-Integrals and Oscillating Successions of Functions," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 308.

Taking the extreme limits of integration π and $-\pi$, and writing as usual,

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{r=1} (a_r \cos rx + b_r \sin rx),$$

(14) becomes

$$\operatorname*{Lt}_{n\to\infty} \left(\frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \log \frac{1}{2(1-\cos x)}\, dx. \quad (15)$$

Thus, as stated in the introduction, in the case of the large class of functions $f(x)$ here considered, the series whose general term is the n-th Fourier cosine constant of $f(x)$ divided by n, converges, and its sum may be expressed by the right-hand side of (15).