

On the Solution of Problems in Diffraction by the Aid of Contour Integration

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XIX. *On the Solution of Problems in Diffraction by the Aid of Contour Integration.* By HENRY DAVIES, B.Sc.,
Technical Institute, Portsmouth *.

THE general problem of diffraction consists of finding solutions of the equation

$$\frac{\delta^2 V}{\delta t^2} = a^2 \nabla^2 V. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solutions must remain finite throughout the space considered and must satisfy certain specified boundary conditions.

This equation is modified when assumptions are made concerning the light vector.

Consider the case of a wedge of angle α , and assume that the electric force—taken as the light vector—is parallel to the edge of the wedge. Assume also that $V \propto e^{ikt}$. Then the general equation reduces to

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + m^2 V = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

provided the origin of co-ordinates is taken in the edge of the wedge, the latter being assumed to occupy the space $\alpha < \theta < 2\pi$.

The boundary conditions are that V shall vanish at $\theta = 0$ and at $\theta = \alpha$, and shall become infinite at a point (r', θ') .

2. The proper solution for unbounded space is

$$V = K_0(mR), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$R = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}$$

and $K_n(x)$ is Bessel's function of the second kind and of order n .

At this point it is necessary to introduce the relations which hold between the various functions which will be used.

The Bessel's functions of the second kind are related to those of the first kind by the equation

$$K_n(x) = \frac{\pi}{2 \sin n\pi} \cdot [J_{-n}(x) - e^{-in\pi} J_n(x)]. \quad . \quad . \quad (4)$$

* Read June 8, 1906.

When n is large the value of $J_n(x)$ is given very approximately by

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Pi(n)}, \quad \dots \quad (5)$$

where $\Pi(n)$ represents an infinite product.

Taking the asymptotic value of $\Pi(n)$ we have

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\sqrt{2\pi n} \epsilon^{n \log n - n}}. \quad \dots \quad (6)$$

From (6) it is evident that $J_n(x)$ vanishes at infinity when the real part of n is positive.

To obtain a value for $J_{-n}(x)$ proceed as follows:—

$$\begin{aligned} \Pi(n) \Pi(-n) &= \frac{n\pi}{\sin n\pi} \\ \therefore \frac{1}{\Pi(-n)} &= \frac{\sin n\pi}{n\pi} \Pi(n). \end{aligned}$$

Therefore

$$\begin{aligned} J_{-n}(x) &= \left(\frac{x}{2}\right)^{-n} \frac{1}{\Pi(-n)} \\ &= \left(\frac{x}{2}\right)^{-n} \frac{\sin n\pi}{n\pi} \cdot \Pi(n) \\ &= \left(\frac{x}{2}\right)^{-n} \frac{\sin n\pi}{n\pi} \cdot \sqrt{2\pi n} \epsilon^{n \log n - n}. \quad \dots \quad (7) \end{aligned}$$

This does not vanish at infinity when the real part of n is positive. This difficulty can be overcome as follows.

From (4) by multiplication by $J_n(x')$

$$\begin{aligned} K_n(x) J_n(x') &= J_n(x') \cdot \frac{\pi}{2 \sin n\pi} \left[J_{-n}(x) - \epsilon^{-in\pi} J_n(x) \right] \\ &= \left(\frac{x'}{2}\right)^n \frac{1}{\Pi(n)} \cdot \frac{\pi}{2 \sin n\pi} \left\{ \left(\frac{x}{2}\right)^{-n} \frac{\sin n\pi}{n\pi} \cdot \sqrt{2\pi n} \cdot \epsilon^{n \log n - n} \right. \\ &\quad \left. - \epsilon^{-in\pi} \left(\frac{x}{2}\right)^n \cdot \frac{1}{\sqrt{2\pi n} \epsilon^{n \log n - n}} \right\} \dots \quad (8) \end{aligned}$$

With substitution of the value of $\frac{1}{\Pi(n)}$ this reduces to

$$\begin{aligned} K_n(x) J_n(x') &= \frac{1}{2n} \cdot \left(\frac{x'}{2}\right)^n \left(\frac{x}{2}\right)^{-n} \\ &\quad - \epsilon^{-in\pi} \left(\frac{x}{2}\right)^n \left(\frac{x'}{2}\right)^n \frac{\pi}{2 \sin n\pi} \left\{ \frac{1}{\sqrt{2\pi n} \epsilon^{n \log n - n}} \right\} \end{aligned}$$

Hence when the real part of n is positive this becomes

$$K_n(x) J_n(x') = \frac{1}{2n} \left(\frac{x'}{x}\right)^n, \quad \dots \quad (9)$$

since the second part vanishes at infinity.

The expression

$$\frac{\cos n\{\pi - (\theta - \theta')\}}{\sin n\pi}$$

vanishes at infinity if $\theta - \theta'$ lies between 0 and 2π .

3. Consider the integral

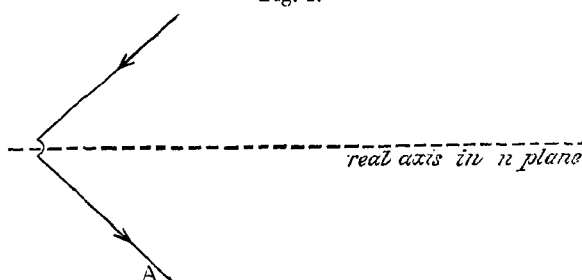
$$\int_A \frac{\cos n(\pi - \theta - \theta')}{\sin n\pi} J_n(mr') K_n(mr) dn, \quad \dots \quad (10)$$

where $r > r'$ and $\theta > \theta'$;

the integral being taken over the path A in the n -plane.

The integral is equal to $2\pi i \Sigma B$, where ΣB is the sum of the residues of this function.

Fig. 1.



The path A (fig. 1) has a small semicircle at the origin.

If, however, we remove the small semicircle then we easily find that

$$\begin{aligned} & \frac{1}{i\pi} \int_{A'} \frac{\cos n(\pi - \theta - \theta')}{\sin n\pi} J_n(mr') K_n(mr) dn \\ &= \frac{1}{\pi} \left\{ J_0(mr') K_0(mr) + 2 \sum_{n=1} J_n(mr') K_n(mr) \right. \\ & \quad \left. \times \cos n(\theta - \theta') \right\}. \quad \dots \quad (11) \end{aligned}$$

The path A' is that with the circle removed. There is a well-known addition theorem for $K_0(mr)$ which gives

$$\begin{aligned} K_0(mR) = & J_0(mr') K_0(mr) + 2 \sum_{n=1} J_n(mr') K_n(mr) \\ & \times \cos n(\theta - \theta'). \quad (12) \end{aligned}$$

From (11) and (12) there results

$$K_0(mR) = \frac{1}{i} \int_{\Delta'} \frac{\cos n(\pi - \bar{\theta} - \bar{\theta}')}{\sin n\pi} J_n(mr') K_n(mr) dn. \quad (13)$$

This equation gives a solution for the unbounded space. It is necessary now to add terms which shall satisfy the boundary equations while introducing no new singularities. After many trials I have found that the following equation satisfies the conditions completely:—

$$\begin{aligned} V = \frac{1}{i} \int_{\Delta'} \left[\cos n(\pi - \bar{\theta} - \bar{\theta}') - \cos n(\pi - \bar{\alpha} - \bar{\theta}') \frac{\sin n\theta}{\sin n\alpha} \right. \\ \left. - \cos n(\pi - \theta') \frac{\sin n(\alpha - \theta)}{\sin n\alpha} \right] \\ \times \frac{J_n(mr') K_n(mr)}{\sin n\pi} dn. \quad (14) \end{aligned}$$

If this is tested term by term it will be found to satisfy the differential equation and the boundary conditions. With some laborious work the trigonometry can be simplified, and the final result appears as

$$V = 2i \int_{\Delta'} \frac{\sin n(\alpha - \theta) \sin n\theta'}{\sin n\alpha} J_n(mr') K_n(mr) dn. \quad (15)$$

$r > r' \quad \text{and} \quad \alpha > \theta > \theta'.$

Since in (15) there is no pole at the origin, then A and A' are identical.

By Cauchy's residue theorem the whole solution is now obtainable as an infinite series for all values of θ in the space $0 > \theta > \alpha$.

The series is

$$V = \frac{4\pi}{\alpha} \cdot \sum \sin \frac{s\pi}{\alpha} \theta \sin \frac{s\pi}{\alpha} \theta' J_{\frac{s\pi}{\alpha}}(mr') K_{\frac{s\pi}{\alpha}}(mr), \quad (16)$$

when $r > r'$, and for the case when $r < r'$ it is only necessary to interchange these quantities.

A solution of the same problem is given by Macdonald in his book on Electric Waves, which depends on a theorem in an earlier portion of his book. That solution is in agreement with the above.

The method can be applied to three-dimensional problems, and some interesting results are being obtained which I hope soon to send in.