# Concerning Continuous Curves in the Plane.

By

Robert L. Moore in Austin (Texas. U. S. A.).

Several years ago I showed[1] that every two points of a continuous curve $M$ can be joined by a simple continuous arc which is a subset of $M$. In the present paper I will establish the more general result expressed below in the statement of Theorem 1. It will also be shown that if, in a plane, two points are separated by a continuous curve $M$ then they are separated by a simple closed curve which is a subset of $M$.

Before proceeding to the proofs of these theorems, I will define certain of the terms to be employed.

The term *connected*, as applied to point-sets which are not necessarily closed, has been used by different authors with different meanings. Lennes[2] calls a point-set connected if it can not be expressed as the sum of two point-sets neither of which contains a limit point[3] of the other. In the present paper such a set will be said to be *connected in the weak sense*. A point-set $M$ will be said to be *connected in the strong sense* if for every two points $A$ and $B$ belonging to $M$ there exists a closed subset of $M$ which contains $A$ and $B$ and is connected in the weak sense. The point-set $K$ will be said to be an *open* subset

[1] R. L. Moore, A theorem concerning continuous curves, Bull. Amer. Math. Soc., 2d. series, **23** (Febr. 1917), S. 233–236. See also H. Tietze, Über stetige Kurven, Jordansche Kurvenbögen und geschlossene Jordansche Kurven, Math. Zeitschr. **5** (1919), S. 284–291; and S. Mazurkiewicz, Sur les lignes de Jordan, Fundamenta Mathematicae **1** (1920), S. 166–209. In this article, Mazurkiewicz establishes numerous results and indicates that some of them were published earlier in a journal (C. R. Soc. Sc. Varsovie) to which I do not at present have access.

[2] N. J. Lennes, Curves in non-metrical analysis situs with an application in the calculus of variations, American Journal of Mathematics **33** (1911), S. 287–326

[3] A point is said to be a limit point of a point-set $M$ if every circle which encloses $P$ encloses at least one point of $M$ distinct from $P$.

of the point-set $M$, or to be open with respect to $M$, if $K$ is a subset of $M$ and $M - K$ is either vacuous or closed.

The truth of Theorem 1 below is not a consequence of the fact that a continuous curve is closed and connected[4]) and that every two of its points can be joined by a simple continuous arc lying wholly in it. To see this, consider the following example. Let $M$ denote the point-set composed of a countable infinity of straight line-intervals $OA, OA_1, OA_2, OA_3, \ldots$, where the points $A, A_1, A_2, A_3, \ldots$ are all on a given circle with center at $O$ and where, for every $n$, the distance from $A_n$ to $A$ is equal to $1/n$. Let $B$ and $C$ denote two distinct points between $O$ and $A$, let $H$ denote the straight line interval $BC$ and let $K$ denote the point-set $M - H$. The closed connected point-set $M$ has the property that every two of its points can be joined by a simple continuous arc lying wholly in it. But, though $K$ is an open subset of $M$ which is connected in the weak sense, not every two points of $K$ can be joined by a simple continuous arc that lies wholly in $K$. Furthermore, $K$ is not connected in the strong sense.

Theorem 1. *If an open subset of a continuous curve $M$ is connected in the weak sense it is also connected in the strong sense, and indeed if $K$ is an open subset of $M$, and $A$ and $B$ are two points which lie in a subset of $K$ which is connected in the weak sense, then $A$ and $B$ can be joined by a simple continuous arc which lies wholly in $K$.*

Proof. Let $H$ denote the point-set $M$-$K$. Since $H$ is closed, if $P$ is a point of $K$ and $n$ is a positive integer there exists a circle $J_{nP}$ with center at $P$ and with radius less than $1/n$ which encloses no point of $H$. Let $K_{nP}$ denote the set of all points $[X]$ such that $X$ and $P$ lie together in a connected subset of $M$ which lies wholly within $J_{nP}$. Let every such $K_{nP}$ be called a region and let the term region be not applied to any point-set which is not a $K_{nP}$ for some positive integer $n$, some point $P$ of $K$ and some $J_{nP}$. The continuous curve $M$ is connected im kleinen[5]). Hence for every $P, n$ and $K_{nP}$ there exists a circle with center at $P$ such that every point of $K$ within this circle belongs to $K_{nP}$. Also, if $X$ is any point belonging to $K_{nP}$, there exists a positive number $\delta$ such that, for every integer $m$ greater than $\delta$, $K_{mX}$ is a subset of $K_{nP}$. With the help of these facts it is easy to see that (1) a point $X$ of $K$

[4]) Obviously every point-set which is connected in the strong sense is also connected in the weak sense and a closed point-set which is connected in the weak sense is also connected in the strong sense. It is accordingly allowable to speak of a "closed, connected point-set" without specifying in which sense the term connected is used.

[5]) Cf **Hans Hahn** and **S. Mazurkiewicz**, loc. cit.

is a limit point of a point-set $T$ belonging to $K$ if, and only if, every region that contains $X$ contains a point of $T$ distinct from $X$, (2) if the space $S$ of my paper on the Foundations of Plane Analysis Situs[6]) be interpreted to mean the set of points $K$, and the term region used therein be interpreted as indicated above, then Axioms 1, 2 and 4 of that paper are all satisfied. But Theorem 15 of F. A. is proved on the basis of Axioms 1, 2 and 4. Furthermore, if $A$ and $B$ are two points which lie in a subset of $K$ which is connected in the weak sense then $K$ contains a subset $D$ which contains $A$ and $B$, is connected in the weak sense, and has the property that if $Y$ is any point of $D$ then there exists a region that contains $Y$ and lies wholly in $D$. Hence, by Theorem 15, $D$ (and therefore $K$) contains a point-set containing $A$ and $B$ which is bounded, closed and connected and contains no proper connected subset that contains both $A$ and $B$. Of course the words closed and connected are to be interpreted here as having the meaning assigned to them in F. A. Since in F. A. a closed point-set is defined in the usual manner as a point-set which contains all its limit points and connectedness is defined in the manner indicated above (for connectedness in the weak sense) in terms of point and limit point, it is clear that the significance to be attached to the words closed and connected (and therefore to the phrase simple continuous arc), as used here, depends on the sense in which the phrase limit point of a point-set is used in F. A. A point $X$ is there defined as the limit point of a point-set $T$ if, and only if, every region containing $X$ contains at least one point of $T$ distinct from $X$. It follows from (1) above that when the terms point and region used in F. A. are given the present interpretation then the phrase *limit point of a point-set* as used in F. A. takes on its ordinary meaning and thus the phrase *simple continuous arc* takes on its ordinary meaning. The truth of Theorem 1 is therefore established.

Definitions. A *domain* is a connected[7]) point-set $K$ such that if $P$ is any point of $K$ then $P$ lies within some circle whose interior is a subset of $K$. The *boundary* of a point-set $M$ is the set of all points [$X$] such that every circle which encloses $X$ encloses at least one point of $M$ and at least one point which does not belong to $M$. A point-set $M$ is said to be *limited*, or *bounded*, if there exists a circle which encloses both $M$ and its boundary. If $M$ is a limited point-set and $C$ is a circle which encloses both $M$ and its boundary then the outer boundary of $M$

[6]) Transactions of the American Mathematical Society 17 (1916), S. 131–164. This paper will be referred to as F. A.

[7]) Hereafter in this paper a set of points will be said to be connected if it is connected in the weak sense.

is the boundary of the point-set composed of all points $[X]$ such that $X$ can be joined to some point of $C$ by a simple continuous arc which contains no point of $M$ or of its boundary. The outer boundary of $M$ according to this definition is evidently a subset of the boundary of $M$. The exterior of a domain $R$ is the complement of $R + \beta$ where $\beta$ is the boundary of $R$. A limited domain $R$ is said to be simply connected if $R$ contains the interior of every simple closed curve which lies wholly in $R$.

Lemma 1. *If a limited domain $R$ has a connected exterior and every point of the boundary of $R$ is a limit point of the exterior of $R$ then $R$ contains every limited point-set whose boundary is a subset of $R$.*

Proof. Let $\beta$ and $E$ respectively denote the boundary and the exterior of $R$ and suppose that $K$ is a limited point-set which is not a subset of $R$. Let $\bar{K}$ denote the set of all those points of $K$ which belong to $\beta + E$. Since $K$ and $R$ are limited, the point-set $(\beta + E) - \bar{K}$ exists. But since $E$ is connected and every point of $\beta$ is a limit point of $E$, therefore $\beta + E$ is connected. Hence one of the sets $\bar{K}$ and $(\beta + E) - \bar{K}$ contains at least one limit point of the other one. Every such point is a boundary point of $K$. Thus if $K$ is not a subset of $R$, its boundary is not a subset of $R$. Lemma 1 is therefore established.

Lemma 2. *The interior $I$ of a simple closed curve $J$ contains every limited point-set whose boundary lies in $I$.*

The truth of Lemma 2 follows from Lemma 1 and the fact that the exterior of $I$ is connected and that every point of $J$ is a limit point of the exterior of $I$.

Theorem 2. *In order that the limited domain $R$ should be simply connected it is necessary and sufficient that the boundary of $R$ should be connected.*

Proof. This condition is necessary. For suppose, on the contrary, that there exists a simply connected domain $R$ whose boundary $\beta$ is not connected. Then $\beta$ is the sum of two mutually exclusive closed point-sets $\beta_1$ and $\beta_2$. Hence[8]) there exists a simple closed curve which lies in $R$ and separates $\beta_1$ from $\beta_2$. Thus the interior of $J$ contains points of $\beta$ and is therefore not a subset of $R$, contrary to the hypothesis that $R$ is simply connected.

The condition in question is also sufficient. For suppose, on the contrary, that there exists a limited domain $R$ which has a connected boundary $\beta$ and contains a simple closed curve $J$ whose interior is not a subset of $R$. Since $J$ itself is a subset of $R$ and every point of $J$ is

---

[8]) Cf. F. Hausdorff, Grundzüge der Mengenlehre, Veit and Co., Leipzig 1914, S. 344, VII.

a limit point of $I$, the interior of $J$, therefore $R$ contains at least one point of $I$. Since the connected point-set $I$ contains a point of $R$ and a point not belonging to $R$, it must contain a point of $\beta$, the boundary of $R$. Since $\beta$ is a connected point-set which contains a point of $I$, but no point of $J$, the boundary of $I$, therefore $\beta$ must lie wholly in $I$ and therefore, by Lemma 2, $R$ is a subset of $I$. But this is contrary to the hypothesis that $R$ contains $J$. Thus the supposition that the condition of Theorem 2 is not sufficient has led to a contradiction.

Theorem 3. *If $R$ is a limited domain and $\gamma$ is the outher boundary of $R$ then $\gamma$ is the complete boundary of some limited domain which contains $R$.*

Proof. Let $P$ denote a point of $R$ and let $K$ denote the set of all points $[X]$ such that $X$ can be joined to $P$ by a simple continuous arc which contains no point of $\gamma$. Suppose $F$ is a point of the boundary of $K$. If $F$ does not belong to $\gamma$ there exists a circle $\alpha$ enclosing $F$ but neither containing nor enclosing any point of $\gamma$. The circle $\alpha$ encloses a point $Y$ which belongs to $K$. There is a simple continuous arc $PY$ lying in $K$. There is a simple continuous arc $YF$ lying within $\alpha$ and therefore containing no point of $\gamma$. The point-set $PY + YF$ contains a simple continuous arc $PF$ which contains no point of $\gamma$. Hence $F$ belongs to $K$ and so does every other point within $\alpha$. Hence $F$ is not a point of the boundary of $K$. Thus the hypothesis that not every point of the boundary of $K$ belongs to $\gamma$ has led to a contradiction. But every point of $\gamma$ is a boundary point of $R$ and therefore of $K$. Hence $\gamma$ is the complete boundary of the domain $K$.

That the domain $K$ is limited may be proved as follows. Since $R$ is limited there exists a circle $C$ which encloses both $R$ and its boundary. Suppose that $K$ contains a point $Z$ without $C$. Then there exists a simple continuous are $PZ$ which contains no point of $\gamma$. Since $P$ is in $R$ and $Z$ is not in $R$, there exists on $PZ$ at least one point of $\beta$, the boundary of $R$. Hence, since $\beta$ is closed, there is a last point of $\beta$ on $PZ$. Call this point $W$. The point $W$ evidently belongs to $\gamma$. Thus the supposition that $K$ contains a point without $C$ leads to a contradiction. It follows that $K$ is limited.

Theorem 4. *If the boundary $\beta$ of a limited domain $R$ is a continuous curve then the outer boundary of $R$ is a simple closed curve.*

Proof. Let $C$ denote a circle which encloses $R$ and its boundary. By Theorem 3, $\alpha$, the outer boundary of $R$, is the complete boundary of some limited domain $K$ which contains $R$ but contains no point in common with $E$, the unlimited point-set which consists of all points $[P]$

such that $P$ can be joined to a point of $C$ by a simple continuous arc which contains no point of $\beta$. By definition, $\alpha$ is the complete boundary of $E$. Since $E$ is one of the domains complementary to the continuous curve $\beta$, it follows by a theorem due to Schoenflies[9]), that every point of $\alpha$ is „allseitig erreichbar" with respect to $E$. Hence, by a theorem due to Miss Torhorst[10]), $\alpha$ is itself a continuous curve. It follows by the above mentioned theorem of Schoenflies, that if $P$ is any point in $E$, or in $K$, and $Z$ is any point of $\alpha$ then there exists a simple continuous arc $PK$ which contains, in common with $\alpha$, only the point $P$.

Suppose now that $A$ and $B$ are any two points of $\alpha$. Then there exists a simple continuous arc $AOB$ which is a subset of $\alpha$. Let $P_1$ and $P_2$ denote points belonging to $K$ and $E$ respectively. There exist simple continuous arcs $P_1A$, $P_1B$, $P_2A$ and $P_2B$ each of which has only one point in common with $\alpha$. The point-set composed of $P_1A$ and $P_1B$ contains, as a subset, a simple continuous arc $A\overline{P}_1B$, while the point-set composed of $P_2A$ and $P_2B$ contains a simple continuous arc $A\overline{P}_2B$. The arcs $A\overline{P}_1B$ and $A\overline{P}_2B$ are clearly subsets of $K$ and of $E$ respectively. Let $J_1$ denote the simple closed curve formed by the two arcs $AOB$ and $A\overline{P}_1B$, and let $J_2$ denote that formed by $AOB$ and $A\overline{P}_2B$. Let $R_1$ and $R_2$ denote the interiors of $J_1$ and $J_2$ respectively. Since a portion of the boundary of $R_1$ belongs to $K$, therefore $R_1$ contains points of $K$. Similarly $R_2$ contains points of $E$. Since the connected point-set $E$ contains points without $R_1$ but no point on the boundary of $R_1$, therefore it contains no point in $R_1$. Hence $R_1$ contains no point of $\alpha$, the boundary of $E$.

Let $J_3$ denote the closed curve formed by the arcs $A\overline{P}_1B$ and $A\overline{P}_2B$ and let $R_3$ denote its interior. Then either $R_3 = R_1 + R_2 + \underset{\smile}{AOB}$ or $R_2 = R_1 + R_3 + \underset{\smile}{A\overline{P}_1B}$.

Case 1. Suppose that $R_3 = R_1 + R_2 + \underset{\smile}{AOB}$. Then there are points of the connected point-set $K$ without $R_2$ and therefore, since $K$ contains no point of the boundary of $R_2$, $K$ contains no point of $R_2$. Hence there is no point of $\alpha$ in $R_2$. So, in this case, $\underset{\smile}{AOB}$[11]) contains every point of $\alpha$ which is in $R_3$ and therefore, since $R_3$ is a domain containing $\underset{\smile}{AOB}$, no point of $\underset{\smile}{AOB}$ is a limit point of $\alpha - AOB$.

[9]) A. Schoenflies, Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Zweiter Teil, Leipzig 1908, S. 237.

[10]) Über den Rand der einfach zusammenhängenden ebenen Gebiete. Math. Zeitschr. 9 (1921), S. 64 (73).

[11]) If $AOB$ is a simple continuous arc, the notation $\underset{\smile}{AOB}$ is used to designate the point-set $AOB - (A + B)$.

Case 2. Suppose that $R_2 = R_1 + R_3 + \underset{\smile}{A\bar{P}_1B}$. In this case there are points of $K$ in $R_2$, but no points of $K$ on the boundary of $R_2$, and therefore, since $K$ is connected, $K$ lies wholly in $R_2$. Hence there is no point of $K$, and therefore no point of $\alpha$, without $R_2$. Hence $\alpha - AOB$ is a subset of $R_2$. But it has already been shown that $R_1$ contains no point of $\alpha$. Hence $\alpha - AOB$ is a subset of $R_3$.

It follows that in Case 2, as well as in Case 1, no point of $\underset{\smile}{AOB}$ is a limit point of $\alpha - AOB$. Thus neither of the sets $\underset{\smile}{AOB}$ and $\alpha - AOB$[12]) contains a limit point of the other one. So if $A$ and $B$ are any two points of the closed, connected, limited point-set $\alpha$ then $\alpha - (A + B)$ is the sum of two mutually exclusive point-sets neither of which contains a limit point of the other one. It follows[13]) that $\alpha$ is a simple closed curve.

Theorem 5. *If the points $A$ and $B$ do not lie on the continuous curve $\beta$, but are separated*[14]) *from each other by $\beta$, then they are separated from each other by a simple closed curve which is a subset of $\beta$.*

Proof. Clearly one of the points $A$ and $B$ lies in a limited domain $R$ which contains no point of $\beta$ but has a portion of $\beta$ as its boundary. This portion of $\beta$ is itself a continuous curve. Hence, by Theorem 4, the outer boundary of $R$ is a simple closed curve. It is clear that this curve separates $A$ from $B$.

[12]) The point-set $\alpha - AOB$ certainly exists. For otherwise the simple continuous arc $AOB$ would be the complete boundary of a limited domain, which is clearly impossible.

[13]) See my paper Concerning simple continuous curves, Transactions of the American Mathematical Society, 21 (1920), S. 342.

[14]) Two points $A$ and $B$ are said to be separated from each other by the closed point-set $M$ if every simple continuous arc from $A$ to $B$ contains a point of $M$ distinct from $A$ and from $B$.

(Eingegangen am 31. Januar 1922.)