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# XXXI. The attraction of ellipsoidal shells and of solid ellipsoids at external and internal points, with some historical notes

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# XXXI. *The Attraction of Ellipsoidal Shells and of Solid Ellipsoids at External and Internal Points, with some Historical Notes.* By ANDREW GRAY, *LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow* *.

1. I WROTE out the greater part of the following paper while on a holiday this summer, and was not aware until my return home that the method given below (§§ 24, 28 to 30) for the determination of the potential of a homothetic ellipsoidal shell had been anticipated by a passage in Thomson and Tait's 'Natural Philosophy' (Part ii., § 525). That also, I have since found, had been anticipated in § 24 of the beautiful memoir by Chasles, "Sur l'attraction d'une couche ellipsoïdale," in the *Journal de l'Ecole Polytechnique* (t. xv., 1835). The investigation contained in §§ 14 to 18 below is, however, quite distinct, and I think new. It gives a complete solution of the problem of the shell and the ellipsoid, and leads naturally to the other discussion already referred to, which differs in some respects from the solution of Chasles. I have, moreover, worked out in detail some particular problems. With all due acknowledgment, therefore, I venture to allow my notes to stand. I am aware of course of the recent discussions of the problem, for various laws of density, by Ferrers, Dyson

* From the 'Proceedings of the Royal Philosophical Society of Glasgow,' 1906. Communicated by the Author.

and others; but perhaps this solution of the older and simpler problem may not be without interest.

2. No problem in the Theory of Attractions has received more attention than that of the attraction of a solid ellipsoid on an external particle. The subject engaged the attention of Newton and Maclaurin *, who dealt with it by a combination of the geometrical methods of which these mathematicians had such a perfect mastery, and the results of Newton's own calculus of fluxions and fluents (differential and integral calculus); and all the celebrated analysts of the end of the eighteenth century and the beginning of the nineteenth, Laplace, Lagrange, Ivory, Gauss, Poisson, Chasles, wrote memoirs on the subject which have become classical. To find the attraction of the ellipsoid at an internal point had been soon found to be a comparatively easy matter. The process adopted was to take the attracted point as origin of polar coordinates by which the positions of the particles of the ellipsoid (supposed of uniform density) were specified, to express the volume of an element of the attracting body by these coordinates, and then the components of attraction on the particle as triple integrals with respect to the radius-vector and the two angular coordinates. As Poisson puts it in the introduction to the very remarkable memoir † read to the Académie des Sciences on October 3, 1833:—"The integration with reference to the radius-vector can be carried out without any difficulty, and in the case of an internal point a second integration is easily effected, so that the three components of attraction are expressed by single integrals, reducible to elliptic integrals of the first and second kinds. These integrals are obtainable in a finite form when the ellipsoid is one of revolution. When, however, the attracted particle lies outside the ellipsoid, the double integrals contain a radical, and have limits which render them much more complicated, so that, instead of carrying out the second integration directly, we have to turn the difficulty by reducing the problem for the external point to that for an internal point, which leaves the problem of the direct integration unsolved. For this reduction the theorem of Ivory leaves nothing to be desired."

3. Ivory's theorem depends on his notion of corresponding points on two ellipsoidal surfaces, the axes of which are coincident. Let $a$, $b$, $c$ be the lengths of the semi-axes of one, say the smaller, ellipsoid, $x$, $y$, $z$ the coordinates of a

* See Newton's *Principia*, Lib. I., ss. xii. and xiii., and Maclaurin's 'Treatise on Fluxions,' vol. ii.

† *Mémoires de l'Académie Royale des Sciences de l'Institut*, t. xiii., 1835.

point P on the surface, and similarly $a'$, $b'$, $c'$, $x'$, $y'$, $z'$ the lengths of the semi-axes of the larger ellipsoid, and the coordinates of a point Q upon it. If the coordinates fulfil the relations $x/a=x'/a'$, $y/b=y'/b'$, $z/c=z'/c'$, the points P, Q are corresponding points. It is easy to show that if P, P′ be two points on the first ellipsoid, Q, Q′ the corresponding points on the other, and the ellipsoids be confocal, the distances PQ′, P′Q are equal.

Now considering attractions in the direction of the principal axes, and taking first the axes $a$, $a'$, it can be proved that the attraction X of the first ellipsoid, A say, on a particle of unit mass at Q on the surface of the other ellipsoid, B, is to the attraction X′ of the ellipsoid B on a unit particle at the corresponding point P on the surface of A, in the ratio $bc/b'c'$. Similarly $Y/Y'=ca/c'a'$, $Z/Z'=ab/a'b'$. The ellipsoids are here supposed to be solid and of uniform density, $\rho$ say, and confocal. If we call the masses of the ellipsoids M, M′, then since $M/M'=abc/a'b'c'$, we may express the theorem in the form (not given by Ivory)

$$\frac{X}{X'}=\frac{a'}{a}\frac{M}{M'},\quad \frac{Y}{Y'}=\frac{b'}{b}\frac{M}{M'},\quad \frac{Z}{Z'}=\frac{c'}{c}\frac{M}{M'}. \quad . \quad . \quad (1)$$

When expressed in this form the theorem is evidently true whether the two ellipsoids have the same density or not, provided that each is homogeneous. For the components of force on a unit particle evidently vary with the masses of the ellipsoids, when their dimensions remain unchanged, and therefore a change in the ratio $M/M'$, caused by varying the density of either ellipsoid, is represented by a corresponding change in each of the ratios $X/X'$, $Y/Y'$, $Z/Z'$.

In the particular case of $M=M'$, the theorem takes the form

$$\frac{X}{X'}=\frac{a'}{a},\quad \frac{Y}{Y'}=\frac{b'}{b},\quad \frac{Z}{Z'}=\frac{c'}{c}. \quad . \quad . \quad . \quad (2)$$

It was first pointed out by Poisson that Ivory's theorem is true for every law of attraction, provided the law is a function of the distance only.

4. Let us now suppose that the problem of finding the attraction of a homogeneous ellipsoid at an internal point has been solved, and that it is required to find the attraction at an external point, Q say. It is only necessary to find for the corresponding point P on the surface of the given ellipsoid the attraction exerted on a unit particle by the confocal ellipsoid, the surface of which contains the point Q. The

components are given by (1), and thus the so-called external problem is reduced to the internal problem of which the solution is known.

5. The external problem was, however, solved directly by Poisson in the memoir above referred to, by the device, which he appears to have been the first to adopt, of imagining the ellipsoid divided into infinitely thin similar and similarly situated ellipsoidal shells, or "elliptic homœoids" as they have been called by Thomson and Tait*, then finding by direct integration the attraction exerted by such a shell on a unit particle at the given external point, and finally passing, by another integration from shell to shell, to the attraction exerted by the solid ellipsoid.

[The term homothetic ellipsoidal shell is used by many writers for the "couche elliptique" on which Poisson based his solution, and is perhaps less open than the term elliptic homœoid to objection on the ground of derivation; but we shall adopt the name elliptic homœoid, or simply homœoid, where there is no risk of ambiguity. Thomson and Tait also give the name "focaloid" to a shell bounded by two confocal ellipsoidal surfaces.]

6. On Ivory's notion of corresponding points Chasles based a very important and elegant theorem of the attraction of confocal homœoids. Imagine two elliptic homœoids, A, B, of infinitesimal thickness, and each of uniform (not necessarily the same) density, the two outer surfaces and the two inner surfaces of which are confocal. Let $a$, $b$, $c$ be the lengths of the semi-axes of A, and $a'$, $b'$, $c'$ the same quantities for B. Also let P and Q be corresponding points on the two shells. The theorem of Chasles affirms that the potential at the point Q due to the homœoid A is to the potential at P due to the homœoid B, as the mass of A is to the mass of B.

For let $ds$ be an element of A and $ds'$ the corresponding element of B, and $p$, $p'$ the length of the perpendiculars from the centre on the tangent plane at the elements. It is easy to prove that

$$\frac{p'ds'}{pds} = \frac{a'b'c'}{abc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The masses of corresponding elements of the shell are $\beta pds$. $\beta' p' ds'$, where $\beta$, $\beta'$ are constants depending on the density and scale of thickness in the two cases. The total masses are $4\pi\beta abc$, $4\pi\beta' a'b'c'$ respectively. Hence it follows that the masses of corresponding elements are in the ratio of the total masses in the two cases. [See § 13 below for the value of $\beta$, $\beta'$.]

* Nat. Phil. Part II. § 494 g.

Now, if $r$ be the distance of Q from any element of the shell A, it is also the distance of P from the corresponding element of B. If $\beta$, $\beta'$ be constant multipliers, as already explained, the potential at Q due to the matter at $ds$ is $\beta p ds/r$, and the potential at P due to the matter at $ds'$ is $\beta' p' ds'/r$ *. The former bears to the latter the ratio $\beta abc/\beta' a'b'c'$. Since this is true of every element the total potentials have the ratio just stated, that is the ratio of the masses of the shells.

It follows from this theorem that if the potential at an internal point is known for a thin elliptic homœoid, the potential at an external point can be found, and *vice versa*, by considering a confocal shell.

7. These results are true whatever the law of attraction may be, if only it is a function of the distance, as may be seen by substituting $f(r)$ for $1/r$ in the expression for the potential of an element. In the case of ordinary matter the law of attraction is that of the inverse square of the distance, and the potential determined is commonly called the Newtonian potential.

It is well known (and it will be referred to again presently) that the Newtonian potential of a homœoid is the same at every internal point, and therefore also at every point of the shell, since the potential is not discontinuous at points within attracting matter of finite volume density, or even at passage across a surface of finite density. Thus in order to find the potential at an external point P, due to a given elliptic homœoid, it is only necessary to imagine a confocal homœoid of equal mass constructed so as to have P on its surface, and find the uniform potential which it produces at every internal point. This is the potential required.

8. It follows from this result that the external confocal ellipsoidal surfaces are the equipotential surfaces of a uniform homœoid, and that such a shell is itself at uniform potential.

* We take here as the specification of the potential

$$V = \Sigma \frac{dm}{r}$$

where $r$ is the distance of the point for which the potential is defined from an element $dm$ of the attracting matter, and $\Sigma$ denotes summation for all elements. Here the unit of mass is that which concentrated at unit distance from an equal mass, also concentrated at a point, is attracted with unit force. When the ordinary unit of mass, the gramme, say, is used, the right-hand side of the equation for V must be multiplied by the value of the force of attraction which exists between two such units placed at unit distance, a centimetre, say, apart. This multiplier is called the "gravitation constant." It is generally omitted (that is taken as unity) in what follows: where it is inserted it is denoted by $\kappa$.

The resultant attraction due to such a shell exerted on a particle at an external point is along the normal through the point to the confocal surface on which the point lies.

It can be seen at once without analysis that an elliptic homœoid exerts no force at any point in the internal hollow, that is, that the potential has there a uniform value. For the shell may be imagined constructed by homogeneously straining a uniform thin spherical shell, within which, of course, the potential is uniform. Such strain is that which elongates all parallel dimensions of the shell in the same ratio, and therefore leaves it of uniform density, though of varying thickness, proportional to the length of the perpendicular from the centre on the tangent plane at each point. If then a cone of small solid angle be drawn with its vertex at any point in the hollow of the spherical shell, so as to intercept two elements of the surface, these two elements exert equal and opposite forces on a particle at the vertex. By the strain the masses of these two elements are not altered, nor the ratio of their distances from the vertex of the new cone into which the former one is changed. The elements, therefore, still exert equal and opposite forces on a particle at the vertex. Hence, as cones can be thus drawn so as to exhaust the shell by pairs of elements, the shell as a whole exerts no force at the common vertex.

9. The same idea of division of a solid ellipsoid into homœoids had, however, occurred to O. Rodrigues, and been used by him in a "Mémoire sur l'attraction des Sphéroïdes," published in the *Correspondance sur l'Ecole Polytechnique* (t. iii., 1816). The method adopted for the solution of the problem of the attraction of a solid ellipsoid seems to have been suggested by a previous paper by Gauss, and consists in finding the variation, $\delta W$, say, in the ratio of the potential of the ellipsoid at the given point, $h, k, l$, say, to the mass of the solid, when the squares of the semi-axes $a^2, b^2, c^2$ are altered by the same small amount, $\delta\phi$, say, that is by the passage from the given ellipsoid to an adjacent confocal ellipsoid. It is shown that for an external point $\delta W=0$, and for an internal point

$$\delta W=\frac{3\,\delta\phi}{4\,abc}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}-1\right). \quad . \quad . \quad . \quad (4)$$

From these results Rodrigues deduced the attraction of the ellipsoid at the given point. It is easy to find from them an expression for the potential. (See § 18 below.)

The determination of $\delta W$ depends on the evaluation of a certain integral taken throughout the volume of the ellipsoid in which the element of volume is expressed in ordinary polar coordinates. The integral is transformed by considering a homothetic shell within the ellipsoid, and taking as the element of volume that intercepted between the two surfaces at an element $ds$ of one of them. Then the integration is effected for the whole shell to which $ds$ belongs, and then carried from shell to shell for the whole ellipsoid.

It is to be noticed that Rodrigues did not determine the attraction or the potential due to a single homothetic shell, but merely imagined the solid ellipsoid divided into such shells in his process of integration.

10. In the volume of the *Mémoires de l'Institut* (t. xv., 1835) already referred to and immediately preceding the memoir cited, is a "Mémoire sur l'attraction des Ellipsoïdes," also by Chasles, in which the mode of division into homothetic shells is used, and is attributed to Poisson. There can be no doubt that Poisson was the first to calculate explicitly the attraction of such a shell at an external point, and to apply it to the problem of the solid ellipsoid; but it is equally clear that the idea of this mode of division is of earlier date. Over this point arose in 1837 a curious reclamation. Another memoir by Chasles, in which the same method was used, was referred by the Académie des Sciences to Libri and Poinsot. The latter reported without mentioning Poisson's memoir of 1835, and thereupon Poisson in the *Comptes Rendus* (t. vi., pp. 83 -840) called attention to this mode of decomposition of a solid ellipsoid, and affirmed that it offered the only means of reducing the double integrals of the problem to single integrals. Poinsot rejoined re-affirming the priority of Rodrigues in this matter, and the discussion was closed by some further remarks by Poisson, and a second rejoinder from Poinsot. These are to be found at the beginning of the next volume of the *Comptes Rendus*: a fairly full account of the controversy is given also by Todhunter in his 'History.'

11. In Poisson's memoir of 1835 it is proved that the resultant attraction of an elliptic homœoid at an external point $f$, $g$, $h$, is directed along the internal axis of the cone drawn from the external point as vertex to touch the homœoid. This is a very remarkable theorem, and attracted very considerable attention. For that axis of the cone is the normal to the confocal ellipsoidal surface through $f$, $g$, $h$; and the theorem at once gives the family of external confocal ellipsoidal surfaces as the equipotential surfaces of the homœoid. The importance of these surfaces was not perceived until later, when the researches of Green, Gauss, Chasles, and others had become known, and had led to new methods of treating

problems of attraction—methods which had become essential for the progress of the theory of electricity.

Poisson resolved the force due to each element of the homœoid along the axis of the cone, and by expressing the component in terms of polar coordinates referred to $f, g, h$ as origin, was able to obtain the resultant force in an integrable form. His process is one of direct integration of the expressions obtained, and involves some troublesome considerations as to the limits of the integration with respect to $\theta$, the angle between the axis of the cone and the line drawn to the element considered, and therefore runs to considerable length.

12. A very different process of calculating this integral is followed by Chasles in his memoir already cited, "Sur l'attraction d'une couche ellipsoïdale." There the theorem of Lamé (given for the steady motion of heat in a uniform solid), that Laplace's differential equation of the potential is integrable when the equipotential surfaces are known, is employed for the family of equipotential surfaces revealed by Poisson's theorem, and the attraction is reduced to the evaluation of a constant left undetermined by the integration of the specialized form of the differential equation for this case. This is effected by considering the particular case of the attracted point on the surface, evaluating the integral for this case, and comparing with the expression obtained from the integration of the differential equation.

13. I shall now show how the integral for the resultant force at $f, g, h$, expressed in terms of the element $ds$, its coordinates $x, y, z$, the perpendicular $p$ from the centre on the tangent plane at $ds$, the distance $r$ from the point $f, g, h$ to $ds$, and the angle $\theta$ between this line and the axis of the cone, can, by means of a simple geometrical theorem of confocal surfaces, which I have not before seen remarked, be transformed to an immediately integrable form, so that the whole calculation can be set forth very briefly.

In order that the result may be at once applicable to the calculation of the potential of a solid ellipsoid, I take as the equation of the outer surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k$$

or in the usual abridged notation

$$\Sigma\frac{x^2}{a^2}=k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $k$ is less than 1. The equation of the inner surface is

$$\Sigma\frac{x^2}{a^2}=k-dk, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as $k$ must diminish from the value unity, for the surface which has the equation

$$\Sigma\frac{x^2}{a^2}=1,$$

to zero when the axes are of infinitesimal length. Further, I shall take as the equation of the confocal surface

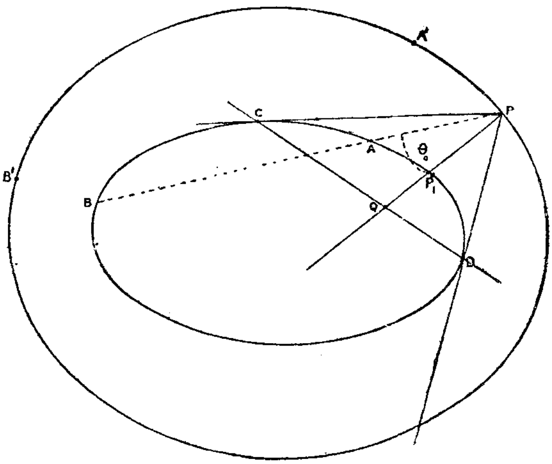
$$\Sigma\frac{x^2}{a^2+u}=k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The thickness of the homœoid at the point A, of coordinates $x$, $y$, $z$, on its outer surface is $\frac{1}{2}pdk/k$, and the mass of an element of area $ds$ at the same point of the shell is $\frac{1}{2}p\rho dsdk/k$, where $p$ is the length of the perpendicular from the centre on the tangent plane at $x$, $y$, $z$, and $\rho$ is the volume density of the matter of the shell. For the thickness of the shell is clearly $\Sigma(xdx/a^2)/\sqrt{\Sigma(x^2/a^4)}$, that is $p\Sigma(xdx/a^2)/k$, and by differentiation of (5) this is at once seen to be $\frac{1}{2}pdk/k$. Thus $\frac{1}{2}\rho dk/k$ is the constant $\beta$ of § 6 above, and similarly $\beta'$ is found.

14. I shall now establish the geometrical lemma referred to in § 13 above, on which will be based a proof of Poisson's theorem that the resultant force exerted by the homœoid on a unit particle at the external point P of coordinates $f$, $g$, $h$ acts along the normal to the confocal through P: then I shall give a very simple and direct calculation of the amount of this resultant force, and finally obtain the potential of the homœoid, and of a solid ellipsoid, at any external point.

Let the enveloping cone be drawn from P as vertex to the external surface (5) of the homœoid. The points of contact lie in the polar plane of P; and the internal axis of the cone, the normal at P to the confocal, meets this plane in a point Q. Now let a line drawn from P at any angle $\theta_0$ to PQ meet the homœoid in the two points A, B (see fig. 1). Consider for the present only one of these, A, and let $x$, $y$, $z$ be its coordinates, and $r$ denote its distance from P. P is on the confocal represented by (7): let $P_1$ (coordinates $f_1$, $g_1$, $h_1$) be the corresponding point on the surface (5) of the homœoid, and A′ (coordinates $x'$, $y'$, $z'$) be the point on the confocal corresponding to A. The distance of A′ from $P_1$ is also $r$ by the property of pairs of corresponding points. Also $f_1, g_1, h_1 = fa/a', gb/b', hc/c'$ and $x', y', z' = xa'/a, yb'/b, zc'/c$, where $a', b', c' = \sqrt{a^2+u}, \sqrt{b^2+u}, \sqrt{c^2+u}$.

Fig. 1.




Now let $p_0, p'$, be the lengths of the perpendicular from the centre on the tangent plane at P, and the tangent plane at A′ respectively, and $\theta'$ denote the angle between the latter perpendicular drawn outwards and the line $P_1A'$. The lemma to be established is expressed by the equation

$$p' \cos\theta_0 = p_0 \cos\theta'.$$

In order to express $\cos\theta_0$ we have the direction cosines of QP and AP. These are $\{f/(a^2+u), g/(b^2+u), h/(c^2+u)\}p_0/k$ and $(f-x, g-y, h-z)/r$. Hence

$$\cos\theta_0 = \frac{p_0}{k}\Sigma\left(\frac{f}{a^2+u}\frac{f-x}{r}\right) = \frac{p_0}{rk}\left(k - \Sigma\frac{fx}{a^2+u}\right), \quad (8)$$

or as we may write it

$$\cos\theta_0 = \frac{p_0}{rk}\left(k - \Sigma\frac{f_1x'}{a^2+u}\right). \quad . \quad . \quad . \quad . \quad (8')$$

Similarly we obtain

$$\cos\theta' = \frac{p'}{rk}\left(k - \Sigma\frac{f_1x'}{a^2+u}\right) \quad . \quad . \quad . \quad . \quad (9)$$

so that

$$p' \cos\theta_0 = p_0 \cos\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which was to be proved. A similar relation of course holds also for the points $P_1$, A on the homœoid.

This theorem, it is to be remarked, is not confined to confocal ellipsoidal surfaces.

15. Now, imagine drawn from P as vertex a cone of small solid angle $d\omega$, intercepting two elements of the homœoid at A and B: let $ds$ be the area of that at A. The length of the perpendicular from the centre on the tangent plane at A being $p$, the mass of the element is $\frac{1}{2}\rho p dk ds/k$; and if $ds'$ be the element at A′ of the surface of the confocal corresponding to $ds$, that is, containing the points corresponding to those contained in $ds$, we know that

$$p ds = \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} p' ds'. \quad . \quad . \quad (11)$$

The attraction at right angles to the axis PQ exerted by the element on a unit particle at P is when $\kappa=1$ (see footnote § 6)

$$\tfrac{1}{2}\rho \frac{dk}{k}\frac{p ds}{r^2}\sin\theta_0$$

which by (11) and (10) may be written

$$\tfrac{1}{2}\rho p_0 \frac{dk}{k}\frac{a\,b\,c}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\frac{\cos\theta'\,ds'}{r^2}\tan\theta_0.$$

In this the only factor which varies from element to element is

$$\frac{\cos\theta'\,ds'}{r^2}\tan\theta_0.$$

Now $\cos\theta'\,ds'/r^2$ is the solid angle $d\omega$ subtended at $P_1$ by the element $ds'$ at A′. We can exhaust the whole of the homœoid by means of elements intercepted by small cones drawn from P as vertex; and to this corresponds precisely an exhaustion of the confocal by small cones radiating from the internal point $P_1$ as vertex. And clearly for every elementary cone of solid angle $d\omega$, there exists another in the same plane through PQ, for which the factor just referred to has the same value with opposite sign. Hence in no plane through PQ is there any force perpendicular to PQ on a unit particle at P; that is the resultant is along PQ. This is Poisson's theorem.

16. The total force F along PQ (in the direction from P to Q) is given by

$$F = \tfrac{1}{2}\rho\frac{dk}{k}\int p\frac{\cos\theta_0\,ds}{r^2}, \quad . \quad . \quad . \quad . \quad (12)$$

in which the expression to be integrated over the homœoid is the component of attraction along PQ due to the single element $ds$. If now we use the transformation already employed above, that is, express the coordinates of A and P in terms of the coordinates of their corresponding points A′ and $P_1$, A′ on the confocal, $P_1$ on the given shell, and substitute for $ds$ its proper value in terms of the area $ds'$ of the corresponding element on the confocal as given by (11), and use the theorem (10), the transformed equation is

$$F=\tfrac{1}{2}\rho\frac{dk}{k}\frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}p_0\int\frac{\cos\theta'}{r^2}\,ds' \quad . \quad (13)$$

in which the integral is now taken over the confocal. The only variable factor is now $ds'\cos\theta'/r^2$, and it is well known that for the complete confocal shell

$$\int\frac{ds'\cos\theta'}{r^2}=4\pi$$

since the point $P_1$ is within it. Hence, if V be the potential at P, (13) becomes

$$F=-\frac{dV}{dn}=2\pi\rho\frac{dk}{k}\frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}p_0 \quad . \quad (14)$$

where $dn$ is an infinitesimal step outward along the normal at P to the confocal. But by the equation of the confocal [see also § 29]

$$dn=\tfrac{1}{2}\frac{k}{p_0}\,du$$

and (14) becomes

$$-dV=\pi\rho abc\,\frac{dk\,du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\,. \quad . \quad (15)$$

The potential V of the homœoid at P, and at any other point of the confocal surface, is thus given by

$$V=\pi\rho abc\,dk\int_{\lambda'}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad . \quad (16)$$

where $u$ is now supposed to vary in value as we go from confocal to confocal outward from P in the integration: the confocal from which the integration starts is that on which P lies, and $\lambda'$ is the positive root of (7) regarded as a cubic in $u$. The value of the expression on the right is $-V_\infty+V$; and as $V_\infty=0$, we have (16).

17. We may proceed in precisely the same way when P is within the homœoid. A confocal ellipsoidal surface of

equation

$$\Sigma \frac{f^2}{a^2-u} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is described through P and the corresponding point $P_1$ on the shell is taken as before. P is joined to the point A on the shell and $P_1$ to the corresponding point A′ on the confocal. The angle $\theta_0$ is as before between AP and the normal at P, and $\theta'$ between $P_1A'$ and the normal at A′; also $AP = A'P_1 = r$. If $p_0$, $p'$ be the lengths of the perpendiculars from the centre on the tangent planes at P and A′, we have as before

$$p' \cos \theta_0 = p_0 \cos \theta'.$$

Precisely as in the former case we get for the attraction on a unit particle at P

$$F = \tfrac{1}{2}\rho \frac{dk}{k} \frac{abc}{\sqrt{(a^2-u)(b^2-u)(c^2-u)}} p_0 \int \frac{\cos \theta'}{r^2} ds', \quad . \quad (18)$$

where the integral is to be taken over the confocal. The integral is the total solid angle at $P_1$ subtended by the confocal, and as $P_1$ is outside that surface, the solid angle is zero. Hence the force is zero at every point within the shell, and the potential is there uniform. Thus the problem of the attraction of a homœoid is completely solved.

We have, in the result stated in (13), the curious theorem that the value of F at P is, to a constant factor, equal to the potential produced, at any point internal to itself, by a uniform magnetic shell coinciding with the confocal surface. The strength of this shell is proportional to the length of the perpendicular from the centre on the tangent plane to the confocal at the point P, and therefore varies with the position of P on the surface, as does the length of that perpendicular.

A similar theorem holds for a distribution upon any surface whatever, which maintains that surface at uniform potential. The potential which that distribution produces at an external point P is equal to the potential which a magnetic shell, coinciding with the equipotential surface through P for the same distribution, produces at any point internal to itself. The strength of the shell varies with the position of P, and is inversely proportional to the distance, $dn$, of P measured along the normal to a chosen neighbouring equipotential surface. This expresses, in fact, the relation (32) below, namely,

$$-\frac{dV}{dn} = 4\pi\sigma.$$

This shows that $\sigma$ is inversely proportional to $dn$ for a constant $dV$. [See § 27.]

18. For a solid homogeneous ellipsoid, which has

$$\Sigma \frac{x^2}{a^2} = 1$$

for the equation of its surface, (15) becomes

$$V = \pi \rho abc \int_\lambda^\infty \int_{f(u)}^1 \frac{du\,dk}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad . \quad (19)$$

where λ [the positive root of the equation in $u$

$$\Sigma \frac{f^2}{a^2+u} = 1]$$

is the lower limit of the integration with respect to $u$, and $f(u)$ is the value of $k$ that fulfils equation (7) for any one particular value that $u$ may have in the range of integration. This value of $k$ is given by

$$k = \Sigma \frac{f^2}{a^2+u}.$$

Thus

$$V = \pi \rho abc \int_\lambda^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad . \quad (20)$$

There is no difficulty in the application of this method of integration by shells to the formal determination of the potential of solid ellipsoids or of thick shells of varying density, if each homœoidal film is of uniform density throughout. If the density $\rho$ vary from shell to shell then for the ellipsoid by (16)

$$V = \pi abc \int_0^1 \rho dk \int_{\lambda'}^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

where $\lambda'$ is again the positive root of (7) for any given value of $k$. But

$$dk = -\left\{ \Sigma \frac{f^2}{(a^2+u)^2} \right\} du$$

and when $k=0$, $u=\infty$; likewise when $k=1$ $u=\lambda$, where λ is the positive root of

$$\Sigma \frac{f^2}{a^2+u} = 1$$

regarded as a cubic in $u$. Therefore

$$V = \pi abc \int_\lambda^\infty \rho \left\{ \Sigma \frac{f^2}{(a^2+u)^2} \right\} du \int_u^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (20')$$

19. The result stated in (20) seems to have been first given by Plana in a "Note sur l'intégrale $\int \frac{dM}{r} = V$" which appeared in *Crelle's Journal*, vol. xx. (1840). It was given also by Lejeune Dirichlet in the same journal in 1846. His process of demonstration is, however, indirect. The result in (20) is first assumed, and then verified by showing that this value of V satisfies Laplace's differential equation of the potential

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

at every point external to the ellipsoid, and that V vanishes at infinity. This latter fact is important. It is fairly obvious for both the homœoid and the solid ellipsoid. The integral (16) for the homœoid, for example, may be compared with the greater integral obtained by putting for each of $a^2, b^2, c^2$ the value of the smallest. This integral is then evaluated, and it is seen that it vanishes at infinity; *a fortiori* so does the integral for the homœoid.

20. If the point $f, g, h$ be on the surface of the ellipsoid the value of $\lambda$ is zero in (16) and (20). In the former case the modified equation gives the potential at every internal point for the homœoid; in the latter case

$$V = \pi\rho abc \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad . \quad (22)$$

for a surface point on the solid ellipsoid. The case of an internal point requires examination in the latter case.

It is easy to show that equation (22) is applicable without change of form to the case in which the point $f, g, h$ lies within the surface of the solid ellipsoid. For let the point be on the homœoidal surface given by

$$\Sigma \frac{f^2}{a^2} = \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

then the potential is made up of two parts, $V_1$ due to the ellipsoid internal to the surface (23), and $V_2$ due to the homœoid of finite thickness external to the point. By (22)

$$V_1 = \pi\rho abc\mu^{\frac{3}{2}} \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2\mu+u}\right) du}{\sqrt{(a^2\mu+u)(b^2\mu+u)(c^2\mu+u)}},$$

which, if we write $u'$ for $u/\mu$ and substitute, becomes (accents omitted)

$$V_1 = \pi\rho abc \int_0^\infty \frac{\left(\mu - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad . \quad (24)$$

Again, as may easily be verified,

$$V_2 = \pi\rho abc \int_\mu^1 dk \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$= \pi\rho abc(1-\mu) \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad . \quad (25)$$

Hence

$$V = V_1 + V_2 = \pi\rho abc \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} . \quad (26)$$

The value of V given by (20) and (26) was verified by Dirichlet by showing that it satisfies Poisson's differential equation

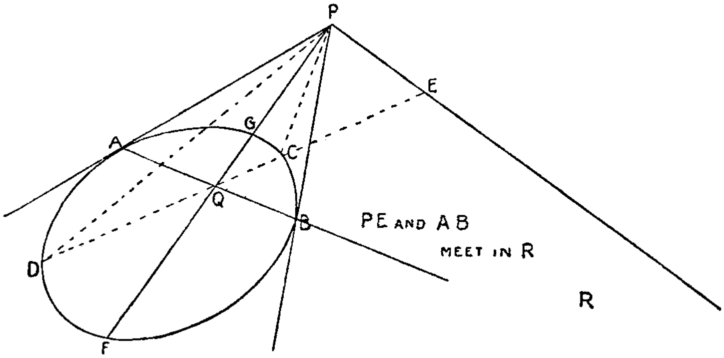
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho \quad . \quad . \quad . \quad . \quad (27)$$

within the attracting matter, and Laplace's equation elsewhere, gives continuous values of the force-components $-dV/dx$, $-dV/dy$, $-dV/dz$ at the surface, and vanishes for $u=\infty$. Thus (20) and (26) give the solution of the differential equation of the potential for the given distribution of matter, and the known family of equipotential surfaces possessed by each homœoidal part. As has already been remarked, it was shown by Lamé that the differential equation could be integrated in these circumstances. It would, however, be outside the scope of the present paper to enter into a discussion of the process. Suffice it to say that any solution which fulfils the conditions indicated above can be proved to be the only one.

21. The lemma stated in § 14 above enables the whole problem of the ellipsoid to be disposed of very simply; but, so far as merely proving Poisson's theorem of the direction of the attraction of a thin homœoid is concerned, nothing more elegant has ever been invented than the demonstration published in *Crelle's Journal* (Bd. 12, 1834) by Steiner, immediately after the theorem was announced by Poisson in the memoir of 1833, to which reference has already been made. Steiner's construction is shown in the adjoining diagram.

P is the external point and PQ is the normal to the confocal through P to the homœoid of which a section through PQ is given in fig. 2. A, B are two of the points in which the enveloping cone touches the homœoid, and therefore the line

Fig. 2.




AB is the polar of P with respect to the elliptic section made by the plane BPA. The line PQ meets the curve in G and produced again meets it in F. G, F are points in which the line PQ, starting from P and ending in F, is divided harmonically. Similarly, if PE be drawn at right angles to PG and meet AQB produced in R, the line RQ is also harmonically divided in B and A. It follows that if EQ meet the curve in C and D, EQ is divided harmonically in C and D. Thus PQ not only bisects the angle APB, but also the angle CPD. Hence DQ/QC = PD/PC.

Now let a cone of small vertical angle be drawn from Q as vertex with its axis along CD. It will intercept two elements of the homœoid at C and D, the masses of which are directly as the squares of their distances from Q, while their attractions, per unit of their mass in each case, are inversely as the squares of these distances. Hence the total attractions on a unit particle at Q are equal and opposite. But it has been seen that PD/PC = DQ/QC ; hence the attractions of the same pair of elements on a particle at P must be of equal amount, and being along PD and PC are equally inclined to PQ, and have therefore a resultant along that line. The same thing is true for any other pair of elements intercepted by a cone with vertex at Q, and the whole homœoid may be exhausted by pairs of elements in this way. Any plane through PQ thus divides the homœoid into two portions which exert attractions at P equal in amount and equally inclined to PQ.

22. Returning now to the results expressed by equations (16), (20), with lower limit $\lambda$ or 0, as the case may be, it is obvious that we can change the lower limit of the integral. Thus, taking (16) we write

$$u+\lambda'_1=u'+\lambda'$$

where $\lambda'_1$ is positive as well as $\lambda$. Then when $u$ is equal to $\lambda'$, $u'$ is equal to $\lambda'_1$, and the equation becomes

$$V=\pi\rho abcdk\int_{\lambda'_1}^{\infty}\frac{du'}{\sqrt{(a^2+u')(b^2+u')(c^2+u')}}, \quad . \quad (28)$$

which is of the same form as (16) and the accents on the $u$'s may be omitted. This proves that two confocal homœoids of equal mass produce the same potential at any point P external to both, that is, that their external fields are identical. If the masses are different, the potentials (and therefore also the field-intensities) at the different points are proportional to the masses.

This of course is a particular case of the very general theorem of the potentials of distributions which was given by George Green, of Nottingham, in his celebrated "Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," published by subscription at Nottingham in 1828.

The substitution used above is also applicable to (20), and proves that any two confocal solid ellipsoids of equal mass produce the same potential at every point external to both. If the masses are different in the two ellipsoids, the potentials at the same external point are proportional to the masses. This is what is usually called Maclaurin's * theorem; but it was only given in its full generality by Laplace many years after Maclaurin's death. It is stated in Maclaurin's 'Treatise of Fluxions' (Edinburgh, 1742), § 653, that the attractions of two confocal ellipsoids are the same at all external points which are on the prolongation of the axes. This was a very remarkable result for the time, and though the theorem was only fully generalised by Laplace in his book entitled 'Théorie du Mouvement et de la Figure Elliptique des Planètes'

* Colin Maclaurin, 1698–1746, Professor of Mathematics in the University of Edinburgh, appointed as assistant and (apparently) successor to James Gregory in 1725. There being a difficulty, through want of funds, in making this arrangement, Newton offered to pay £20 a year if Maclaurin were appointed. Thus Maclaurin was appointed, *Newtono suadente*, as stated in the inscription on his monument.

(Paris, 1784)*, there is some justification for continuing, as has been very generally done, to associate it, even in its extended form, with the name of Maclaurin.

23. From (14) in § 16 above we obtain the components of attraction at the point P. The cosine of the angle which the normal at P to the confocal makes with the semi-axis, of which the length is $a$, is $p_0 f/(a^2+u)k$. [See (7) § 13.] Hence for the component X of force in the direction of $x$ increasing on a unit particle at P (14) gives

$$X = -2\pi\rho \frac{dk}{k^2} \frac{abcf}{\sqrt{(a^2+u)^3(b^2+u)(c^2+u)}} p^2_0 \quad . \quad (29)$$

Y and Z are obtained by substituting $g/(b^2+u)$, $h/(c^2+u)$ for $f/(a^2+u)$ in this equation. The same results are deducible from (16) by differentiating with respect to $u$, and multiplying the result by the value of $du/df$ drawn from the equation

$$\Sigma \frac{f^2}{a^2+u} = k$$

of the confocal.

In the same way (20) gives

$$X = -2\pi\rho abcf \int_\lambda^\infty \frac{du}{\sqrt{(a^2+u)^3(b^2+u)(c^2+u)}} \quad . \quad (30)$$

and similar expressions for Y, Z, which may be written down by symmetry. In the case of $\lambda=0$, the factors which multiply $f$, $g$, $h$ in these expressions for X, Y, Z are independent of the values of these coordinates. Hence for an internal point $f$, $g$, $h$ of a solid ellipsoid

$$X = Af, \quad Y = Bg, \quad Z = Ch \quad . \quad . \quad . \quad . \quad (31)$$

where A, B, C are constants, the values of which are given by (30) and the other two similar equations

24. I shall now indicate the method which the theory of equivalent distributions affords for the solution of the problem of the ellipsoid. It was shown by Coulomb, for the case of an electrical distribution, that the normal force just outside a closed conductor is proportional to the surface

* This book is referred to by Todhunter in his 'History of Attractions,' and he quotes Professor de Morgan as to its rarity. I have not seen it, and give the reference above only at second hand. The University Library has no copy, and though the magnificent edition of Laplace's works, which is now being published in Paris, has reached vol. xiii., this book has not been included, though much of later date and on similar subjects has already appeared.

density (amount of attracting or repelling matter per unit area) in the neighbourhood. In the language of the potential this is expressed by the equation

$$-\frac{dV}{dn}=4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where $\sigma$ is the surface-density, and $dV/dn$ denotes the rate of variation of potential per unit of distance outwards along the normal. This equation is at once transferable to gravitational attraction, and $\sigma$ becomes the density of a thin stratum of ordinary matter. Here $\sigma$ of course denotes the surface-density of ordinary matter; *e. g.* the mass $\frac{1}{2}p\rho dk/k$ of a homœoid taken per unit area at an element just outside which the normal force $-dV/dn$ is taken.

25. I shall not discuss the properties of level surfaces here; but merely apply some of the properties I have mentioned to the problem of the ellipsoid. But a theorem of Bertrand may be referred to of which our process will afford an illustration. Let there be a family of surfaces represented by the equation

$$f(x, y, z)=\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $\alpha$ is a variable parameter; and let them be such that if a distribution of matter be placed on the surface, S say, characterized by any chosen value of $\alpha$, and be made of surface-density inversely proportional to the distance from that surface to an adjacent one of the family, the whole space within the surface is at uniform potential. Then the surfaces external to S are level surfaces for the distribution specified.

The truth of this theorem may be seen as follows. Let the distribution specified be made on an inner surface, $S_1$ say, of the family: the space within is at uniform potential, and therefore so also are all points of the surface. But if $\sigma$ be the density at any point of the surface, then just outside

$$-\frac{dV}{dn}=4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Now the step from the surface $S_1$ to an adjacent one $S_2$ may be taken as $dn$, and being inversely proportional to $\sigma$ gives a constant difference of potential between $S_1$ and $S_2$. Hence $S_2$ is a level surface for the distribution on $S_1$. Let now the distribution be transferred to $S_2$, and be made according to the law set forth in (32) for the values of $dV/dn$ which exist at $S_2$ with the distribution on $S_1$. Since $S_2$ is a level surface for the distribution on $S_1$, the transference thus effected will bring the whole space within $S_2$ to uniform potential equal to that which existed at $S_2$ before the transference. But the surface $S_2$ is by hypothesis one over which a distribution can be made of density inversely proportional at the different points to the normal step to an adjacent surface $S_3$, and producing uniform potential in the interior. If the mass in this latter distribution be made the same as that which has been transferred from $S_1$, the potentials at $S_2$ and within it produced in the two cases will be the same. For it can be proved * that there cannot be two distributions of a given charge of matter over a surface so as to produce uniform potential at all parts on or within the surface.

The distribution therefore transferred from $S_1$ must have the same density as in the other case supposed, that is $\sigma$ must be inversely proportional to the step from $S_2$ to $S_3$. $S_3$ is a level surface and the charge can now be transferred to $S_3$, when $S_4$ will be found to be a level surface and so on. Hence Bertrand's theorem is proved. It is easy to construct (see Picard, *Traité d'Analyse*, tome i.) an analytical proof of the theorem, founding on Lamé's theorem of the integration of Laplace's equation for a given system of level surfaces.

26. In the transference of matter imagined in the last article the particles may be regarded as carried out along trajectories, cutting the successive surfaces at right angles. Thus, if we draw these trajectories from points in the periphery of $ds_1$, they will mark out elements $ds_2$, $ds_3$, etc., on the successive surfaces. The matter first on $ds_1$ will be carried to $ds_2$, then to $ds_3$, and so on; and this law will hold however small $ds_1$, $ds_2$, etc. may be made.

Let now this process be applied to the elliptic homœoid discussed above. It is plain that we may take as $\sigma$ the value $\frac{1}{2}p\rho dk/k$, which gives us the result

$$-\frac{dV}{dn}=2\pi p\rho\frac{dk}{k}. \qquad (35)$$

But if $m$ denote the total mass of the homœoid

$$m=\frac{1}{2}\rho\frac{dk}{k}\int pds=2\pi\rho abck^{\frac{1}{2}}dk, \qquad (36)$$

since $\int pds$ is three times the volume of the homœoid, that is, $4\pi abck^{\frac{3}{2}}$. Thus we obtain $dk/k=m/2\pi\rho abck^{\frac{3}{2}}$ and

$$-\frac{dV}{dn}=\frac{m}{k^{\frac{3}{2}}}\frac{p}{abc}. \qquad (37)$$

* This is one of a set of theorems as to the uniqueness of solutions of potential problems. The proof will be found in treatises on Electricity or on Gravitational Attraction.

From (37) it follows that the work done against attraction in carrying a unit particle outwards through a small distance $dn$ along the normal is

$$-dV=\frac{1}{k^{\frac{3}{2}}}\frac{m}{abc}pdn. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

27. If then $dn$ be taken for different points of the surface so that $pdn$ is constant, there will be the same step of potential at every point, and the surface on which lie the extremities of these elements, $dn$, of the normals will be, like the surface of the homœoid, an equipotential or level surface. The distance between this surface and that of the homœoid is inversely proportional to $p$. It can in fact be shown very easily that the shell of space between the surfaces is a *focaloid*, to use the name given by Thomson and Tait to a space bounded by confocal ellipsoidal surfaces. For the equation of an ellipsoidal surface external to and near to the external surface of the homœoid $\Sigma(x^2/a^2)=k$, is

$$\Sigma\frac{x^2}{a^2+du}=k,$$

where $du$ is small, and $x$, $y$, $z$ are the coordinates of a point on the new surface. If for $x$, $y$, $z$ in the last equation we write $x+dx$, $y+dy$, $z+dz$, and subtract from the result $\Sigma(x^2/a^2)=k$, we obtain for the thickness $dn$ at $x$, $y$, $z$ of the shell of space

$$\frac{p}{k}\Sigma\left(\frac{xdx}{a^2}\right)=dn=\tfrac{1}{2}\frac{k}{p}du. \quad . \quad . \quad . \quad . \quad (39)$$

Thus the equipotential surface given by $dn$ thus chosen is confocal with the surface of the homœoid.

28. Let us now imagine the mass of the homœoidal shell carried out along the normal at each point, and distributed on the near confocal surface, so that the mass on any element $ds$ of the shell is placed on the element $ds'$ which is marked off by normals drawn from the periphery of $ds$. The shell thus formed will, by Green's principle of equivalent distributions, be a new homœoid which will give the same field external to itself as was produced by the original shell. For, take the tubular space marked out by normals drawn from the periphery of $ds$, and terminated by two caps, one just inside $ds$, but otherwise coinciding with it, the other outside $ds'$, and fitting closely to that element. The surface of this portion of space is the lateral surface, the inner cap, and the outer. Now take the integral of normal force $\int Nds$ over the whole surface of this space. The inner cap contributes nothing to it since there is no force within the homœoid,

the lateral surface also contributes nothing, the outer cap gives $F'ds'$ if $F'$ be the field-intensity there. The matter within the space is $\sigma ds$, and therefore by Green's (or Gauss's) theorem of the surface integral of normal force over a closed surface $F'ds' = -4\pi\sigma ds$. But if matter were distributed over $ds'$, and similarly over the rest of the surface of the confocal, so that the field remained unaltered, the surface integral over the short tube just described would still be $F'ds'$, and the matter on $ds'$ would now be $\sigma ds$. This distribution is unique for the given level surface, and the given field external to it. Since the surface S′ is ellipsoidal, and the potential is constant within it, the distribution upon it effected as described, by carrying the matter out from the initial homœoid, is also homœoidal ; otherwise the distribution over the surface producing uniformity of potential would not be, as it can be proved to be, unique.

It is easy to verify this latter point as to the nature of the distribution. The matter on $ds'$ is now $\sigma ds$, and for $\sigma$ we may write $\beta p$ where $\beta$ is a constant. Hence the new surface-density $\sigma' = \beta p ds/ds'$. But if $p'$ be the length of the perpendicular from the centre for $ds'$, and $a', b', c' = \sqrt{a^2+du}$, $\sqrt{b^2+du}$, $\sqrt{c^2+du}$, we have $p'ds' = pds \,.\, a'b'c'/abc$, so that $\sigma' = \beta p' abc/a'b'c'$, thus $\sigma'$ varies as $p'$, that is, the distribution is homœoidal.

We can now imagine a further step of potential taken from the surface S′ to a succeeding confocal and so on, until we have carried the whole distribution of matter to a surface, every point of which is at an infinite distance from the original surface. There the surface-density will be zero, and the potential at infinity, which has not been altered by the transference, will be zero. It is thus seen that the equipotential surfaces of the external field of the original homœoid are ellipsoids confocal with the original homœoid, a well-known result which has been otherwise established above.

The matter in the transference passes from confocal to confocal, so that the matter on an element of one is carried to the corresponding element on the next, and so on. Thus at any stage of the transference when the distribution is on a given confocal the matter which was originally on the element $ds$ of the original homœoid is situated on the element of the confocal which corresponds to $ds$. The transference is along the hyperbolas which are the orthogonal trajectories of the confocals, or, as they are often called, the lines of force of the field.

29. Another way of dealing with this problem of equivalent

distributions is to apply the theorem of Bertrand discussed in § 27. The initial distribution is an elliptic homœoid, and we know that the family of confocal surfaces surrounding it fulfil the condition stated in Bertrand's theorem. For upon any one of them a homœoidal distribution can be placed so as to produce a constant potential throughout its interior. Hence the confocals are the level surfaces of the field of the original homœoid. Hence also we can suppose the whole distribution carried out from confocal to confocal, so that at each instant the distribution is homœoidal on one of the level surfaces, and the path of each particle is along the line of force at the inner extremity of which it was originally situated.

30. We have found [§ 28, equation (38)] the step of potential from the initial surface to an adjacent one of which the equation is

$$\Sigma\frac{x^2}{a^2+du}=k$$

and it is proved in § 29 that $dn=\frac{1}{2}kdu/p$. Thus (38) becomes

$$-dV=\frac{1}{2\sqrt{k}}\frac{m}{abc}du$$

In the same way the step of potential from the level surface

$$\Sigma\frac{x^2}{a^2+u}=k$$

to the adjoining surface for which $u$ has been increased by $du$ is

$$-dV=\frac{1}{2\sqrt{k}}\frac{mdu}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad . \quad . \quad (40)$$

Integrating from $u=0$ to $u=\infty$, and observing that the integral must vanish at infinity, we have

$$V=\frac{m}{2\sqrt{k}}\int_0^\infty\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad . \quad (41)$$

If in this we insert the value of $m$ stated in (36) it becomes

$$V=\pi\rho abcdk\int_0^\infty\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad . \quad (42)$$

If in (42) we change $u$ to $u-u'$ we obtain

$$V=\pi\rho abcdk\int_{u'}^\infty\frac{du}{\sqrt{(a^2+u-u')(b^2+u-u')(c^2+u-u')}} \quad (43)$$

which is to be interpreted as the total step of potential involved in carrying the matter, supposed initially in a thin homœoid (mass $m$) of which the equation is $\Sigma\{x^2/(a^2-u')\}=k$ from the level surface $\Sigma x^2/a^2=k$ to infinity in the manner described. Thus matter $m$ distributed in a homœoid on the latter surface has the same potential at the surface and at all external points as the same matter had when in the original homœoid. This is for confocal homœoids what Maclaurin's theorem is for solid ellipsoids, and indeed Maclaurin's theorem flows from it at once since a solid ellipsoid may be supposed built up of a succession of homœoids. It is to be observed, however, that this theorem of equivalence of homœoids on confocal surfaces is only a particular case of Green's very general theorem of equivalence. Maclaurin's theorem and the analogous theorems for shells have been explained in § 22, and it is not necessary to pursue the subject here. In § 18 the extension for a shell to a solid ellipsoid has also been fully discussed.

31. The potential of a thin homœoid at a point on its surface or anywhere in the interior is given in (42). From this we can obtain the potential of a thick homœoid at a point within the interior hollow. We shall suppose that the equations of the outer and inner surfaces are respectively

$$\Sigma\frac{x^2}{a^2}=1, \quad \text{and} \quad \Sigma\frac{x^2}{a^2}=h.$$

We have therefore only to integrate (42) with regard to $k$ from $k=h$ to $k=1$. Thus for the thick homœoid

$$V=\pi\rho abc(1-h)\int_0^\infty\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad (44)$$

The potential produced by this homœoid at an external point may be found as follows: The squares of the semi-axes of the interior ellipsoidal hollow are $a^2h$, $b^2h$, $c^2h$, where $h<1$. The potential $V'$ at the external point $f, g, h$ (here $h$ is a coordinate), due to an ellipsoid of density $\rho$ filling the hollow is

$$V'=\pi\rho abch^{\frac{3}{2}}\int_{\lambda_1}^\infty\frac{\left(1-\Sigma\frac{f^2}{a^2h+u_1}\right)du_1}{\sqrt{(a^2h+u_1)(b^2h+u_1)(c^2h+u_1)}} \qquad (45)$$

where $\lambda_1$ is the positive root of the equation

$$\Sigma\frac{f^2}{a^2h+u_1}=1.$$

If in (45) we write $uh$ for $u_1$, we get

$$V'=\pi\rho abc\int_{\lambda'}^{\infty}\frac{\left(h-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad . \quad . \quad (46)$$

where $\lambda'=\lambda_1/h$. The potential at the external point $f$, $g$, $h$ due to the complete ellipsoid is given by (20) [§ 18]. Subtracting (46) from (20) we obtain for the thick homœoid

$$V=\pi\rho abc\left[\int_{\lambda}^{\lambda'}\frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\right.$$

$$\left.+(1-h)\int_{\lambda'}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\right]. \quad (47)$$

When the internal hollow is contracted to zero, that is when $h=0$, $\lambda'$ becomes $\infty$, and the second integral on the right vanishes. The equation then coincides with (20) as it ought to do.

32. We can now find the potential produced by a thick focaloid at an external or internal point. First for an external point: let two confocal ellipsoids have the same density, and let the shell between their surfaces be the focaloid to be considered. The potentials which they produce at a point external to both are in the ratio of the masses of the ellipsoids. For the potential of that which has $\Sigma(x^2/a^2)=1$ for the equation of its surface the equation is (20) [§ 18 above], and for the potential of the other which has, let us say, $\Sigma\{x^2/(a^2-s)\}=1$ for the equation of surface, the equation is the same with as multiplier of the integral $\pi\rho\sqrt{(a^2-s)(b^2-s)(c^2-s)}$ instead of $\pi\rho abc$. Thus for the potential of the focaloid at an external point $f$, $g$, $h$ we obtain

$$V=\tfrac{3}{4}m'\int_{\gamma}^{\infty}\frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad . \quad (48)$$

where $m'$ is put for the mass of the focaloid, that is

$$\tfrac{4}{3}\pi\rho\left\{abc-\sqrt{(a^2-s)(b^2-s)(c^2-s)}\right\}.$$

This equation is precisely the same as (20) which gives the potential at an external point for a solid ellipsoid: and just as in the case of ellipsoids it follows that :—any two uniform focaloids which are confocal with the same ellipsoid produce, at any point external to both, potentials which are in the ratio of the masses of the focaloids.

This theorem was practically given for points in the axes of an ellipsoid of revolution by Maclaurin. For he states [*Fluxions*, Art. 650], with the same limitations, the remarkable result that an ellipsoid made up of confocal shells each of uniform density, differing from shell to shell, and an ellipsoid of the same size and of uniform density, exert attractions on an external particle which are in the same direction, and have values in the same ratio as the masses of the ellipsoids.

The theorem becomes generalized by the extension of Maclaurin's theorem by Laplace to any form of ellipsoid and any external point. Obviously in the theorem just stated for a heterogeneous ellipsoid, the ellipsoids compared need not be of the same size but only confocal.

33. If the point $f$, $g$, $h$ considered be in the hollow within the focaloid, the potential can be found by subtracting from the expression for the potential at the point due to the complete ellipsoid, the potential at the same point due to the solid ellipsoid of the same density bounded by the surface $\Sigma\{x^2/(a^2-s)\}=1$, the internal surface of the focaloid. Making this calculation by (22), and putting in the result $m'$ (see § 32) for the mass of the focaloid, we get

$$V=\tfrac{3}{4}m'\int_0^\infty \frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$-\tfrac{3}{4}(m-m')\int_{-s}^0 \frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (49)$$

where $m$ is the mass of the complete ellipsoid.

In the remaining case, that in which the point $f$, $g$, $h$ is within the mass of the focaloid, the procedure is exactly the same as that just described. The form of the result is slightly different: it is

$$V=\tfrac{3}{4}m'\int_0^\infty \frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$-\tfrac{3}{4}(m-m')\int_{\lambda'-s}^0 \frac{\left(1-\Sigma\frac{f^2}{a^2+u}\right)du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (50)$$

where $\lambda'$ is the positive root of $\Sigma\{f^2/(a^2-s+u)\}=1$, regarded as an equation in $u$. When $\lambda'=0$, the point $f$, $g$, $h$ lies on the internal surface of the focaloid. Then (50) agrees with (49).

34. It is interesting to compare equations (42) and (49). The first gives the potential produced at any internal point by a thin elliptic homœoid of mass $m$, and (49) gives that of a focaloid of mass $m$ at an internal point $f$, $g$, $h$. If we put $f=g=h=0$, we get the potentials at the centre in the two cases. Let it be supposed that both shells are thin, and that both have the same external surface. The thickness of the homœoid being directly, and that of the focaloid inversely, as the length of the perpendicular let fall from the centre on the tangent plane at the point considered, the potential at the centre must be greater for the focaloid than for the homœoid. The first term on the right of (49) involves for the centre the same integral as does (41) for the homœoid, but this integral is multiplied by $\frac{3}{4}m'$ in (49) as against $\frac{1}{2}m$ in (41). The excess is diminished by the second term in (49) which varies with the deviation of the surface from sphericity; and also with the thickness of the focaloid on the whole.

In the particular case in which the surface is spherical the second term just makes the potential at the centre the same for a thin focaloid as for a thin homœoid, as the reader may verify by evaluation. If we take the case of a solid ellipsoid the second term in (49) vanishes, and the potential at the centre is $\frac{3}{2}$ of that which would be produced in the interior of a thin homœoid of the same mass and coincident with the surface of the ellipsoid.

35. From the expression given in (44) for the potential in the interior of a homœoid of any thickness, we can readily calculate the work done by gravitational attraction in bringing together from infinite dispersion in space the matter composing an ellipsoid, or a homœoid of finite thickness, whether uniform or made up of homœoidal shells of different densities. For the case of uniform density, let mass of amount $2\pi\rho abch^{\frac{1}{2}}dh$, be brought from infinity to the homœoid to which (44) refers, and be placed as an additional thin homœoid on the interior surface. The work done by gravitational attraction in bringing this matter into position is V$m$. Hence

$$\mathrm{V}m=2\pi^2\rho^2a^2b^2c^2h^{\frac{1}{2}}(1-h)dh\int_0^\infty\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (51)$$

If then W denote the whole work done in building up the homœoid by adding to the interior until $h$ is changed from 1 to H,

$$W = -2\pi^2\rho^2a^2b^2c^2\int_H^1 h^{\frac{1}{2}}(1-h)\,dh\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$= 2\pi^2\rho^2a^2b^2c^2\{\tfrac{4}{15}-\tfrac{2}{3}H^{\frac{3}{2}}(1-\tfrac{3}{5}H)\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (52)$$

If H=0, this becomes the result for a solid ellipsoid.

For this case

$$W = \tfrac{8}{15}\pi^2\rho^2a^2b^2c^2\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad (53)$$

In the particular case of a uniform sphere this becomes

$$W = \tfrac{3}{5}\frac{M^2}{a}$$

if the unit of mass is grvavitational (see § 6, footnote), or

$$W = \tfrac{3}{5}\kappa\frac{M^2}{a}$$

if the ordinary unit of mass is used and $\kappa$ is the proper value of the gravitational constant. This is the result given by Helmholtz, from which the rate of shrinking of the sun necessary to supply the energy radiated may be calculated.

If the density vary from shell to shell (53) becomes

$$W = 2\pi^2a^2b^2c^2\int_0^1 \rho\frac{dh}{h}\int_h^1 \rho dk\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (54)$$

as the reader may verify by finding the work done in building up a thick homœoid by adding shells of varying density $\rho$, and then varying the constant $h$ of this homœoid from 0 to 1.

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## XXXII. *On a New Principle of Relativity in Electromagnetism.* *By* A. H. Bucherer, *D.Sc.*, *Privatdocent in the Bonn University**.

§ 1. IT is needless to dwell on the serious difficulties which the Maxwellian theory has encountered by the well established experimental fact that terrestrial optics is not influenced by the earth's motion. The endeavours of some distinguished physicists, notably of H. A. Lorentz, to

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