ON THE DEGREE OF APPROXIMATION TO DISCONTINUOUS FUNCTIONS BY TRIGONOMETRIC SUMS.

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The object of this paper is to discuss the degree of accuracy with which functions of real variables having simple discontinuities can be represented by means of certain special approximating functions. The approximating functions used are the following: the sum of the first $n+1$ terms of the FOURIER's expansion of the given function; FEJÉR's arithmetic mean of the first n partial sums of this expansion; and finally a related trigonometric sum previously used by JACKSON [1]). Occasion is taken to introduce a theorem on approximation by means of polynomials of given degree.

The first part of the paper deals with functions of a single variable, while the second part gives some similar theorems with regard to functions of two variables. The generalization of these latter theorems to a greater number of variables is immediate. The theorems here given are merely representative of a large number that might be similarly obtained.

I.

Functions of a single variable.

By a trigonometric sum in x, of order n, is meant a sum of the form

$$\sum_{i=0}^{n} (a_i \cos i x + b_i \sin i x),$$

where, in particular, the coefficients of the terms of highest order may be zero.

THEOREM I. — *Let $f(x)$ be a function of x, of period 2π, which has in the interval from $-\pi$ to π no other discontinuities than a finite number of finite jumps, and in any closed interval not including a point of discontinuity satisfies a* LIPSCHITZ *condition,*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

[1]) See e. g. D. JACKSON, *On approximation by trigonometric sums and polynomials* [Transactions of the American Mathematical Society, vol. XIII (1912), pp. 491-515], p. 492.

where λ is a constant, and the same λ holds for all such intervals. Then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of order n, such that at any point x whose distance from the nearest point of discontinuity is at least δ, the function $f(x)$ is approximately represented by means of $T_n(x)$ with an error not exceeding

$$\frac{1}{n}\left(C_1\lambda + C_2\frac{\nu}{\delta}\right),$$

where C_1 and C_2 are absolute constants, and ν is the oscillation [2]) *of $f(x)$ in the interval $-\pi \leqq x \leqq \pi$.*

The truth of this statement in the case that $f(x)$ is everywhere continuous appears from a theorem of JACKSON [3]) which will be used below; it may be assumed, for the purposes of the proof, that $f(x)$ has at least one discontinuity, so that $\delta \leqq \pi$.

Let $\varphi(x)$ be the function whose graph is obtained by joining with straight lines the ends of each continuous part [4]) of the graph of $f(x)$. It will first be shown that there is a trigonometric sum $\tau_n(x)$ of order n such that

$$|\tau_n(x) - \varphi(x)| \leqq \frac{\pi\nu}{n\delta}.$$

The arithmetic mean of the first n partial sums of the FOURIER's expansion of $\varphi(x)$ is

$$S_n(x) = \frac{1}{2n\pi}\int_0^{2\pi}\varphi(\alpha)\left[\frac{\sin n\frac{\alpha - x}{2}}{\sin\frac{\alpha - x}{2}}\right]^2 d\alpha.$$

It is, of course, a trigonometric sum of order $n - 1$. By a simple change of variable, with attention to the periodicity of the functions involved, this can be written

$$S_n(x) = \frac{1}{n\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varphi(x + 2u)\left[\frac{\sin nu}{\sin u}\right]^2 du,$$

and if u is replaced by $-u$ in the negative part of the interval this becomes

$$S_n(x) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}}[\varphi(x + 2u) + \varphi(x - 2u)]\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

This relation holds not merely for the particular function φ which we are considering, but for all integrable functions $\varphi(x)$, and in particular for $\varphi(x) = 1$; from which it follows that

$$1 = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} 2\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

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[2]) By *oscillation* is meant the difference between the upper and lower limits of $f(x)$ in the interval.

[3]) D. JACKSON, loc. cit. [1]), p. 492.

[4]) The definition adopted for $\varphi(x)$ at the points of discontinuity of $f(x)$ is immaterial.

This identity can be multiplied through by $\varphi(x)$, whence

$$\varphi(x) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} 2\varphi(x)\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

So the difference $S_n(x) - \varphi(x)$ may be written

$$S_n(x) - \varphi(x) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} [\varphi(x+2u) + \varphi(x-2u) - 2\varphi(x)]\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

Now in the interval $-\frac{1}{2}\delta < u < \frac{1}{2}\delta$, the function $\varphi(x+2u)$ is a linear function of u, and so the first factor of the integrand is zero, and the above relation reduces to

$$S_n(x) - \varphi(x) = \frac{1}{n\pi}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} [\varphi(x+2u) + \varphi(x-2u) - 2\varphi(x)]\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

Taking absolute values,

$$|S_n(x) - \varphi(x)| \leqq \frac{1}{n\pi}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} |\varphi(x+2u) + \varphi(x-2u) - 2\varphi(x)|\left[\frac{\sin nu}{\sin u}\right]^2 du$$

$$\leqq \frac{1}{n\pi}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} [|\varphi(x+2u) - \varphi(x)| + |\varphi(x-2u) - \varphi(x)|]\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

The oscillation of $\varphi(x)$ in the interval $-\pi \leqq x \leqq \pi$ is not greater than that of $f(x)$, from which it follows that

$$|\varphi(x+2u) - \varphi(x)| + |\varphi(x-2u) - \varphi(x)| \leqq 2\nu.$$

Moreover, for the interval $0 \leqq u \leqq \frac{1}{2}\pi$,

$$\sin u \geqq \frac{2}{\pi} u.$$

From these inequalities, it follows that

$$|S_n(x) - \varphi(x)| \leqq \frac{2\nu}{n\pi}\left(\frac{\pi}{2}\right)^2 \int_{\frac{\delta}{2}}^{\frac{\pi}{2}} \frac{\sin^2 nu}{u^2} du \leqq \frac{\pi\nu}{2n}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} \frac{du}{u^2}$$

$$= \frac{\pi\nu}{2n}\left(\frac{2}{\delta} - \frac{2}{\pi}\right) \leqq \frac{\pi\nu}{n\delta}.$$

If we set $\tau_n(x) = S_{n+1}(x)$, then $\tau_n(x)$ will be a trigonometric sum of order n, for which

$$|\varphi(x) - \tau_n(x)| \leqq \frac{\pi\nu}{(n+1)\delta} \leqq \frac{\pi\nu}{n\delta}.$$

The difference $f(x) - \varphi(x) = F(x)$, which is a continuous function, if suitably defined at the points of discontinuity of $f(x)$, satisfies a Lipschitz condition everywhere. For if x_1 and x_2 are any two points such that $f(x)$ is continuous for $x_1 \leqq x \leqq x_2$, then the inequality

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

is satisfied, and from the definition of $\varphi(x)$ also the inequality

$$|\varphi(x_2) - \varphi(x_1)| \leqq \lambda |x_2 - x_1|,$$

with the same value of λ. Hence

$$\begin{aligned} |F(x_2) - F(x_1)| &= |f(x_2) - \varphi(x_2) - f(x_1) + \varphi(x_1)| \\ &\leqq |f(x_2) - f(x_1)| + |\varphi(x_2) - \varphi(x_1)| \\ &\leqq 2\lambda |x_2 - x_1|. \end{aligned}$$

If the interval (x_1, x_2) contains points of discontinuity of $f(x)$, it is readily shown by consideration of the partial intervals into which these points divide (x_1, x_2) that the inequality just obtained is still true.

Jackson [5]) has proved that if $f(x)$ is any continuous function of period 2π, satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

then there exists for every positive integral value of n a trigonometric sum which we shall call $t_n(x)$, of order n, such that the inequality

$$|f(x) - t_n(x)| \leqq \frac{K_1 \lambda}{n}$$

is satisfied for all values of x; here K_1 is an absolute constant.

By this theorem, there exists a function $t_n(x)$ such that

$$|t_n(x) - [f(x) - \varphi(x)]| \leqq \frac{2 K_1 \lambda}{n},$$

and consequently

$$|t_n(x) - [f(x) - \tau_n(x)]| \leqq \frac{2 K_1 \lambda}{n} + \frac{\pi \nu}{n \delta}.$$

Here $t_n(x) + \tau_n(x) = T_n(x)$ is a trigonometric sum of order n, and the relation

$$|f(x) - T_n(x)| \leqq \frac{1}{n}\left(2 K_1 \lambda + \frac{\pi \nu}{\delta}\right)$$

establishes the theorem.

It might be pointed out that if we set $\delta = \frac{1}{\sqrt{n}}$, this theorem, supplemented by a brief consideration of the behavior of the approximating functions in the neighborhood of the points of discontinuity of $f(x)$, shows that the graph of $T_n(x)$ lies within a strip, the width of which is of the order of $\frac{1}{\sqrt{n}}$, about the curve which is formed if the ends of the continuous parts of the graph of $f(x)$ are connected by vertical lines.

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[5]) D. Jackson, loc. cit. [1]), p. 492.

From Theorem I can also be obtained the following theorem with regard to approximation by means of polynomials of given degree.

THEOREM II. — *Let $f(x)$ be any function of x, which in a closed interval of length l has no other discontinuities than a finite number of finite jumps, and in any closed sub-interval not including a point of discontinuity satisfies the* LIPSCHITZ *condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|.$$

Then there exists for every positive integral value of n a polynomial $\Pi_n(x)$, of degree [6]) *n, which approximates to $f(x)$, at any point x whose distance from the nearest point of discontinuity is at least as great as δ, with an error not exceeding*

$$\frac{l}{2n}\left(C_1 \lambda + C_2 \frac{\nu}{\delta}\right),$$

where C_1 and C_2 are the constants of Theorem I, and ν is the oscillation of $f(x)$ in the interval.

If the end points of the interval are d_1 and d_2, the transformation

$$x = \frac{(d_2 - d_1)x' + (d_2 + d_1)}{2}$$

carries $f(x)$ into a function $g(x')$ defined in the interval $-1 \leqq x' \leqq 1$, and for any of the continuous portions of $g(x')$

$$|g(x'_2) - g(x'_1)| \leqq \lambda \left| \frac{(d_2 - d_1)x'_2 + d_2 + d_1}{2} - \frac{(d_2 - d_1)x'_1 + d_2 + d_1}{2} \right| = \frac{\lambda l}{2} |x'_2 - x'_1|.$$

The function $g(\cos y) = h(y)$ is a function of period 2π in y, and

$$|h(y_2) - h(y_1)| \leqq \frac{\lambda l}{2} |\cos y_2 - \cos y_1| \leqq \frac{\lambda l}{2} |y_2 - y_1|.$$

Hence Theorem I can be applied to $h(y)$, and it follows that the trigonometric sum $T_n(y)$ of order n exists such that

$$|h(y) - T_n(y)| \leqq \frac{1}{n}\left(C_1 \frac{\lambda l}{2} + C_2 \frac{\nu'}{\delta'}\right),$$

where ν' is the oscillation of $h(y)$ and δ' is not greater than the distance from the point y to the nearest point of discontinuity of $h(y)$.

Since $h(y)$ is an even function we may assume that $T_n(y)$ is also even [7]). Then $T_n(y)$ is a polynomial of degree n in $\cos y = x'$, and so it is also a polynomial of degree n in x, which can be called $\Pi_n(x)$, and

$$|f(x) - \Pi_n(x)| \leqq \frac{1}{n}\left(C_1 \frac{\lambda l}{2} + C_2 \frac{\nu'}{\delta'}\right).$$

The letters ν' and δ' refer, of course, to the function $h(y)$, and should be replaced by ν and δ, which refer to the function $f(x)$. Now ν and ν' are the same, and δ

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[6]) This language is not intended to imply that the coefficient of x^n is necessarily different from zero.

[7]) See e. g. D. JACKSON, loc. cit. [1]), p. 494. His assumption that the function represented is continuous, is obviously unnecessary.

may be taken so that $\frac{2}{l}\delta \leqq \delta'$. To recognize the latter fact, let a be the point of discontinuity of $f(x)$ nearest to the point x under consideration, and let $a = \cos b$. Then

$$|a - x| = \frac{l}{2}|a' - x'| = \frac{l}{2}|\cos b - \cos y| \leqq \frac{l}{2}|b - y|.$$

Hence

$$|f(x) - \Pi_n(x)| \leqq \frac{l}{2n}\left(C_1 \lambda + C_2 \frac{\nu}{\delta}\right).$$

Other theorems with regard to approximation by means of polynomials of given degree can be deduced from the theorems that follow, but this one is sufficient to give an idea of their nature.

A theorem similar to I, but applying to a far more general class of functions, is the following, in which it is to be noted that since δ is raised to the third power in the expression for the limit of error, this theorem does not include Theorem I as a special case.

THEOREM III. — *Let $f(x)$ be a function of x, of period 2π, finite and integrable in the interval $-\pi \leqq x \leqq \pi$. Then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of order n, such that for any point x, lying in the middle of a closed interval of length $2\delta \leqq 2\pi$, in which $f(x)$ satisfies a* LIPSCHITZ *condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

the difference between $f(x)$ and $T_n(x)$ is less than or at most equal to

$$\frac{1}{n}\left(C_3 \lambda + C_4 \frac{\nu}{n^2 \delta^3}\right),$$

where C_3 and C_4 are absolute constants, and ν is the oscillation of $f(x)$ in the interval $-\pi \leqq x \leqq \pi$.

The λ may be different for different intervals, and the smaller the value of λ for the neighborhood of a point, the better is the approximation at that point. It is sufficient for the proof of this theorem if the condition imposed on the function is satisfied when one of the points x_1, x_2, coincides with the point x at which the approximation is to be measured. This fact is perhaps brought out more clearly in Theorem VI, which is based on essentially the same reasoning as this one.

To prove this theorem we use JACKSON's approximating function [8])

$$(1) \qquad I_m(x) = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + 2u)\left[\frac{\sin mu}{m \sin u}\right]^4 du}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\sin mu}{m \sin u}\right]^4 du},$$

[8]) D. JACKSON, loc. cit. [1]), p. 492.

which is a trigonometric sum of order $2(m-1)$. Using the same device as in the proof of Theorem I, the difference between $f(x)$ and $I_m(x)$ can be written as a similar quotient, in which $f(x+2u)$ under the sign of integration is replaced by $[f(x+2u)-f(x)]$. If in the negative part of the interval u is replaced by $-u$, this may be written

$$I_m(x)-f(x)=\frac{\int_0^{\frac{\pi}{2}}[f(x+2u)+f(x-2u)-2f(x)]\left[\frac{\sin mu}{m\sin u}\right]^4 du}{2\int_0^{\frac{\pi}{2}}\left[\frac{\sin mu}{m\sin u}\right]^4 du}.$$

The integral in the denominator satisfies the inequalities

$$(2)\qquad \int_0^{\frac{\pi}{2}}\left[\frac{\sin mu}{m\sin u}\right]^4 du>\int_0^{\frac{\pi}{2}}\left[\frac{\sin mu}{mu}\right]^4 du=\frac{1}{m}\int_0^{\frac{m\pi}{2}}\frac{\sin^4 u}{u^4}du\geqq\frac{1}{m}\int_0^{\frac{\pi}{2}}\frac{\sin^4 u}{u^4}du,$$

and the last integral is an absolute constant, K_2.

The numerator integral can be divided up into the sum of the integrals from 0 to $\frac{1}{2}\delta$ and from $\frac{1}{2}\delta$ to $\frac{1}{2}\pi$. It the first of these intervals,

$$|f(x+2u)+f(x-2u)-2f(x)|\leqq 4\lambda u,$$

because of the LIPSCHITZ condition. In the other the following holds:

$$|f(x+2u)+f(x-2u)-2f(x)|\leqq 2\nu.$$

Consequently

$$|I_m(x)-f(x)|\leqq\frac{m}{2K_2}\left[4\lambda\int_0^{\frac{\delta}{2}}u\left[\frac{\sin mu}{m\sin u}\right]^4 du+2\nu\int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\left[\frac{\sin mu}{m\sin u}\right]^4 du\right].$$

The integral over the interval $(0, \frac{1}{2}\delta)$ is certainly less than the integral of the same integrand from 0 to $\frac{1}{2}\pi$. Moreover, in the interval $0\leqq u\leqq\frac{1}{2}\pi$,

$$\sin u\geqq\frac{2}{\pi}u,$$

so that

$$(3)\qquad \begin{cases}\int_0^{\frac{\pi}{2}}u\left[\frac{\sin mu}{m\sin u}\right]^4 du<\left(\frac{\pi}{2}\right)^4\int_0^{\frac{\pi}{2}}u\left[\frac{\sin mu}{mu}\right]^4 du\\ =\frac{1}{m^2}\left(\frac{\pi}{2}\right)^4\int_0^{\frac{m\pi}{2}}\frac{\sin^4 u}{u^3}du<\frac{1}{m^2}\left(\frac{\pi}{2}\right)^4\int_0^{\infty}\frac{\sin^4 u}{u^3}du,\end{cases}$$

and here also the last integral is an absolute constant. In the other integral the inequalities are somewhat different,

$$(4)\qquad \begin{cases}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\left[\frac{\sin mu}{m\sin u}\right]^4 du\leqq\left(\frac{\pi}{2}\right)^4\int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\left[\frac{\sin mu}{mu}\right]^4 du\leqq\left(\frac{\pi}{2}\right)^4\frac{1}{m^4}\int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\frac{du}{u^4}\\ =\frac{1}{3m^4}\left(\frac{\pi}{2}\right)^4\left[\left(\frac{2}{\delta}\right)^3-\left(\frac{2}{\pi}\right)^3\right]<\frac{1}{3m^4}\left(\frac{\pi}{2}\right)^4\left(\frac{2}{\delta}\right)^3.\end{cases}$$

To sum up the result of these inequalities:

$$|I_m(x) - f(x)| \leqq \frac{m}{2K_2}\left[4\lambda\frac{K_3}{m^2} + 2\nu\frac{K_4}{m^4\delta^3}\right],$$

in which the K's are absolute constants. From this is at once deduced the form used in the statement of the theorem [9]).

If the arithmetic mean of the first n partial sums of the FOURIER's expansion of a function is used as the approximating function, the following theorem is obtained:

THEOREM IV. — *Let $f(x)$ be a finite and integrable function of x, of period 2π. Then if x is any point lying in the middle of an interval of length $2\delta \leqq 2\pi$ in which $f(x)$ satisfies a* LIPSCHITZ *condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

the difference at this point between $f(x)$ and the arithmetic mean $S_n(x)$ of the first n partial sums of its FOURIER's *expansion, $n > 1$, is not greater than* [10])

$$\frac{\log n}{n}\left(C_5\lambda + C_6\frac{\nu}{\delta\log n}\right),$$

where C_5 and C_6 are absolute constants, and ν is the oscillation of $f(x)$ in the interval $-\pi \leqq x \leqq \pi$.

The difference can be formulated as in the proof of Theorem I:

$$S_n(x) - f(x) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} [f(x+2u) + f(x-2u) - 2f(x)]\left[\frac{\sin nu}{\sin u}\right]^2 du.$$

It is again convenient to divide the interval of integration into two parts, the interval from 0 to $\frac{1}{2}\delta$, and that from $\frac{1}{2}\delta$ to $\frac{1}{2}\pi$. By the LIPSCHITZ condition and the definition of ν,

$$|S_n(x) - f(x)| \leqq \frac{1}{n\pi}\left[4\lambda\int_0^{\frac{\delta}{2}} u\left[\frac{\sin nu}{\sin u}\right]^2 du + 2\nu\int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\left[\frac{\sin nu}{\sin u}\right]^2 du\right].$$

Since the integrand is positive, we can replace the first integral by the integral from 0 to $\frac{1}{2}\pi$, and for this we have

$$\int_0^{\frac{\pi}{2}} u\left[\frac{\sin nu}{\sin u}\right]^2 du < \left(\frac{\pi}{2}\right)^2\int_0^{\frac{\pi}{2}}\frac{\sin^2 nu}{u}du = \left(\frac{\pi}{2}\right)^2\int_0^{\frac{n\pi}{2}}\frac{\sin^2 u}{u}du$$

$$< \left(\frac{\pi}{2}\right)^2\left[\int_0^1 du + \int_1^{\frac{n\pi}{2}}\frac{du}{u}\right] = \left(\frac{\pi}{2}\right)^2\left[1 + \log\frac{n\pi}{2}\right]$$

$$\leqq K_5 \log n \qquad (n > 1),$$

9) The fact that the order of the trigonometric sum $I_m(x)$ is not m, but $2(m-1)$, is obviously inessential.

10) For the case that $f(x)$ is everywhere continuous, see S. BERNSTEIN, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné* [Mémoires couronnés et autres mémoires publiés par l'Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique (Bruxelles), Series II, Vol. IV (1912), fasc. I, pp. 1-104], pp. 88, 89.

where K_5 is an absolute constant. The integral from $\frac{1}{2}\delta$ to $\frac{1}{2}\pi$, as in the proof of Theorem I, is less than or equal to $\frac{\pi^2}{2\delta}$, and therefore

$$|S_n(x) - f(x)| \leqq \frac{1}{n\pi}\left[4\lambda K_5 \log n + 2\nu\frac{\pi^2}{2\delta}\right] \qquad (n > 1),$$

which proves the theorem.

The next theorem, which deals with the FOURIER's expansion itself, is restricted to the same class of functions as are dealt with in Theorem I.

THEOREM V. — *Let $f(x)$ be any function of x, of period 2π, which in the interval $-\pi \leqq x \leqq \pi$ is continuous except for μ finite jumps, and in any closed interval not including a point of discontinuity satisfies the* LIPSCHITZ *condition*

$$|f(x_2) - f(x_1)| \leqq \lambda|x_2 - x_1|.$$

At any point x whose distance from the nearest point of discontinuity is greater than or equal to δ, the difference between $f(x)$ and the sum of the first $n+1$ terms of its FOURIER's *expansion, $n > 1$, is not greater than*

$$\frac{\log n}{n}\left(C_7\lambda + C_8\frac{\mu\nu}{\delta\log n}\right),$$

where C_7 and C_8 are absolute constants, and ν is the oscillation of $f(x)$ in the interval $-\pi \leqq x \leqq \pi$.

If the end points of the interval are points of discontinuity, we need count only one of them in estimating μ.

Let us first consider the function $\psi(x)$, of period 2π, which is equal to $\frac{1}{2}(\pi - x)$ in the interval $0 < x < 2\pi$. The remainder after the first $n+1$ terms of its FOURIER's expansion may be written

$$R_n(x) = \frac{\pi}{2} - \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}dx$$

$$= \frac{\pi}{2} - \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin u}{u}du + J_n(x),$$

where

$$|J_n(x)| < \frac{2M}{n+\frac{1}{2}} \qquad (0 < x \leqq \pi),$$

and M is an absolute constant [11]). Since the following is an alternating series with terms decreasing in absolute value:

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin u}{u}du = u_0 + u_1 + \cdots,$$

$$u_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin u}{u}du,$$

[11]) For proof of these statements see M. BÔCHER, *Introduction to the Theory of* FOURIER's *Series* [Annals of Mathematics, vol. VII (1906), pp. 81-152], pp. 123 ff.

the remainder after k terms is less in absolute value than the $(k+1)$th term. If then k is determined so that the point $\left(n+\frac{1}{2}\right)x$ lies in the interval from $k\pi$ to $(k+1)\pi$, we have

$$\begin{aligned}\left|\frac{\pi}{2}-\int_0^{\left(n+\frac{1}{2}\right)x}\frac{\sin u}{u}du\right| &= \left|\int_{\left(n+\frac{1}{2}\right)x}^{(k+1)\pi}\frac{\sin u}{u}du+u_{k+1}+u_{k+2}+\cdots\right.\\ &< \left|\int_{\left(n+\frac{1}{2}\right)x}^{(k+1)\pi}\frac{\sin u}{u}du\right|+|u_{k+1}|\\ &\leqq \int_{\left(n+\frac{1}{2}\right)x}^{\left(n+\frac{1}{2}\right)x+\pi}\left|\frac{\sin u}{u}\right|du+|u_{k+1}|\\ &< 2\int_{\left(n+\frac{1}{2}\right)x}^{\left(n+\frac{1}{2}\right)x+\pi}\frac{du}{u}<2\int_{\left(n+\frac{1}{2}\right)x}^{\left(n+\frac{1}{2}\right)x+\pi}\frac{du}{nx}=\frac{2\pi}{nx}.\end{aligned}$$

Hence

$$|R_n(x)|\leqq\frac{2\pi}{nx}+\frac{2M}{n}\leqq\frac{C_8\pi}{nx} \qquad (0<x\leqq\pi),$$

where C_8 is an absolute constant. Because of the periodicity of $\psi(x)$ and the fact that it is an odd function, a corresponding inequality holds at points of continuity outside the interval $(0, \pi)$.

The function $\frac{\sigma}{\pi}\psi(x-a)$, where σ is a constant, has a finite jump of magnitude σ at the point a. The remainder in its FOURIER's series, at a point x distant not less than δ from the nearest point of discontinuity, can not exceed $\frac{C_8\sigma}{n\delta}$. If

$$F(x)=\frac{\sigma_1}{\pi}\psi(x-a_1)+\frac{\sigma_2}{\pi}\psi(x-a_2)+\cdots+\frac{\sigma_\mu}{\pi}\psi(x-a_\mu),$$

the function $F(x)$ is approximately represented with an error not exceeding the sum of the errors corresponding to the various terms, i. e., with an error not greater than

$$\frac{C_8\sigma_1}{n\delta_1}+\frac{C_8\sigma_2}{n\delta_2}+\cdots+\frac{C_8\sigma_\mu}{n\delta_\mu},$$

where δ_i is the distance from the point x to the corresponding discontinuity. Each of these quantities can be replaced by the smallest of them, δ; and each σ can be replaced by ν, the oscillation of $F(x)$. So the error is not greater than

$$C_8\frac{\mu\nu}{n\delta}.$$

Now let $\varphi(x)$ be the function whose graph is obtained by joining by straight lines the ends of each continuous part of the graph of $f(x)$. Then $\varphi(x)$ is except for an additive constant a function of the type of $F(x)$. The difference $\Omega(x)=f(x)-\varphi(x)$

is a continuous function satisfying the LIPSCHITZ condition [12])

$$|\Omega(x_2) - \Omega(x_1)| \leqq 2\lambda |x_2 - x_1|$$

everywhere. If then $s_{n1}(x)$ is the sum of the first $n + 1$ terms of its FOURIER's development [13])

$$|s_{n1}(x) - \Omega(x)| \leqq \frac{C_7 \lambda \log n}{n} \qquad (n > 1),$$

where C_7 is an absolute constant. Let $s_{n2}(x)$ be the sum of the first $n + 1$ terms of the FOURIER's expansion for $\varphi(x)$. By the work just performed,

$$|s_{n2}(x) - \varphi(x)| \leqq \frac{C_8 \mu \nu}{n \delta},$$

and we can consider ν to be the oscillation of $f(x)$, since that is not less than the oscillation of $\varphi(x)$. Combining these inequalities,

$$|s_n(x) - f(x)| \leqq \frac{\log n}{n}\left(C_7 \lambda + C_8 \frac{\mu \nu}{\delta \log n}\right) \qquad (n > 1).$$

In the following theorems the constants involved are no longer absolute, but will depend on the point x and on the function in a way not defined; the essential thing is that they are independent of n.

THEOREM VI. — *Let $f(x)$ be any finite and integrable function of x, of period 2π. Then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of order n, such that for any point x at which $f(x)$ has four finite derived numbers,*

$$|f(x) - T_n(x)| \leqq \frac{c_1}{n},$$

where c_1 is a constant independent of n.

We use the approximating function (1), and by the aid of (2) we have the relation

$$|I_m(x) - f(x)| \leqq \frac{m}{2K_2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(x + 2u) - f(x)| \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

Since $f(x)$ remains finite there is some constant G such that $|f(x)| \leqq G$ for all values of x, hence $|f(x + 2u)| \leqq G$, and

$$\left|\frac{f(x + 2u) - f(x)}{2u}\right| \leqq \frac{G}{u}$$

for $u \neq 0$. Let N be the greatest of the absolute values of the derived numbers at a particular point x. Then if a small quantity $\varepsilon > 0$ is given, $\delta > 0$ can be determined

[12]) For proof of this fact see the corresponding passage in the proof of Theorem I.

[13]) See H. LEBESGUE, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de* LIPSCHITZ [Bulletin de la Société Mathématique de France, vol. XXXVIII (1910), pp. 184-210], p. 201; D. JACKSON, loc. cit. [1]), p. 502.

such that when $|2u| \leqq \delta$,

$$\left|\frac{f(x+2u)-f(x)}{2u}\right| < N + \varepsilon.$$

Let k_1 be the greater of the numbers $N + \varepsilon$ and $\frac{2G}{\delta}$. Then for all values of $2u$,

$$|f(x+2u)-f(x)| \leqq 2k_1 u.$$

Hence

$$|I_m(x) - f(x)| \leqq \frac{k_1 m}{K_2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |u| \left[\frac{\sin mu}{m \sin u}\right]^4 du,$$

and from the relations (3)

$$|I_m(x) - f(x)| \leqq \frac{k_2}{m},$$

where k_2 is a constant independent of m. The truth of the theorem is an immediate consequence.

The proofs of the next three theorems are so similar to the one just given, that the mere statement of the theorems seems all that is necessary.

THEOREM VII. — *Let $f(x)$ be any finite and integrable function of x, of period 2π. Then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of order n, such that if x is a point for which the quotient*

$$\frac{f(x+2u) + f(x-2u) - 2f(x)}{u^2}$$

remains finite,

$$|f(x) - T_n(x)| \leqq \frac{c_2}{n^2},$$

where c_2 is a constant independent of n.

It is to be noted that in this case the function $f(x)$ itself may be discontinuous at the point x.

THEOREM VIII. — *Let $f(x)$ be any finite and integrable function of x, of period 2π. Then if x is any point at which $f(x)$ has four finite derived numbers, the order of the approximation to $f(x)$ given by the arithmetic mean of the first n partial sums of its* FOURIER'*s expansion is subject to the inequality*

$$|S_n(x) - f(x)| \leqq \frac{c_3 \log n}{n},$$

where c_3 is a constant independent of n.

THEOREM IX. — *Let $f(x)$ be a finite and integrable function of x, of period 2π. Then it is approximately represented by the arithmetic mean of the first n partial sums of its* FOURIER'*s expansion at any point x of the sort described in Theorem VII, with an error not exceeding $\frac{c_4}{n}$, where c_4 is a constant independent of n.*

II.

Functions of two variables.

Several of the preceding theorems, together with the simpler theorems referring to continuous functions, can be generalized to functions of two variables. The approximating functions used are the analogues of those used with a single variable.

By a trigonometric sum of order m in x and n in v is meant a sum of the form

$$T_{mn}(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n}(a_{ij}\cos ix\cos jy + b_{ij}\cos ix\sin jy + c_{ij}\sin ix\cos jy + d_{ij}\sin ix\sin jy).$$

The partial sum of the double FOURIER's series for a function $f(x, y)$, of period 2π in each variable, to terms of order m in x and n in y, can be written

$$s_{mn}(x, y) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\alpha, \beta)\frac{\sin(2m+1)\left(\frac{\alpha-x}{2}\right)}{\sin\left(\frac{\alpha-x}{2}\right)}\frac{\sin(2n+1)\left(\frac{\beta-y}{2}\right)}{\sin\left(\frac{\beta-y}{2}\right)}\,d\alpha\, d\beta.$$

The order of approximation to a given continuous function $f(x, y)$ by means of this function s_{mn} has been investigated by BERNSTEIN [14]).

An arithmetic mean of these partial sums, $S_{mn}(x, y)$, is given by the formula

$$(5)\quad \begin{cases} S_{mn}(x, y) = \dfrac{1}{mn}\displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s_{ij}(x, y) \\ \quad = \dfrac{1}{4mn\pi^2}\displaystyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\alpha, \beta)\left[\dfrac{\sin m\left(\frac{\alpha-x}{2}\right)}{\sin\left(\frac{\alpha-x}{2}\right)}\right]^2\left[\dfrac{\sin n\left(\frac{\beta-y}{2}\right)}{\sin\left(\frac{\beta-y}{2}\right)}\right]^2 d\alpha\, d\beta. \end{cases}$$

By a simple change of variable this can be written in the form

$$S_{mn}(x, y) = \frac{1}{4mn\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x+2u, y+2v)\left[\frac{\sin mu}{\sin u}\right]^2\left[\frac{\sin nv}{\sin v}\right]^2 du\, dv.$$

This approximating function is used in Theorems XI and XIII.

The LIPSCHITZ condition in the case of two variables may be written

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

This form brings out the fact that the condition on the function at two points depends merely on their distance apart. It is equivalent to the condition

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda(|x_2-x_1| + |y_2-y_1|),$$

except possibly for a difference in the value of λ.

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[14]) S. BERNSTEIN, loc. cit. [10]), pp. 98-100.

THEOREM X. — *Let $f(x, y)$ be any function of x and y having the period 2π in each variable, and satisfying the* LIPSCHITZ *condition*

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then there exists for every pair of positive integral numbers m, n, a trigonometric sum $T_{mn}(x, y)$, of order m in x and n in y, such that

$$|f(x, y) - T_{mn}(x, y)| \leqq C_9 \lambda \left(\frac{1}{m} + \frac{1}{n} \right),$$

where C_9 is an absolute constant.

Form the approximating function

$$I_{mn}(x, y) = h_{mn} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + 2u, y + 2v) \left[\frac{\sin m u}{m \sin u} \right]^4 \left[\frac{\sin n v}{n \sin v} \right]^4 du\, dv,$$

where

$$\frac{1}{h_{mn}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\sin m u}{m \sin u} \right]^4 \left[\frac{\sin n v}{n \sin v} \right]^4 du\, dv.$$

If the transformation $\alpha = x + 2u$, $\beta = y + 2v$ is made, and the resulting expression for $I_{mn}(x, y)$ is expanded, it will be found that it is a trigonometric sum of order $2(m - 1)$ in x and $2(n - 1)$ in y. The difference $I_{mn}(x, y) - f(x, y)$ is equal to

$$h_{mn} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [f(x + 2u, y + 2v) - f(x, y)] \left[\frac{\sin m u}{m \sin u} \right]^4 \left[\frac{\sin n v}{n \sin v} \right]^4 du\, dv.$$

Inasmuch as the double integral for the reciprocal of h_{mn} breaks up at once into a product of two simple integrals, it follows from formula (2) that

$$h_{mn} < \frac{m n}{4 K_2^2}.$$

And from the LIPSCHITZ condition it follows that

$$|f(x + 2u, y + 2v) - f(x, y)| \leqq \lambda \sqrt{(2u)^2 + (2v)^2} \leqq \lambda (|2u| + |2v|),$$

so that

$$|I_{mn}(x, y) - f(x, y)| \leqq \frac{m n}{2 K_2^2} \lambda \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (|u| + |v|) \left[\frac{\sin m u}{m \sin u} \right]^4 \left[\frac{\sin n v}{n \sin v} \right]^4 du\, dv.$$

This may be treated as was the corresponding integral in the proof of Theorem III, thus obtaining

$$|I_{mn}(x, y) - f(x, y)| \leqq \frac{\lambda m n}{2 K_2^2} \left(\frac{2 K_3 K_6}{m^2 n} + \frac{2 K_3 K_6}{n^2 m} \right),$$

where K_6 is a new absolute constant; hence follows the theorem.

THEOREM XI. — *Let $f(x, y)$ be any function of period 2π in each variable, which satisfies the* LIPSCHITZ *condition*

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then if $S_{mn}(x, y)$ is the mean of the partial sums of the FOURIER'S *expansion of $f(x, y)$ to those of order $m-1$ in x and $n-1$ in y, the following inequality holds for all values of x and y:*

$$|S_{mn}(x, y) - f(x, y)| \leqq C_{10} \lambda \left(\frac{\log m}{m} + \frac{\log n}{n} \right),$$

where C_{10} is an absolute constant.

The proof of this theorem is deduced from the inequalities used in the proof of Theorem IV, in precisely the same way as the proof of the preceding theorem is deduced from those used in proving Theorem III.

THEOREM XII. — *Let $f(x, y)$ be any finite and integrable function of period 2π in each variable. Then there exists for every pair of positive integers m, n, a trigonometric sum $T_{mn}(x, y)$, of order m in x and n in y, such that if (x, y) is any point having a neighborhood of radius $\delta \leqq \pi$ in which $f(x, y)$ satisfies the* LIPSCHITZ *condition*

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

then for this point

$$|T_{mn}(x, y) - f(x, y)| \leqq \left(\frac{1}{m} + \frac{1}{n} \right) C_{11} \lambda + \left(\frac{1}{m^3} + \frac{1}{n^3} \right) C_{12} \frac{\nu}{\delta^3},$$

where ν is the oscillation of $f(x, y)$ in the square $-\pi \leqq x \leqq \pi$, $-\pi \leqq y \leqq \pi$, and C_{11} and C_{12} are absolute constants.

The same approximating function is used as in the proof of Theorem X. So we have

$$|I_{mn}(x, y) - f(x, y)|$$
$$\leqq \frac{mn}{4K_2^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(x + 2u, y + 2v) - f(x, y)| \left[\frac{\sin mu}{m \sin u} \right]^4 \left[\frac{\sin nv}{n \sin v} \right]^4 du\, dv.$$

Let this integral be expressed as the sum of the integral over a circle C of radius $\frac{1}{2}\delta$ about the origin and the integral over the rest of the square. Let us denote the square by S, and by R the part of the square that remains after the removal of C. In the circle the LIPSCHITZ condition is satisfied, and so

$$\left| \int\!\!\int_C [f(x + 2u, y + 2v) - f(x, y)] \left[\frac{\sin mu}{m \sin u} \right]^4 \left[\frac{\sin nv}{n \sin v} \right]^4 du\, dv \right|$$
$$\leqq 2\lambda \int\!\!\int_C (|u| + |v|) \left[\frac{\sin mu}{m \sin u} \right]^4 \left[\frac{\sin nv}{n \sin u} \right]^4 du\, dv$$
$$\leqq 2\lambda \int\!\!\int_S (|u| + |v|) \left[\frac{\sin mu}{m \sin u} \right]^4 \left[\frac{\sin nv}{n \sin v} \right]^4 du\, dv$$
$$\leqq 2\lambda \left(\frac{2K_3K_6}{m^2 n} + \frac{2K_3K_6}{n^2 m} \right).$$

On the other hand,

$$\left|\iint_R [f(x+2u,\ y+2v)-f(x,\ y)]\left[\frac{\sin m u}{m \sin u}\right]^4\left[\frac{\sin n v}{n \sin v}\right]^4 du\, dv\right|$$
$$\leqq \nu \iint_R \left[\frac{\sin m u}{m \sin u}\right]^4\left[\frac{\sin n v}{n \sin v}\right]^4 du\, dv.$$

The last integral is increased if the circle is replaced by the inscribed square with sides parallel to the sides of S. The resulting integral can be written as the sum

$$4\int_{\frac{\delta}{2\sqrt{2}}}^{\frac{\pi}{2}}\int_{\frac{\delta}{2\sqrt{2}}}^{\frac{\pi}{2}} + 4\int_{\frac{\delta}{2\sqrt{2}}}^{\frac{\pi}{2}}\int_0^{\frac{\delta}{2\sqrt{2}}} + 4\int_0^{\frac{\delta}{2\sqrt{2}}}\int_{\frac{\delta}{2\sqrt{2}}}^{\frac{\pi}{2}} \left[\frac{\sin m u}{m \sin u}\right]^4\left[\frac{\sin n v}{n \sin v}\right]^4 du\, dv = 4(A_1 + A_2 + A_3).$$

Then

$$A_1 < \int_0^{\frac{\pi}{2}}\int_{\frac{\delta}{2\sqrt{2}}}^{\frac{\pi}{2}} \left[\frac{\sin m u}{m \sin u}\right]^4\left[\frac{\sin n v}{n \sin v}\right]^4 du\, dv.$$

This double integral is equal to the product of two simple integrals of forms already considered; one does not exceed an absolute constant divided by n, and the other [cf. (4)] does not exceed an absolute constant divided by $m^4\delta^3$. Hence

$$A_1 < \frac{K_7}{m^4 n \delta^3},$$

where K_7 is an absolute constant; or, equally well, by symmetry,

$$A_1 < \frac{K_7}{m n^4 \delta^3}.$$

It is seen similarly that

$$A_2 < \frac{K_7}{m n^4 \delta^3}, \qquad A_3 < \frac{K_7}{m^4 n \delta^3}.$$

Combining these results, we have the theorem.

THEOREM XIII. — *Let $f(x, y)$ be a finite and integrable function, of period 2π in each variable. If (x, y) is any point having a neighborhood of radius $\delta \leqq \pi$ in which f satisfies the* LIPSCHITZ *condition*

$$|f(x_2,\ y_2) - f(x_1,\ y_1)| \leqq \lambda \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

then for this point

$$|S_{mn}(x,\ y) - f(x,\ y)| \leqq \left(\frac{\log m}{m} + \frac{\log n}{n}\right) C_{13}\lambda + \left(\frac{1}{m} + \frac{1}{n}\right) C_{14}\frac{\nu}{\delta},$$

where $S_{mn}(x, y)$ is the arithmetic mean defined by (5), ν is the oscillation of f, and C_{13} and C_{14} are absolute constants.

This theorem is proved by treating the integral (5) in exactly the same way as

$I_{n:n}(x, y)$ was treated in the proof of Theorem XII. The inequalities needed are suggested in the proof of Theorem IV.

These theorems with regard to periodic functions of two variables will, of course, give theorems analogous to Theorem III for the approximate representation of functions of two variables by means of polynomials.

From the manner in which the proofs of these theorems can be referred back to the proofs in the case of one variable, it is evident that the demonstrations of similar theorems for any number of variables would involve no new methods.

Harvard University, Cambridge, Mass., May 1914.

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