

## On the Behaviour of an Air-Core Transformer when the Frequency is below a certain Critical Value

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JANUARY 1894.

- I. *On the Behaviour of an Air-Core Transformer when the Frequency is below a certain Critical Value.* By E. C. RIMINGTON\*.

[Plates I. & II.]

It is usually supposed in the case of a transformer whose primary is connected to terminals having an alternating potential-difference of constant value between them, that the apparent impedance of the primary is diminished on closing the secondary. Under certain conditions, however, this is not the case, as the following investigation will show.

Let  $r_1$  be the resistance of the primary circuit ;

L its inductance ;

$r_2$  the resistance of the secondary circuit ;

N its inductance ;

M the mutual inductance between the two coils.

The coefficients of induction are assumed constant in the following investigation, a result that can only be obtained in practice when coils not containing iron cores are employed. A pure sine-function alternating P.D. is also assumed.

Let  $p = 2\pi n$ , where  $n$  is the frequency of alternation.

Let  $e$  be the value of the P.D. at any instant  $t$ , and  $E$  its maximum value.

\* Read October 27, 1893.

Let  $c_1$  and  $c_2$  be the currents in the primary and secondary circuits respectively,  $C_1$  and  $C_2$  being their maxima.

Let  $I_1 = \sqrt{r_1^2 + p^2 L^2}$ , the impedance of the primary ;  
and  $I_2 = \sqrt{r_2^2 + p^2 N^2}$ , the impedance of the secondary.

We have the well-known equations :—

$$L \frac{dc_1}{dt} + M \frac{dc_2}{dt} + c_1 r_1 = e ; \quad . \quad . \quad . \quad (1)$$

$$N \frac{dc_2}{dt} + M \frac{dc_1}{dt} + c_2 r_2 = 0. \quad . \quad . \quad . \quad (2)$$

Differentiate (1) with respect to  $t$ , and multiply by  $N$  ;  
differentiate (2) and multiply by  $M$  ; then on subtraction we obtain

$$(LN - M^2) \frac{d^2 c_1}{dt^2} + N r_1 \frac{dc_1}{dt} - M r_2 \frac{dc_2}{dt} = N \frac{de}{dt}. \quad . \quad (3)$$

Multiply (1) by  $r_2$  and add to (3). This gives

$$(LN - M^2) \frac{d^2 c_1}{dt^2} + (N r_1 + L r_2) \frac{dc_1}{dt} + r_1 r_2 c_1 = r_2 e + N \frac{de}{dt}. \quad (4)$$

Similarly we obtain

$$(LN - M^2) \frac{d^2 c_2}{dt^2} + (N r_1 + L r_2) \frac{dc_2}{dt} + r_1 r_2 c_2 = -M \frac{de}{dt}. \quad (5)$$

Now it is obvious, if the P.D. be a pure sine function and the coefficients constants, that the currents must also be pure sine functions differing only in phase from the P.D.

Assume \*, then,

\* This assumption will evidently give a particular solution to equation (4), viz.  $c_1 = C_1 \sin pt$ .

The complete solution is obtained by adding to this the solution of equation (4), assuming the right-hand member zero. So that the complete solution to (4) is

$$c_1 = C_1 \sin pt + A e^{-\frac{k+h}{2} t} + B e^{-\frac{k-h}{2} t},$$

where

$$k = \frac{N r_1 + L r_2}{LN - M^2},$$

and

$$h = \frac{\sqrt{(N r_1 - L r_2)^2 + 4 r_1 r_2 M^2}}{LN - M^2}.$$

The constants  $A$  and  $B$  depend on the phase of the P.D. at the instant the coil is switched on. The exponential terms (since they are both real and negative) rapidly die away, so that practically  $c_1 = C_1 \sin pt$  after a short time has elapsed. The same remarks apply to the value of  $c_2$ .

$$c_1 = C_1 \sin pt, \quad c_2 = C_2 \sin (pt + \theta), \quad \text{and} \quad e = E \sin (pt + \phi),$$

$$\frac{dc_1}{dt} = pC_1 \cos pt \quad \text{and} \quad \frac{d^2c_1}{dt^2} = -p^2C_1 \sin pt;$$

$$\text{also} \quad \frac{de}{dt} = pE \cos (pt + \phi).$$

Inserting these values in equation (4) gives

$$\begin{aligned} C_1 [ \{ r_1 r_2 - p^2 (LN - M^2) \} \sin pt + p(Nr_1 + Lr_2) \cos pt ] \\ = E r_2 \sin (pt + \phi) + E p N \cos (pt + \phi). \quad \dots \quad (6) \end{aligned}$$

For shortness, let

$$a \text{ denote } r_1 r_2 - p^2 (LN - M^2), \text{ and}$$

$$b \text{ denote } p(Nr_1 + Lr_2).$$

Then (6) may be written

$$C_1 \sqrt{a^2 + b^2} \sin (pt + \psi) = EI_2 \sin (pt + \phi + \chi), \quad \dots \quad (7)$$

where

$$\tan \psi = \frac{b}{a} \quad \text{and} \quad \tan \chi = \frac{pN}{r_2}.$$

As (7) must hold for all values of  $t$ , it follows that

$$C_1 \sqrt{a^2 + b^2} = EI_2,$$

or

$$C_1 = \frac{EI_2}{\sqrt{a^2 + b^2}}, \quad \dots \quad (8)$$

and that  $\psi = \phi + \chi$ , or  $\phi = \psi - \chi$ .

Hence

$$\tan \phi = \frac{\frac{b}{a} - \frac{pN}{r_2}}{1 + \frac{bpN}{ar_2}} = \frac{br_2 - apN}{ar_2 + bpN} = \frac{pL - \frac{p^3 NM^2}{I_2^2}}{r_1 + \frac{p^2 r_2 M^2}{I_2^2}} \quad \dots \quad (9)$$

From (9) it is evident that the difference in phase between the primary current and the P.D. is always diminished on closing the secondary, since, when the latter is open

$$\tan \phi = \frac{pL}{r_1}.$$

In the same manner from (5) we obtain

$$\begin{aligned} C_2 \sqrt{a^2 + b^2} \sin (pt + \theta + \psi) &= -pME \cos (pt + \phi) \\ &= pME \sin \left( pt + \phi + \frac{3\pi}{2} \right). \end{aligned}$$

Hence

$$C_2 = \frac{pME}{\sqrt{a^2 + b^2}}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\theta + \psi = \phi + \frac{3\pi}{2},$$

but

$$\psi = \phi + \chi.$$

Hence

$$\theta = \frac{3\pi}{2} - \chi = \pi + \left(\frac{\pi}{2} - \chi\right),$$

or  $\theta$  is greater than  $\pi$  and less than  $\frac{3\pi}{2}$ . Also

$$\tan \theta = \cot \chi = \frac{r_2}{pN}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now from equation (8),

$$C_1 = \frac{E}{\frac{\sqrt{a^2 + b^2}}{I_2}}.$$

Call  $I$  the apparent impedance of the primary when the secondary is closed ( $I_1$  is its impedance with secondary open). Then

$$I = \frac{\sqrt{a^2 + b^2}}{I_2},$$

$$\begin{aligned} \text{or } I^2 I_2^2 &= a^2 + b^2 \\ &= r_1^2 r_2^2 + p^4 (LN - M^2)^2 - 2p^2 r_1 r_2 (LN - M^2) \\ &\quad + p^2 N^2 r_1^2 + p^2 L^2 r_2^2 + 2p^2 r_1 r_2 LN \\ &= I_1^2 I_2^2 - p^2 M^2 \{ p^2 (2LN - M^2) - 2r_1 r_2 \}, \end{aligned}$$

or

$$I^2 = I_1^2 - \frac{p^2 M^2}{I_2^2} \{ p^2 (2LN - M^2) - 2r_1 r_2 \}. \quad . \quad . \quad (12)$$

Equation (12) shows that  $I$  will only be less than  $I_1$  when the quantity inside the brackets is positive; so that, if  $2r_1 r_2 > p^2 (2LN - M^2)$ ,  $I^2$  is greater than  $I_1^2$ , and hence  $I > I_1$ , or the impedance of the primary is *increased* on closing the secondary.

For convenience let  $\alpha_1 = \frac{pL}{r_1}$ , and  $\alpha_2 = \frac{pN}{r_2}$ .  $\alpha_1$  is of course the tangent of the angle of lag of the primary current behind the P.D. when the secondary is unclosed, while  $\frac{\pi}{2} + \tan^{-1} \alpha_2$  is the phase-angle between the primary and secondary currents when the secondary is closed.

Let  $M = \beta \sqrt{LN}$ , so that  $\beta$  represents the ratio of the magnetic induction passing through the secondary to that through the primary, and is of course less than unity, also  $100(1-\beta)$  is the percentage magnetic leakage\*. Substituting these values in (12) it becomes

$$\left(\frac{I}{I_1}\right)^2 = 1 + \frac{\beta^2 \alpha_1 \alpha_2 \{2 - \alpha_1 \alpha_2 (2 - \beta^2)\}}{(1 + \alpha_1^2)(1 + \alpha_2^2)} \dots \quad (13)$$

To make  $\frac{I}{I_1}$  greater than unity, obviously

$$\alpha_1 \alpha_2 \text{ must be less than } \frac{2}{2 - \beta^2},$$

\* This is only the case when the two coils are equal in dimensions and similar in shape; otherwise the ratio of the total lines of magnetic induction through the secondary to those through the primary, when a current flows in the primary, will not be the same as the ratio of the lines through the primary to those through the secondary when there is a current in the latter.

$\beta$  is the geometrical mean of these two ratios. Thus: let  $n_1$  be the number of turns in the primary and  $G_1$  some constant depending on its shape and size, then the magnetic induction through the primary  $= G_1 n_1$ , and  $L = G_1 n_1^2$ . The induction through the secondary  $= G_2 n_2$ , and  $N = G_2 n_2^2$ , where the constant  $G_2$  depends on the shape and size of the secondary. Let  $\beta_1$  be the fraction of the primary induction that threads the secondary, and  $\beta_2$  the fraction of the secondary induction that threads the primary. Then

$$M = G_1 n_1 \beta_1 n_2 = G_2 n_2 \beta_2 n_1.$$

Hence

$$M^2 = \beta_1 \beta_2 G_1 G_2 n_1^2 n_2^2,$$

or

$$M = \sqrt{\beta_1 \beta_2} \sqrt{LN};$$

so that

$$\beta = \sqrt{\beta_1 \beta_2}.$$

Also

$$G_1 \beta_1 = G_2 \beta_2, \text{ or } \frac{\beta_1}{\beta_2} = \frac{G_2}{G_1}.$$

So that for coils of the same shape and size,

$$\beta_1 = \beta_2 = \beta, \text{ since } G_1 = G_2.$$

When the coils are of different shapes or sizes,

$$\frac{\beta_1}{\beta_2} = \frac{G_2}{G_1} = \frac{N}{L} \cdot \left(\frac{n_1}{n_2}\right)^2;$$

also if the number of turns in the primary and secondary is equal,

$$\frac{\beta_1}{\beta_2} = \frac{N}{L}.$$

or the critical value of

$$\alpha_1 = \frac{2}{\alpha_2(2-\beta^2)}; (\alpha_2 \text{ given});$$

and that of

$$\alpha_2 = \frac{2}{\alpha_1(2-\beta^2)}; (\alpha_1 \text{ given}).$$

When the primary and secondary are identical or have the same shape and coil-volume \*, and the secondary when closed is short-circuited,

$$\alpha_1 = \alpha_2 = \alpha.$$

Then

$$\left(\frac{I}{I_1}\right)^2 = 1 + \frac{\beta^2 \alpha^2 \{2 - \alpha^2(2 - \beta^2)\}}{(1 + \alpha^2)^2}, \dots (14)$$

and the critical value of  $\alpha = \sqrt{\frac{2}{2 - \beta^2}}$ .

When  $\beta = 1$ , or there is no magnetic leakage, the critical value of  $\alpha = \sqrt{2}$ .

To find the value of  $\alpha_1$  that will make  $\frac{I}{I_1}$  a maximum,  $\alpha_2$  and  $\beta$  being given.

Obviously from (13)  $\frac{I}{I_1}$  is a maximum when  $\left(\frac{I}{I_1}\right)^2$  is a maximum, that is, when  $\frac{\alpha_1 \{2 - \alpha_1 \alpha_2 (2 - \beta^2)\}}{1 + \alpha_1^2}$  is a maximum, since  $\alpha_2$  and  $\beta$  are constants.

Differentiating with respect to  $\alpha_1$  and equating to zero gives

$$\alpha_1 = \frac{\sqrt{4 + \alpha_2^2(2 - \beta^2)^2} - \alpha_2(2 - \beta^2)}{2}. \dots (15)$$

If  $\beta = 1$ , or there be no magnetic leakage,

$$\alpha_1 = \frac{\sqrt{4 + \alpha_2^2} - \alpha_2}{2} \quad \text{or} \quad \frac{\alpha_1}{1 - \alpha_1^2} = \frac{1}{\alpha_2}. \dots (15a)$$

If the coils have the same time-constants, or  $\alpha_1 = \alpha_2 = \alpha$ ,

$$\alpha = \frac{1}{\sqrt{3 - \beta^2}} \dots (15b)$$

\* Relative difference of thickness of insulation being supposed negligible, if different-sized windings are used,

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and

$$\frac{I}{I_1} = \sqrt{1 + \frac{\beta^2}{4 - \beta^2}} = \frac{2}{\sqrt{4 - \beta^2}}$$

if also  $\beta = 1$ ,

$$\alpha = \frac{1}{\sqrt{2}}, \text{ and } \frac{I}{I_1} = \sqrt{1.5} = 1.155.$$

Obviously, if  $\alpha_1$  and  $\beta$  are known the value of  $\alpha_2$  that makes  $\frac{I}{I_1}$  a maximum is by symmetry,

$$\alpha_2 = \frac{\sqrt{4 + \alpha_1^2(2 - \beta^2)^2} - \alpha_1(2 - \beta^2)}{2} \quad \text{or} \quad \frac{1 - \alpha_2^2}{\alpha_2(2 - \beta^2)} = \alpha_1,$$

or in the case of no magnetic leakage,  $\frac{\alpha_2}{1 - \alpha_2^2} = \frac{1}{\alpha_1}$ .

If  $\alpha_1$  and  $\alpha_2$  are both variables, we have the two equations to be satisfied, viz.:—

$$\alpha_2 = \frac{1 - \alpha_1^2}{\alpha_1(2 - \beta^2)} \quad \text{and} \quad \alpha_1 = \frac{1 - \alpha_2^2}{\alpha_2(2 - \beta^2)},$$

$$\therefore \alpha_1 \alpha_2 (2 - \beta^2) = 1 - \alpha_1^2 = 1 - \alpha_2^2,$$

$$\therefore \alpha_1 = \alpha_2 = \alpha \text{ say,}$$

and

$$\alpha = \frac{1 - \alpha^2}{\alpha(2 - \beta^2)} \quad \text{or} \quad \alpha = \frac{1}{\sqrt{3 - \beta^2}}.$$

So that to get  $\frac{I}{I_1}$  a maximum, the primary and secondary should have the same value of  $\alpha$ , each equal to  $\frac{1}{\sqrt{3 - \beta^2}}$ , or

in the case of no magnetic leakage  $= \frac{1}{\sqrt{2}}$ ; in which case  $\frac{I}{I_1}$  will be 1.155, or a  $15\frac{1}{2}$  per cent. increase in impedance caused by short-circuiting the secondary, and this is the greatest that can be obtained.

Consider, then, the case of a transformer having coils with equal time-constants, and suppose there is no magnetic leakage.

For values of  $\alpha$  below  $\frac{1}{\sqrt{2}}$  the impedance is increased, and putting  $\beta = 1$  in equation (14) gives

$$\frac{I}{I_1} = \frac{\sqrt{1 + 4\alpha^2}}{1 + \alpha^2}.$$





Obviously  $P = P_2 + H$ , unless there are masses of metal present, in which eddy currents are developed, or the frequency is so great that appreciable energy is radiated.

$$\begin{aligned} P_1 &= \frac{p}{2\pi} \cdot \frac{E^2}{I_1} \int_0^{2\pi} \sin pt \cdot \sin (pt + \phi) dt \\ &= \frac{E^2 r_1}{2I_1^2} = \frac{E^2}{2r_1} \cdot \frac{1}{1 + \alpha^2}. \end{aligned} \quad (19)$$

Hence, from (16) and (19),

$$\frac{P}{P_1} = \frac{(1 + 2\alpha^2)(1 + \alpha^2)}{1 + 4\alpha^2} = \frac{1 + 3\alpha^2 + 2\alpha^4}{1 + 4\alpha^2}, \quad (20)$$

and from this equation (20) the curve marked PPP (Plate I.) is plotted.

The curves for  $P$ ,  $H$ ,  $P_2$ , and  $P_1$  are plotted in Plate II. for values of  $\alpha$  up to 2.

### *Magnetizing Effect of the Coils.*

Let  $g$  be the number of effective current-turns at any instant when the secondary is closed, and  $G$  its maximum value.

$$\text{Then} \quad g = n_1 c_1 + n_2 c_2,$$

where  $n_1$  and  $n_2$  are the number of turns in the primary and secondary respectively, and

$$G = \sqrt{n_1^2 C_1^2 + n_2^2 C_2^2 + 2n_1 n_2 C_1 C_2 \cos \theta},$$

where  $\theta$  is the phase-angle between the two currents.

If we assume the primary and secondary to occupy equal volumes, and we can neglect the relative difference in thickness

of the insulation of the two coils,  $\frac{n_1}{n_2} = \sqrt{\frac{r_1}{r_2}}$ ,

$$\begin{aligned} \therefore G &= n_1 \sqrt{C_1^2 + C_2^2 \frac{r_2}{r_1} + 2C_1 C_2 \sqrt{\frac{r_2}{r_1}} \cos \theta}, \\ &= \frac{n_1 E}{\sqrt{a^2 + b^2}} \sqrt{I_2^2 + p^2 M^2 \frac{r_2}{r_1} - 2p^2 MN \sqrt{\frac{r_2}{r_1}}}; \end{aligned}$$

or (since  $M^2$  is assumed  $=LN$ )

$$G = \frac{n_1 E r_2}{\sqrt{a^2 + b^2}} = \frac{n_1 E}{r_1 \sqrt{1 + 4\alpha^2}}.$$

Call  $G_1$  the maximum value of  $g$  when the secondary is open ; then

$$G_1 = \frac{n_1 E}{I_1} = \frac{n_1 E}{r_1 \sqrt{1 + \alpha^2}},$$

$$\therefore \frac{G}{G_1} = \sqrt{\frac{1 + \alpha^2}{1 + 4\alpha^2}} \quad \dots \dots \dots (21)$$

Evidently, then, the magnetizing effect is always diminished on closing the secondary.

The curve GGG (Plate I.) is plotted from equation (21).

It will be seen, on inspecting Plate I., that in the cases of the impedance-curve and the current-curve the critical value for  $\alpha = \sqrt{2} = 1.414$ , while the value of  $\alpha$  that makes them a maximum and a minimum respectively is  $\alpha = \frac{1}{\sqrt{2}} = .707$ . This latter value of  $\alpha$  is the critical value for the curve PPP, while, as may be seen by differentiating equation (20) and equating to zero, the value of  $\alpha$  that makes this curve a minimum is  $\frac{\sqrt{\sqrt{3}-1}}{2}$ , or .4278.

Consider, now, the case of primary and secondary having the same time-constants but with magnetic leakage.

Then we know the *critical value* of  $\alpha = \sqrt{\frac{2}{2-\beta^2}}$ .

The value of  $\alpha$  to make  $\frac{I}{I_1}$  a maximum from (15b)  $= \frac{1}{\sqrt{3-\beta^2}}$ .

Let  $\gamma$  represent the ratio of the former value of  $\alpha$  to the latter, then

$$\gamma = \sqrt{\frac{6-2\beta^2}{2-\beta^2}};$$

also the maximum value of  $\frac{I}{I_1} = \frac{2}{\sqrt{4-\beta^2}}$ .

From equation (11),

$$\tan \theta = \frac{r_2}{pN} = \frac{1}{\alpha};$$

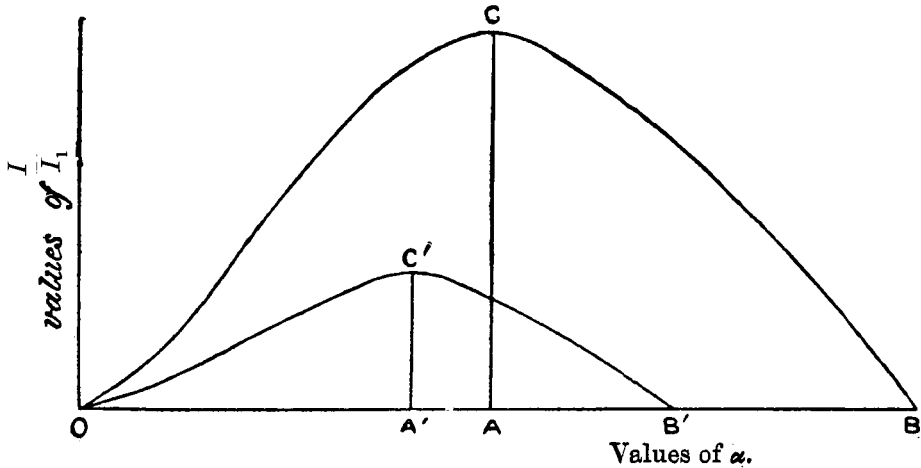
and  $\theta$  is the angle by which the secondary current is in

advance of the primary, and it lies between  $\pi$  and  $\frac{3\pi}{2}$ ; it therefore follows that the angle by which the secondary current lags behind the primary lies between  $\frac{\pi}{2}$  and  $\pi$ , and is  $\frac{\pi}{2} + \tan^{-1} \alpha$ . The following Table gives values of the above quantities for values of  $\beta$  from 1 to  $\cdot 1$ .

$\beta$ .	Critical values of $\alpha$ .	Values of $\alpha$ for max. $\frac{I}{I_1}$ .	$\gamma$ .	Max. values of $\frac{I}{I_1}$ .	Critical phase-angle.	Phase-angle for max. $\frac{I}{I_1}$ .
1.0	$1.414 = \sqrt{2}$	$.707 = \frac{1}{\sqrt{2}}$	2	1.155	$144^\circ 44'$	$125^\circ 16'$
.9	1.300	.676	1.920	1.120	$142^\circ 26'$	$124^\circ 4'$
.8	1.210	.650	1.865	1.090	$140^\circ 26'$	$123^\circ 2'$
.7	1.150	.630	1.825	1.070	$139^\circ 0'$	$122^\circ 13'$
.6	1.105	.615	1.795	1.050	$137^\circ 51'$	$121^\circ 36'$
.5	1.070	.605	1.770	1.032	$136^\circ 56'$	$121^\circ 11'$
.4	1.045	.595	1.760	1.020	$136^\circ 16'$	$120^\circ 45'$
.3	1.023	.586	1.750	1.010	$135^\circ 39'$	$120^\circ 22'$
.2	1.010	.581	1.740	1.007	$135^\circ 17'$	$120^\circ 10'$
.1	1.002	.578	1.730	1.001	$135^\circ 4'$	$120^\circ 2'$

It will be seen from the column of values for  $\gamma$  that when  $\beta=1$ , or there is no magnetic leakage, the critical value of  $\alpha$  is twice its value for maximum  $\frac{I}{I_1}$ ; and the effect of leakage is to diminish this number, so that when  $\beta=.1$  it is reduced to 1.73. This is shown in the subjoined figure.

Fig. 1.



The curve OCB represents the critical part of the curve III (Plate I.) when there is no leakage, and A comes midway between O and B, so that  $OA = \frac{1}{2} OB$ ; also  $OB = \sqrt{2} = 1.414$ ,  $OA = .707$ , and, if the point O represent 100 divisions,  $AC = 15\frac{1}{2}$  divisions.

The curve OC'B' represents the critical part when  $\beta = .1$ ;  $OB' = 1.002$  and  $OA' = .578 = .577 OB'$ ; also  $A'C' = .12$  division. Hence we see that the effect of magnetic leakage is to shift the point A' corresponding to the maximum values of  $\frac{I}{I_1}$  from midway between O and B' and towards B'.

The author was enabled, through the kindness of Dr. Fleming, to try an experiment in the meter-testing room of the Electric Supply Company, using a Kelvin balance for measuring the currents through the primary coil. The alternating P.D. was obtained from the terminals of a transformer capable of giving over  $100^A$ ; and as the maximum current ever taken was  $6\frac{1}{2}^A$  about, the P.D. may be assumed constant. The frequency was 83.3.

The air-core transformer used for the experiment consisted of two coils wound one inside the other, and of No. 20 B.W.G. cotton-covered wire. The inner coil was used as primary and the outer as secondary.

Each coil consisted of 5 layers, of 125 turns per layer.

Calculating the time-constants or values of  $\frac{L}{R}$  for the coils by Perry's approximate formula, they worked out as .00121 for the inner coil or primary, and .00152 for the outer coil or secondary.

$$p = 2\pi \times 83.3 = 523.$$

Hence

$$\alpha_1 = .00121 \times 523 = .633,$$

$$\alpha_2 = .00152 \times 523 = .795.$$

The values of  $\alpha_1$  and  $\alpha_2$  were probably smaller than these values, as the coils became fairly warm from working; moreover the primary had the leads and the resistance of a Kelvin balance in series with it.

Take, then,  $\alpha_1$  as .5, and  $\alpha_2$  as .7.

The observed value of  $\frac{I}{I_1}$  was 1.032.

Substituting these values in equation (13) makes  $\beta = .57$ , or a magnetic leakage of 43 per cent., if the P.D. were a true sine function. This seems a rather large value for the leakage; and it is probable that the leakage was considerably less than this, but that the P.D. was not a pure sine function, and on this account the ratio  $\frac{I}{I_1}$  was less than it otherwise would have been.

The above experiment was only a rough one, but it showed an increase of 3.2 per cent. in the impedance of the primary on closing the secondary: moreover the time-constants of the coils were not suited for giving the best effect with the frequency employed; but, as is seen from the previous theoretical investigation, by employing a primary and a secondary having equal time-constants suitably related to the frequency, and a pure sine function P.D., an increase of impedance of from 10 to 12 per cent. ought to be obtained;  $15\frac{1}{2}$  per cent. increase could never be obtained in practice, as there must always be some magnetic leakage.

In transformers with iron cores this effect would be likely to escape notice, as the values of  $\frac{L}{R}$  would be so large that the critical frequency would be very small, so that for all frequencies employed in practice the impedance of the primary would diminish on closing the secondary. The iron core would also distort the current from a pure sine function.

The above investigations may be with advantage treated by the geometrical or clock-face diagram method, as it shows at once to the eye the relations between the different quantities and the phase-angles.

Consider first the case of an identical primary and secondary and no magnetic leakage; or  $L = N = M$ , and  $r_1 = r_2$ .

As before, call  $\frac{pL}{r_1} = \frac{pN}{r_2} = \alpha$ , and let  $\tan^{-1} \alpha = \theta$ , or

$$\tan \theta = \alpha = \frac{pL}{r_1} = \frac{pN}{r_2}.$$

Draw a line  $OA = C_1 r_1$ . Draw  $OB$  at right angles to  $OA$  and  $90^\circ$  behind  $OA$  as regards the direction of rotation, shown by the arrow; make  $OB = pLC_1$ . Then  $AB$  represents

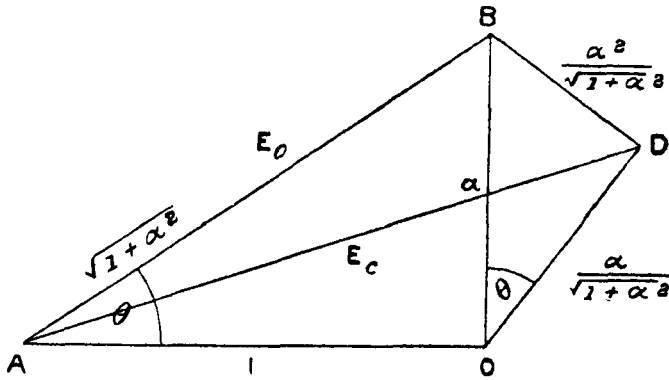


Now, as the diagram is drawn,  $E_c$  is greater than  $E_0$ , the primary current being constant, so that if the terminal P.D. were kept constant the primary current with the secondary closed divided by the same with the secondary open would equal  $\frac{E_0}{E_c}$ , or the impedance ratio  $\frac{I}{I_1} = \frac{E_c}{E_0}$ .

The actual lines that need be drawn are shown in full, the dotted ones being superfluous except for explanation, and they will in future be omitted.

If we take  $C_1 r_1$  equal to unity we shall have the diagram fig. 3.

Fig. 3.



$$\text{Now } AD^2 = AO^2 + OD^2 - 2AO \cdot OD \cos AOD$$

$$= 1 + \frac{\alpha^2}{1 + \alpha^2} - \frac{2\alpha}{\sqrt{1 + \alpha^2}} \cos \left( \frac{\pi}{2} + \theta \right)$$

$$= 1 + \frac{\alpha^2}{1 + \alpha^2} + \frac{2\alpha}{\sqrt{1 + \alpha^2}} \sin \theta.$$

$$\text{Now } \tan \theta = \alpha, \text{ hence } \sin \theta = \frac{\alpha}{\sqrt{1 + \alpha^2}};$$

$$\therefore AD^2 = 1 + \frac{3\alpha^2}{1 + \alpha^2} = \frac{1 + 4\alpha^2}{1 + \alpha^2}, \text{ or } AD = \sqrt{\frac{1 + 4\alpha^2}{1 + \alpha^2}}.$$

Now AD represents  $E_c$  and AB represents  $E_0$ . So that

$$E_c > = < E_0 \text{ accordingly as } \sqrt{\frac{1 + 4\alpha^2}{1 + \alpha^2}} > = < \sqrt{1 + \alpha^2},$$

or as

$$\begin{aligned} 1 + 4\alpha^2 &> = < (1 + \alpha^2)^2, \\ 1 &> = < \pm (1 - \alpha^2). \end{aligned}$$



Taking the minus sign,

$$E_c > < E_0 \text{ as } \alpha < = > \sqrt{2}.$$

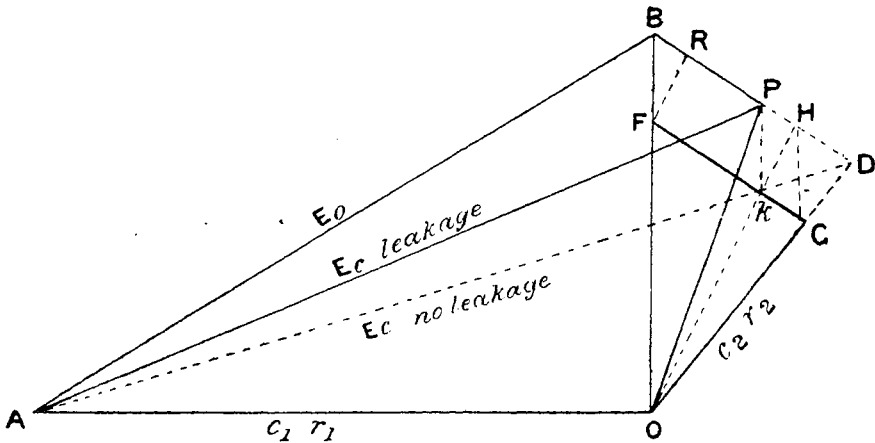
Taking the plus sign,

$$E_c > < E_0 \text{ as } \alpha > = < 0;$$

that is  $E_c > E_0$  if  $\alpha$  lies between  $\sqrt{2}$  and 0, not including these values.

*Case of an identical Primary and Secondary with Magnetic Leakage.*

Fig. 4.



As before make  $OA = C_1 r_1$ ,  $OB = p L C_1$ , then  $AB = E_0$ . Let  $M = \beta L = \beta N$ .

In  $OB$  take a point  $F$  such that  $OF = \beta \times OB = p M C_1$ . Draw  $OD$  making angle  $\theta = \text{angle } OAB$  with  $OB$  and drop perpendicular  $BD$ . Drop perpendicular  $FG$  on  $OD$ , then  $FG = p N C_2$ , and  $OG = C_2 r_2$ .

Through  $G$  draw  $GH$  parallel to  $OB$  and cutting  $BD$  in  $H$ . Join  $OH$  and let it cut  $FG$  in  $K$ , through  $K$  draw  $KP$  parallel to  $OB$  and cutting  $BD$  in  $P$ .

Through  $F$  draw  $FR$  parallel to  $OH$  and meeting  $BD$  in  $R$ .

Then 
$$\frac{BH}{RH} = \frac{OB}{OF} = \frac{1}{\beta}.$$

Therefore  $RH = \beta \cdot BH$ , but  $BH = FG$ , so

$$RH = \beta \cdot FG = \beta p N C_2 = p M C_2.$$

Again,  $RH = FK = BP$  since  $FR$  is parallel to  $OH$  and  $KP$  to  $OB$ .

Now  $OP$  is the resultant of  $OB = pLC_1$ , and  $BP = pMC_2$ . Join  $AP$ .

Then  $AP$  represents  $E_c$ .

If there were no leakage  $AD$  would represent  $E_c$ . Now  $AP$  is obviously less than  $AD$ , or the effect of leakage is to diminish the ratio  $\frac{E}{E_0}$ , that is the ratio of  $\frac{I}{I_1}$ .

Obviously from the figure the greater the leakage the lower  $F$ , the smaller  $FG$  and  $BH$ , the smaller also the ratio  $\frac{BP}{BH}$  and consequently the nearer  $P$  to  $B$ , that is the nearer the value of  $E_c$  to that of  $E_0$ .

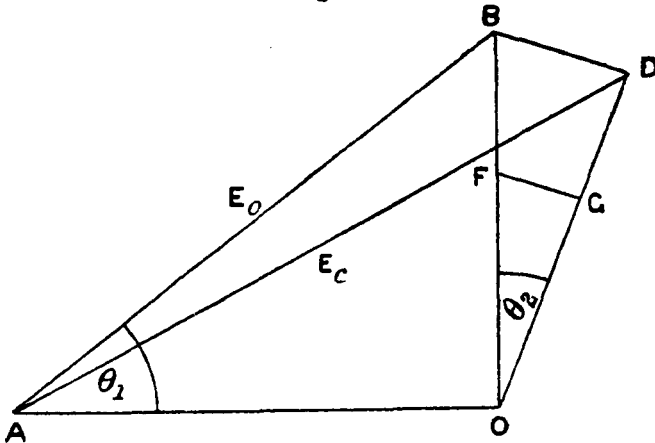
*Case of Two Coils with different Time-Constants without Magnetic Leakage.*

Primary  $L, r_1$ , and  $\alpha_1 = \frac{pL}{r_1}$ ;  $\tan \theta_1 = \alpha_1$ .

Secondary  $N, r_2$ , and  $\alpha_2 = \frac{pN}{r_2}$ ;  $\tan \theta_2 = \alpha_2$ .

$M^2 = LN$ , so that  $\frac{M}{N} = \frac{L}{M}$ .

Fig. 5.



$OA = C_1 r_1$ ,  $OB = p L C_1$ ,  $AB = E_0$ .

Make  $\frac{OF}{OB} = \frac{M}{L}$ .  $\therefore OF = p M C_1$ .

Draw OD making angle  $\theta_2 = \tan^{-1} \alpha_2$  with OB, and drop perpendiculars FG and BD on to OD.

Then  $FG = pNC_2$  and  $OG = C_2r_2$ .

Now

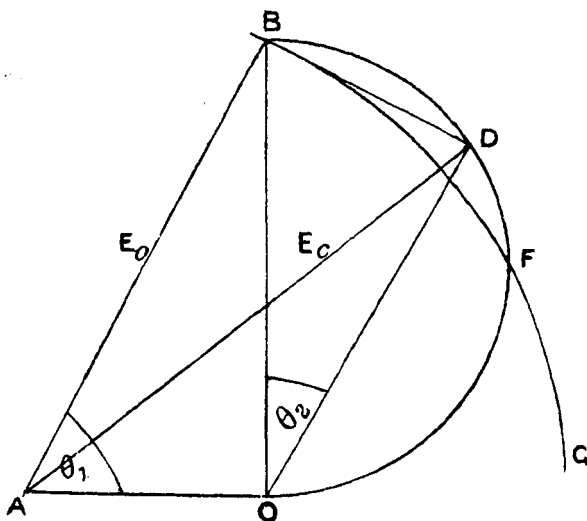
$$\frac{M}{N} = \frac{L}{M} = \frac{OB}{OF} = \frac{BD}{FG} \text{ (by similar triangles).}$$

$$\therefore BD = FG \times \frac{M}{N} = pNC_2 \times \frac{M}{N} = pMC_2.$$

Hence OD represents the back E.M.F. in primary due to self and mutual inductions, and  $AD = E_c$ .

However large the value of  $\alpha_1$  (unless it were infinite, which is of course not possible) it is always possible by making  $\alpha_2$  sufficiently small to obtain  $E_c$  greater than  $E_0$ . In fig. 6 the

Fig. 6.

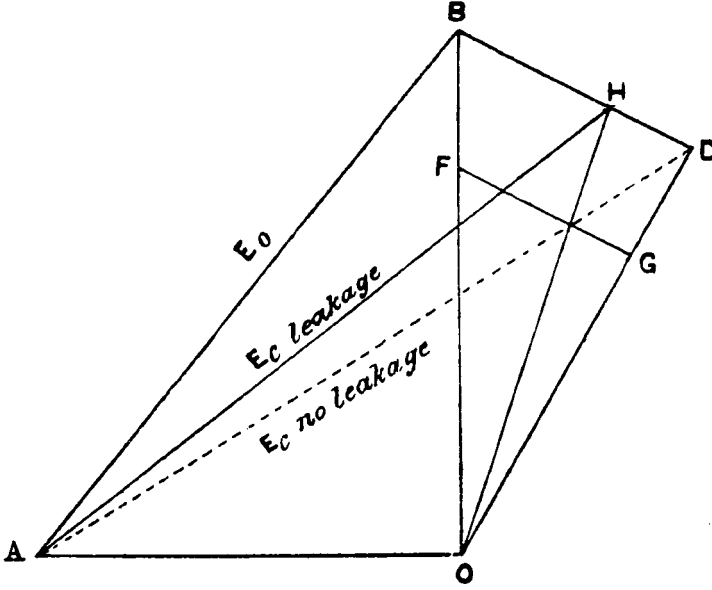


angle  $OAB = \theta_1 = \tan^{-1} \alpha_1$  and  $AB = E_0$ . On OB describe the semicircle BDO. With centre A and radius AB describe the circle BFG cutting BDO in F. Now by taking any point D in the circle BDO between B and F and joining OD and AD, the latter, which represents  $E_c$ , will be greater than AB or  $E_0$ , and the tangent of the angle BOD represents  $\alpha_2$ .

Obviously when D coincides with F,  $E_c = E_0$ , or closing the secondary makes no difference to the impedance of the primary; while if D be between F and O,  $E_c$  is less than  $E_0$ .

*Case of Two Coils with different Time-Constants with  
Magnetic Leakage.*

Fig. 7.



Let  $M^2 = \beta^2 LN$  so that  $\frac{M}{N} = \beta^2 \frac{L}{M}$ .

Construction is the same as in the last case, or  $\frac{OF}{OB} = \frac{M}{L}$ ;

H is taken so that  $\frac{BH}{BD} = \beta^2$ .

Now  $\frac{BD}{FG} = \frac{L}{M}$ ;  $\therefore \frac{BH}{FG} = \beta^2 \frac{L}{M} = \frac{M}{N}$ ,

so that

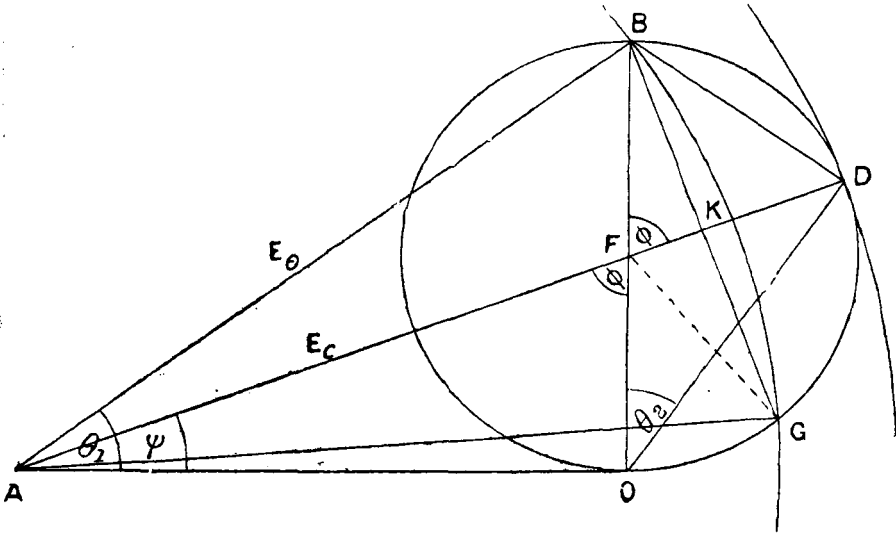
$$BH = \frac{M}{N}, \quad FG = \frac{M}{N} \cdot pNC_2 = pMC_2.$$

$\therefore$  OH is the back E.M.F. in the primary due to self and mutual inductions, and consequently  $AH = E_c$ .

$AD = E_c$  if there be no leakage.

PROBLEM I. *A transformer with a given primary (or  $\alpha_1$  known), to find the value of  $\alpha_2$  for secondary so as to make  $\frac{E_c}{E_0}$  or  $\frac{I}{I_1}$  a maximum. Also to find the critical value of  $\alpha_2$ . No magnetic leakage.*

Fig. 8.



Let  $\theta_1 = \tan^{-1} \alpha_1$  which is known.

Draw  $OA = C_1 r_1$ ; draw  $OB$  at right angles to  $OA$  and equal to  $p L C_1$  or  $\alpha_1 C_1 r_1$ . Then  $AB = E_0$  and the angle  $OAB = \theta_1$ .

On  $OB$  as diameter describe the circle  $ODB$  with centre  $F$ ; then  $OF = \frac{1}{2}OB$ .

Join  $AF$  and produce it to cut the circle in  $D$ .

Then  $AD$  is the greatest straight line that can be drawn from  $A$  to cut the circle  $ODB$ , since if two circles touch the line through their centres passes through the point of contact

Join  $BD$  and  $OD$ , then the angle  $BOD = \theta_2$  and  $\frac{BD}{OD} = \tan \theta_2 =$  the required value of  $\alpha_2$ .

Let the angle  $BFD = \phi$  and the angle  $OAD = \psi$ .

Then

$$\theta_2 = \frac{1}{2}\phi = \frac{1}{2} \left( \frac{\pi}{2} - \psi \right) \text{ or } 2\theta_2 = \frac{\pi}{2} - \psi.$$

Now  $\frac{\tan \psi}{\tan \theta} = \frac{OF}{OB} = \frac{1}{2}$ , or  $\cot \psi = \frac{2}{\tan \theta_1}$ ;  
also

$$\tan 2\theta_2 = \tan\left(\frac{\pi}{2} - \psi\right) = \cot \psi = \frac{2}{\tan \theta_1}.$$

Hence  $\frac{2 \tan \theta_2}{1 - \tan^2 \theta_2} = \frac{2}{\tan \theta_1}$ ,

or  $\frac{\alpha_2}{1 - \alpha_2^2} = \frac{1}{\alpha_1}$ ,

or  $\alpha_2 = \frac{\sqrt{4 + \alpha_1^2} - \alpha_1}{2}$

Obviously from the figure, however small  $\theta_1$  may be,  $\theta_2$  is never greater than  $45^\circ$ , or  $\alpha_2$  greater than unity.

*To find the Maximum Value of  $\frac{E_c}{E_0}$  or  $\frac{I}{I_1}$ .*

Call  $OA=1$ , then  $OB=\alpha_1$ , and  $AB=\sqrt{1+\alpha_1^2}$ .

Now

$$AD = AF + FD = \sqrt{1 + \frac{\alpha_1^2}{4}} + \frac{\alpha_1}{2} = \frac{\sqrt{4 + \alpha_1^2} + \alpha_1}{2}.$$

But

$$\frac{E_c}{E_0} = \frac{AD}{AB} = \frac{\sqrt{4 + \alpha_1^2} + \alpha_1}{2 \sqrt{1 + \alpha_1^2}},$$

or the maximum value of  $\frac{I}{I_1} = \frac{\sqrt{4 + \alpha_1^2} + \alpha_1}{2 \sqrt{1 + \alpha_1^2}}$ .

Hence

$$\begin{aligned} \left(\frac{I}{I_1}\right)_{\max.}^2 &= \frac{4 + \alpha_1^2 + \alpha_1^2 + 2\alpha_1 \sqrt{4 + \alpha_1^2}}{4(1 + \alpha_1^2)} \\ &= 1 + \frac{\alpha_1(\sqrt{4 + \alpha_1^2} - \alpha_1)}{2(1 + \alpha_1^2)}. \end{aligned}$$

Now since  $\sqrt{4 + \alpha_1^2}$  is always greater than  $\alpha_1$ , whatever value  $\alpha_1$  has (except  $\infty$ )  $\left(\frac{I}{I_1}\right)_{\max.}^2$  is always greater than

unity, or  $\left(\frac{I}{I_1}\right)_{\max.}$  is always greater than unity.



*Construction.* Draw a circle with centre F, and take any diameter OB, draw OA at right angles to OB.

With centre F and radius 3OF describe a circle cutting OA in A.

Join AB and AF and produce AF to meet the circle in D. Join BD and OD.

Then the angle OAB = the angle BOD =  $\theta$ , and the required value of  $\alpha$

$$= \tan \theta = \frac{OB}{OA} = \frac{BD}{OD}$$

*Proof.* Call the angle BOD =  $\theta$ .

Join OH and produce it to meet AB in G : join FG.

Now AH = HD by construction, and OH is parallel to BD.

Hence GH is parallel to BD, and therefore AG = GB.

Again, since AG = GB, and OF = FB, GF is parallel to OA.

$\therefore$  the angle BFG = the angle BOA = a right angle.

Now OF = FB and FG is common : hence

$$OG = GB = AG.$$

$\therefore$  the angle GAO = GOA =  $\frac{\pi}{2} - \text{GOB} = \text{BOD} = \theta$ , since

the angle HOD being in a semicircle =  $\frac{\pi}{2}$ .

$\therefore$  the angle BAO = BOD =  $\theta = \tan^{-1} \alpha$ , or the coils have the same time-constant.

Now from the solution of Problem I. the construction is such as to give a maximum value for  $\frac{I}{I_1}$ , but it is so arranged that while giving a maximum  $\frac{I}{I_1}$  the time-constant of the secondary is the same as that of the primary. Q.E.F.

*To find  $\alpha$  or  $\tan \theta$ .*

$$AF = 3OF \text{ (by construction).}$$

$$\text{Now } AO^2 = AF^2 - OF^2 = 8OF^2.$$

$$\therefore AO = 2OF\sqrt{2}.$$

$$\text{Now } OB = 2OF \text{ and } \alpha = \tan \theta = \frac{OB}{OA} = \frac{2OF}{2OF\sqrt{2}} = \frac{1}{\sqrt{2}}.$$



To find the maximum value of  $\frac{I}{I_1}$ , that is of  $\frac{E_c}{E_0}$ .

$$AO^2 = 8OF^2 \text{ and } AD = 4OF.$$

Now

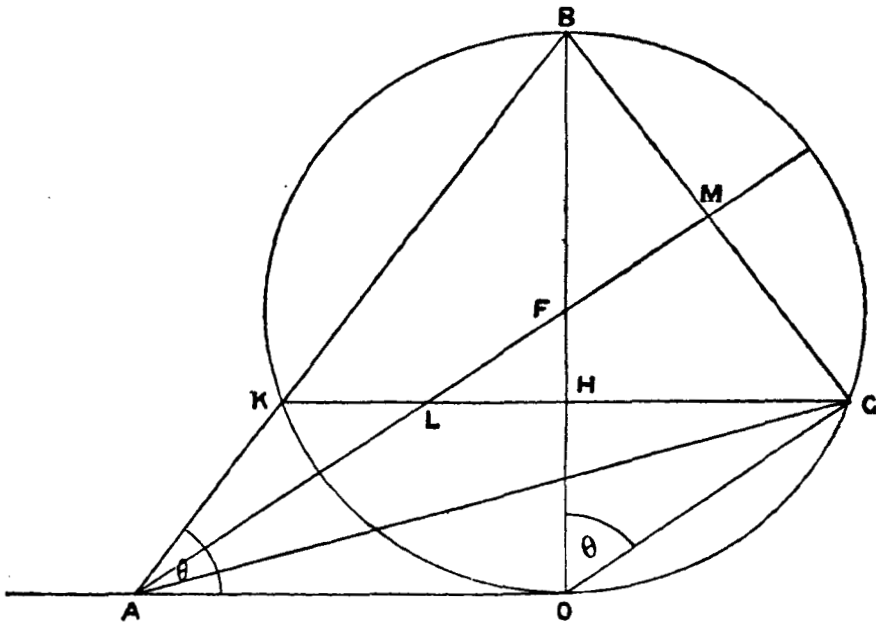
$$AB = \sqrt{AO^2 + OB^2} = \sqrt{8OF^2 + 4OF^2} = \sqrt{12OF^2} = 2OF\sqrt{3}.$$

Also

$$\frac{I}{I_1} = \frac{E_c}{E_0} = \frac{AD}{AB} = \frac{4OF}{2OF\sqrt{3}} = \frac{2}{\sqrt{3}} = 1.155.$$

PROBLEM III. *A transformer with primary and secondary having the same time-constants, to find the critical value of  $\alpha$ . No magnetic leakage.*

Fig. 10.



*Construction.* Draw any circle with centre F and a diameter OB, draw OA at right angles to OB. In OB take a point H such that  $BH = 2OH$  or  $OH = \frac{1}{3}OB$ .

Through H draw KHG parallel to OA and meeting the circle in K and G.

Join BK and produce it to meet OA in A. Join OG and BG.

Then the angle OAB = the angle BOG =  $\theta$  = the critical angle required, and the critical value of  $\alpha = \tan \theta = \frac{OB}{OA}$ .

*Proof.* Firstly, to prove the angles OAB and BOG to be equal.

Since KG is parallel to OA, the angles KHB and GHB are right angles, and KH=HG, also BH is common.

Therefore the angle KBH=GBH.

But AOB is a right angle by construction, and BGO because it is in a semicircle.

Hence the angle BAO=BOG= $\theta$ .

Secondly, to prove that AG=AB or  $E_c=E_0$ , in which case from Problem I. it is known that  $\theta$  is the critical angle. Join AF and produce it to meet BG in M.

Since KG is parallel to AO by construction,

$$\frac{FH}{FO} = \frac{LH}{AO};$$

but  $FH = \frac{1}{3}FO$  (since  $OH = \frac{1}{3}OB = \frac{2}{3}FO$ );

$$\therefore LH = \frac{1}{3}AO.$$

Again,  $\frac{AO}{KH} = \frac{OB}{HB} = \frac{3}{2}$ ; hence  $AO = \frac{3}{2}KH$ ;

$$\therefore LH = \frac{1}{2}KH = \frac{1}{2}HG \quad \text{or} \quad LG = 3LH.$$

But  $AO = 3LH$ ;  $\therefore AO = LG$ ;

or AL is parallel to OG; and the angle AMB=AMG=a right angle.

Also BM=MG and AM is common.

$$\therefore AB=AG \quad \text{or} \quad E_c=E_0.$$

*To find the Critical Value of  $\alpha$  or  $\tan \theta$ .*

Since AF is parallel to OG the angle BFM= $\theta$ ;

$$\therefore \text{the angle LFH} = \theta = \text{angle BKH}.$$

Hence 
$$\frac{LH}{FH} = \frac{BH}{KH}$$

Now 
$$KH = 2LH;$$

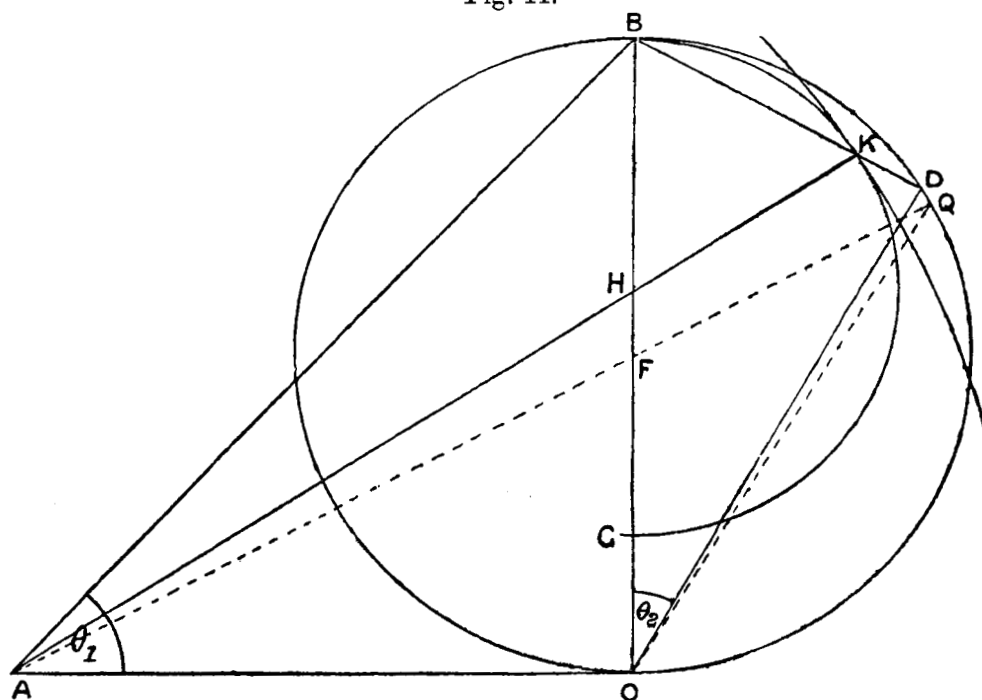
$$\therefore 2LH^2 = FH \cdot BH = \frac{OF}{3} \cdot \frac{4OF}{3} = \frac{4OF^2}{9},$$

or 
$$LH = \frac{OF\sqrt{2}}{3}.$$

Now 
$$\alpha = \tan \theta = \frac{LH}{FH} = \frac{OF\sqrt{2}}{3FH} = \frac{OF\sqrt{2}}{3 \cdot \frac{OF}{3}} = \sqrt{2}.$$

PROBLEM IV. *A primary with given time-constant, or  $\alpha_1$  known, magnetic leakage or  $\beta$  known, to find the value of  $\alpha_2$  so as to make  $\frac{I}{I_1}$  a maximum.*

Fig. 11.



*Construction.* Draw a circle BDO with diameter OB and centre F; draw OA at right angles to OB and make OA of such length that  $\frac{OB}{OA} = \alpha_1$ .

Join AB. Then the angle OAB represents  $\theta_1$ .

From B mark off BH such that

$$\frac{BH}{BF} = \beta^2 \quad (M = \beta \sqrt{LN}).$$

With centre H and radius HB describe circle BKG.

Join AH and produce it to cut circle BKG in K.

Join BK and produce it to cut circle BDO in D.

Join OD, then the angle BOD is the required value of  $\theta_2$ ,

and  $\tan BOD = \frac{BD}{OD}$  the required value of  $\alpha_2$ .

Also  $\frac{AK}{AB}$  is the maximum value of  $\frac{E}{E_0}$  or of  $\frac{I}{I_1}$ .

*Proof.* Since  $BH = \beta^2 BF$ ,  $BG = \beta^2 BO$ , and  $BK = \beta^2 BD$  (for join KG, then KG is parallel to DO since the angles at

K and D are both right angles, being in semicircles, and therefore by similar triangles  $\frac{BK}{BD} = \frac{BG}{BO}$ .

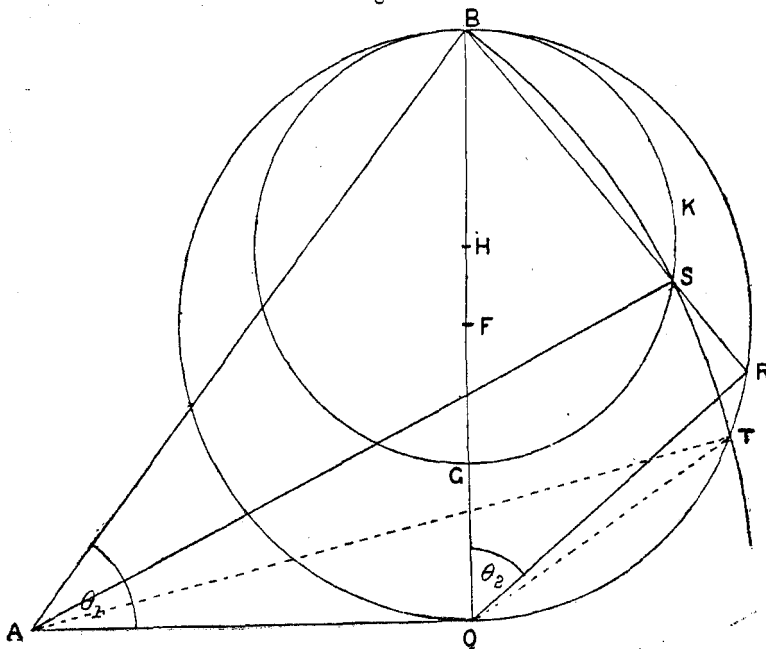
It follows then that, whatever the value of  $\theta_1$  the point K lies on the circle BKG when BK represents the back E.M.F. in the primary due to mutual induction, OK the resultant back E.M.F. due to mutual and self-induction, and  $AK = E_o$  the impressed potential difference.

With centre A describe a circle touching the circle BKG, then K is the point of contact, since AK passes through H the centre of circle BKG. Therefore AK is the maximum value of  $E_o$ , and the angle BOD the value of  $\theta_2$ , which gives this maximum value for  $E_o$ . If there were no leakage the dotted line AQ would represent the maximum value of  $E_o$ , and the angle BOQ the corresponding value of  $\theta_2$ .

It is easily seen from the figure that the effect of leakage is to diminish the maximum value of  $\frac{E_o}{E_i}$  or of  $\frac{1}{I_1}$ , and to diminish also the value of  $\theta_2$ , and consequently that of  $\alpha_2$ , necessary to obtain it.

PROBLEM V. *Same conditions as Problem IV.; to obtain the critical value of  $\alpha_2$ .*

Fig. 12.



The first part of the construction is the same as in Problem IV.

With centre A and radius AB describe circle BST cutting circles BKG and BTO in S and T respectively. Join BS.

Produce BS to meet circle BTO in R and join OR.

Then the angle BOR is the critical value of  $\theta_2$ , and that of  $\alpha_2$

$$= \tan \theta_2 = \frac{BR}{OR}.$$

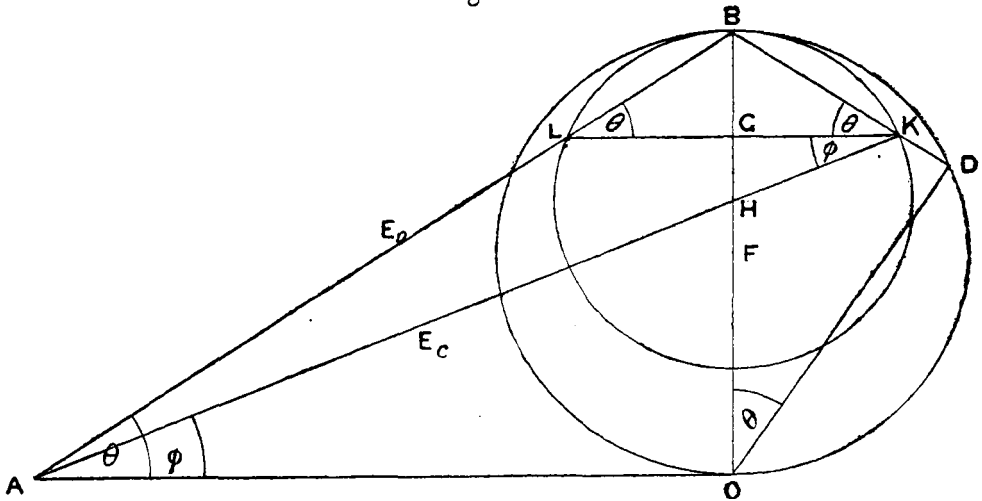
If there were no leakage the angle BOT would be the critical value of  $\theta_2$ .

The proof is obvious from that of Problem IV.

Here, again, the effect of magnetic leakage is to diminish the critical value of  $\theta_2$ , and consequently that of  $\alpha_2$ .

PROBLEM VI. *A transformer with primary and secondary having the same time-constants, to find the value of  $\alpha$  to obtain  $\frac{I}{I_1}$  a maximum, the magnetic leakage or value of  $\beta$  being known.*

Fig. 13.



This is the same as Problem II., only there is magnetic leakage.

*Construction.* Draw a circle BDO with centre F and diameter BO.

Draw OA at right angles to OB.

In OB take a point H such that  $\frac{BH}{BF} = \beta^2$ .

Take a point G in BH such that  $\frac{HG}{GB} = \frac{OH}{OB}$ .

(For method of obtaining G geometrically see Problem VII.)

With centre H and radius HB describe circle LBK.

Through G draw LGK parallel to OA and meeting the circle LBK in L and K.

Join BL and produce it to meet OA in A.

Join BK and produce it to meet circle BDO in D. Join OD and AK.

Then  $\frac{AK}{AB}$  is the maximum value of  $\frac{I}{I_1}$ ; and the angle

$BAO = BOD = \theta$  is the required angle;

and

$\tan \theta = \frac{OB}{OA}$  is the required value of  $\alpha$ .

*Proof.* Obviously the angle LBG = the angle KBG.

$\therefore$  the angle BAO = the angle BOD =  $\theta$ .

Call H the point where AK cuts OB.

Now since LK is parallel to OA, the angle BAO = the angle BLK = BKL =  $\theta$ ; and the angle KAO = LKA =  $\phi$  say.

Now

$$\frac{\tan \phi}{\tan \theta} = \frac{OH}{OB} = \frac{HG}{GB}.$$

But by construction, when  $\frac{OH}{OB} = \frac{HG}{GB}$ , H is the centre of the circle LBK.

$\therefore$  AK is the longest line that can be drawn from A to the circle LBK.

$$\therefore \frac{AK}{AB} = \frac{E_c}{E_0} = \frac{I}{I_1} \text{ is a maximum.}$$

PROBLEM VII.—Same conditions as Problem VI.; to obtain the critical value of  $\alpha$ .

*Construction.* Draw with centre F a circle BDO and take a diameter BO, draw OA at right angles.

In BO take H such that  $\frac{BH}{BF} = \beta^2$ .

With H as centre and radius HB describe circle BNGL.



*Proof.* Obviously the angle  $BAO = BOD$  as before.  
Join  $AH$  and produce it to meet  $BD$  in  $R$ . Join  $GN$ .  
Now

$$\frac{GK}{KB} = \frac{\tan GKN}{\tan KNB};$$

but

$$KNB = BLK = BAO = \theta.$$

Again,

$$\frac{OH}{OB} = \frac{\tan OAH}{\tan BAO};$$

but

$$\frac{GK}{KB} = \frac{OH}{OB} \text{ by construction.}$$

$$\therefore \frac{\tan OAH}{\tan \theta} = \frac{\tan GKN}{\tan \theta};$$

$\therefore$  the angle  $GKN =$  the angle  $OAH$ , or  $AR$  is parallel to  $GN$ ;

$\therefore$  the angles at  $R$  are right angles and  $BR = RN$ .

Also  $AR$  is common.

$\therefore AB = AN$  or  $E_c = E_0$ .

In all the above geometrical constructions the phase-angle between the secondary and primary currents is represented by  $\frac{\pi}{2} + \theta_2$ , or, when the primary and secondary have the same time-constant, by  $\frac{\pi}{2} + \theta$ .

#### DISCUSSION.

Prof. MINCHIN showed that the impedances might be represented by two hyperbolas, having  $p$  as abscissæ and the squares of the impedance as ordinates. These could be readily constructed from the data given. A line representing the primary inductance, drawn on the same diagram, intersects one hyperbola, showing that the impedance has always a maximum value. By a simple construction the phase-angle between the primary and secondary currents could be determined for any given conditions.

Dr. SUMPNER observed that increased impedance on closing the secondary necessarily meant a decrease in the lag of the primary current behind the primary P.D.



Mr. BLAKESLEY was pleased to see the geometrical method of such service, and thought it much simpler than the analytical one. The reason why increased impedance on closing the secondary of ordinary transformers had not been noticed, was because their lag-angles were very large. In a figure published some years ago to represent the actions of transformers, the angles he had chosen were such as would make the primary impedance increase on closing the secondary. Giving an expression connecting the primary currents on open and closed secondary respectively, he now showed that to get increased impedance the sum of the lag-angles in primary and secondary must exceed  $90^\circ$ . To get large power in the secondary, the primary lag should be nearly  $90^\circ$  and the secondary about  $45^\circ$ . He also pointed out that some of the figures in the paper might be simplified considerably.

Prof. PERRY said he had long had the impression that if a sufficiently small current were taken from the secondary, increased impedance would be observable in all cases, and he quoted some numbers he had given in the Phil. Mag. for 1891, showing a decided increase.

Mr. RIMINGTON, in reply, said he was not aware that the effect he had now brought forward had been observed previously. The result was completely worked out analytically before using geometrical methods.

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## II. *A New Mode of making Magic Mirrors.*

*By J. W. KEARTON\*.*

THE first explanation that occurred to me on seeing the Japanese mirror about fourteen months ago was that the face might bear directly invisible differences in polish, which a powerful beam of light would probably convert into visible ones by reflexion on to the screen. To produce such minute differences, it was my intention to take pairs of different metals closely agreeing in colour and reflective power, as silver and platinum, and to deposit electrically in the form of some easily recognizable figure a thin coating of the one

\* Read January 26, 1894.