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463. On Some Arithmetical Conventions
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with peculiar satisfaction that one so closely identified with the movement for the Higher Education of Women found himself able to give the benefit of his enthusiasm and his experience to his fellow governors of Newnham. His work upon the Councils of the London Mathematical Society and of the Mathematical Association, of which he was for many years a distinguished and active member, was highly appreciated. Pupils were constantly passing through his hands to take up their posts as teachers of Mathematics, and he was thus enabled indirectly to sow the good seed which bore fruit eventually in the success of the movement for reform in the teaching of this subject. His own teaching was sound and thorough, and based upon principles which, as a writer in a contemporary has remarked, "received remarkable vindication in the record of his family." His only son showed promise of being one of the most remarkable men of his time, and the three daughters who have survived him are also evidence of the combined effect of nature and nurture upon the members of a highly gifted family. His life was not without its tragedies, but he faced the decrees of Fate with signal fortitude, and in this, as in all else, was a noble example to those who were admitted to the privilege of his friendship.

MATHEMATICAL NOTES.

463. [V. 1. a. δ, ε.] *On Some Arithmetical Conventions.*

A contention that the value of $7-3\times 2$ may be either 1 or 8 recently surprised me, and has led me to look somewhat carefully into the conventions as to the sequence in which arithmetical operations are to be performed.

Many arithmetical books, in their chapter on Fractions, lay down three rules of interpretation. Stated in their baldest form these rules are :

1. Multiplications and divisions must be performed before additions and subtractions. This assigns a meaning to $27\pm 5\times 3$, say $a\pm b\times c$.
2. Multiplications and divisions must be performed in order (from left to right). This assigns a meaning to $a\div b\times c$, viz. $\frac{ac}{b}$.
3. The word 'of' is, however, equivalent to a bracket. According to this,

$$a\div b \text{ of } c \text{ means } \frac{a}{bc}.$$

It will be convenient to state at once the conclusions I have reached, before entering into the arguments on the subject.

These are, that the Rule 1, though not always happily expressed, is a rule of fundamental importance, and is essential to the harmony of arithmetic and algebra ; but that Rules 2 and 3 are of an artificial character, that they are not necessary and that they cannot be defended.

There is little doubt that Rule 1 has suffered from being found in bad company. The case for its separate existence seems to be (apart from mere authority, though Tannery and Workman both adopt it) :

(*a*) Algebraic and arithmetical expressions consist of terms.

(*b*) Apart from brackets + and − separate terms.

(*c*) There is no essential distinction between ab, $a.b$ and $a\times b$. Each denotes the product of a and b.

If a and b are numbers expressed in figures, the sign × must be used [for . may be confused with a decimal point, *Enc. des Sc. Math.* I, 1, i, p. 40]. Contrast $2{\cdot}4\times 3{\cdot}1$, $2{\cdot}4\,.\,3{\cdot}1$ and $2{\cdot}43{\cdot}1$.

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The rule can now be stated in the following way. Apart from brackets, terms must be evaluated before they can be combined by addition or subtraction. A term may contain several factors.

Examples illustrating the necessity of the rule.

(i) Evaluate $a-bc^2$ when $a=10$, $b=3.14$, $c=1{\cdot}4$.

$$\text{The expression}=10-3.14\times(1{\cdot}4)^2, \text{ etc.}$$

(ii) A square plate, side 10 cm., has a rectangular projection 3·5 by 1·7 cm. What is the total area of the plate and projection?

$$\text{Area}=100+3{\cdot}5\times 1{\cdot}7, \text{ etc.}$$

Observe how the beginner will naturally build up the result term by term.

It will be found on examination of the current text-books that many of them, though they state the Rule 1 in the chapter on Fractions, employ it at a very much earlier stage; *e.g.* in the chapter on Multiplication they may ask pupils to find the value of such expressions as

$$3\times 20+17, \quad 18\times 112+28, \quad 7\times 12+9.$$

[Some apparently would maintain that $3\times 20+17$ is clear, but $17+3\times 20$ ambiguous—another odd consequence of abandoning Rule 1.]

The objectors to the rule contend that it is always possible to avoid any shadow of doubt as to the meaning of an expression by the free use of brackets:

$$(ax)+(by)+c, \quad (18\times 112)+(28\times 19), \text{ etc.}$$

To this I reply that few experienced teachers will be found to maintain that piling up brackets makes things easier for a beginner.

Moreover, when the young student comes to algebraic expressions of the form $a-bc$ or $1+xy$, a nasty little trap has been laid for him if he has been led to suppose that $17-3\times 4$ or $11+4{\cdot}1\times 3{\cdot}2$ are capable of two interpretations.

As regards Rule 2 it is to be noted that it is a peculiarly English rule. The sign of division, $\div$, is practically unknown abroad. Thus Tannery, *Leçons d'Arithmétique*, lays down Rule 1 explicitly, but has no Rule 2, for he does not need it. $\frac{ab}{c}$ is shorter and plainer than $a\times b\div c$, and *a fortiori* than $a\div c\times b$.

Rule 3 is still more obviously unnecessary, is equally insular, and seems a mere device for catching the unwary.

I agree in principle that it is an important part of education in mathematics to learn not to be unwary—but the subject is one of considerable extent, and offers other opportunities for the inculcation of care. C. S. Jackson.

I cordially endorse all that Mr. Jackson says. (1) The recognition of terms in such an expression as $a\times b+c\times d+e\times f$ is absolutely fundamental, and brackets are unnecessary and superfluous. There are so many cases in which brackets are needed that they should be avoided when their presence is not required. (2) Such an expression as $a\div b\times c$ is really ambiguous, because if it means $a\times c\div b$ it should be so written, or be written $(a\div b)\times c$. The first mode of writing it is too much like a deliberate trap. Similarly, if it means $a\div(b\times c)$ it should be so written, unless of course the fractional form $\frac{a}{b\times c}$ is adopted, which, however, when b and c are themselves fractional, has a needlessly complicated appearance.

Brackets should be used to prevent real ambiguities, not to spoon-feed careless pupils.

(3) In some quarters there is apparently a desire to ignore Rule 1, and to extend Rule 2 to a series of numbers connected by the four signs $+$, $-$, $\times$, $\div$. This would make a hopeless antagonism between arithmetic and algebra, and would make an arithmetical expression such as $a\times b+c\times d+e\times f$ equivalent to $abdf+cdf+ef$ instead of merely $ab+cd+ef$. No words seem strong enough

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to use in repudiating such an interpretation. Even if restricted to × and ÷, Rule 2 is uncomfortably like a trap, and, if extended to + and − as well, it leads to an absurdity. I maintain therefore that the rule that operations should be performed from left to right (which might be called the chain-rule) is erroneous in some cases and a mere trap in others, and should be expunged.

P.S.—May I add that I am sorry to see so great an authority on printing as Professor G. H. Bryan advocating that $n(n+1)(n+2)$ should be printed as $n \,.\, n+1 \,.\, n+2$ [see *Gazette*, Jan. 1916, p. 220, § 4]. This is neither the chain-rule nor any other rule, and the saving in expense is negligible. It is a case in which brackets really are needed.

I am glad to see, on the other hand, from § 6 on the same page, that he does not apply the chain-rule to the interpretation of $p+q/2$, or $r/3+s/4$, but obeys the Term Rule. ALFRED LODGE.

464. [D. z. d.; V. a. ε.] *On the Successive Convergents of a Continued Fraction.*

The following may be of interest to teachers of algebra. In proving the relations between successive convergents of a continued fraction

$$a_1 + \frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{+b_n}+\cdots$$

it is customary to proceed as follows. We first prove

$$p_1/q_1 = a_1/1\,;\quad p_2/q_2 = (a_1b_2+a_2)/b_2\,;$$
$$p_3/q_3 = (a_1b_2b_3+a_1a_3+a_2b_3)/(b_2b_3+a_3) = (b_3p_2+a_3p_1)/(b_3q_2+a_3q_1).$$

We thus assume

$$p_n = b_np_{n-1}+a_np_{n-2}\,;\quad q_n = b_nq_{n-1}+a_nq_{n-2},$$

and proceed to prove it by induction.

This is rather a big jump for the beginner. In any case, a direct proof is always to be preferred to an induction. The following proof is simple and direct. It is of course assumed that there is no cancelling.

It is obvious that

$$p_n/q_n = (b_nu+a_nv)/(b_nu'+a_nv'),\quad n > 1,$$

u, v, u', v' being some numbers. To get p_{n+1}/q_{n+1} we use

$$\frac{a_n}{b_n+}\frac{a_{n+1}}{b_{n+1}},\quad \textit{i.e.}\ a_nb_{n+1}/(b_nb_{n+1}+a_{n+1}),$$

instead of a_nb_n. Hence

$$p_{n+1}/q_{n+1} = \{(b_nb_{n+1}+a_{n+1})u+a_nb_{n+1}v\}/\{(b_nb_{n+1}+a_{n+1})u'+a_nb_{n+1}v'\}$$
$$= (b_{n+1}p_n+a_{n+1}u)/(b_{n+1}q_n+a_{n+1}u').$$

To get p_{n+2}/q_{n+2}, we change a_{n+1}/b_{n+1} into $a_{n+1}b_{n+2}/(b_{n+1}b_{n+2}+a_{n+2})$.

Hence

$$p_{n+2}/q_{n+2} = \{(b_{n+1}b_{n+2}+a_{n+2})p_n+a_{n+1}b_{n+2}u\}/\{(b_{n+1}b_{n+2}+a_{n+2})q_n+a_{n+1}b_{n+2}u'\}$$
$$= (b_{n+2}p_{n+1}+a_{n+2}p_n)/(b_{n+2}q_{n+1}+a_{n+2}q_n).$$

Thus for $n=3, 4, 5, \ldots$, we have

$$p_n = b_np_{n-1}+a_np_{n-2},$$
$$q_n = b_nq_{n-1}+a_nq_{n-2}.$$

S. BRODETSKY.

465. [V. 1. a. λ.] *A simple method of applying the equation*

$$y = a_0+a_1x+a_2x^2+a_3x^3,\ \textit{etc.},$$

to curves, with applications.

Ordinarily where the first, second or higher differentials of curves have to be plotted, at least for the higher derivatives, the only satisfactory

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