Stokes' law to liquid spheres of diameters varying from 30 to 50 times the mean free path of air molecules.

3. The results obtained by this method taken in connexion with Rutherford's experiments seem to constitute experimental verification of Stokes' law for these drops.

4. Positively charged drops of water and alcohol are found by direct measurement to carry charges which are multiples of $4{\cdot}65 \times 10^{-10}$, and all of the multiples from 2 to 6 inclusive have been obtained.

5. The mean of the five most reliable determinations of e is $4{\cdot}69 \times 10^{-10}$. The corresponding value of n (the number of molecules in 1 cubic cm. of gas at 0° C., 76 cm. pressure) is $2{\cdot}76 \times 10^{19}$: that of N (the number of molecules in a gram-molecule) is $6{\cdot}18 \times 10^{23}$: that of ϵ ($=3/2 \frac{RT}{N}$, the kinetic energy of agitation in ergs of a molecule at 0° C., 76 cm. pressure) is $2{\cdot}01 \times 10^{-16}$; that of m (the mass in grams of an atom of hydrogen) is $1{\cdot}62 \times 10^{-24}$.

Ryerson Laboratory,
University of Chicago,
October 9, 1909.

XXIII. *The Asymptotic Expansions of Bessel Functions.* *By* J. W. Nicholson, *M.A., D.Sc., Isaac Newton Student in the University of Cambridge**.

MANY physical problems depend, for their final solution, upon a knowledge of the approximate values of Legendre and Bessel functions for a large range of their argument and order. In the case of the Bessel functions, investigators† have almost entirely confined their attention to those special types in which the order n is small, though the argument z may be large or small.

A treatment of the more general problem presented when n is also large has been given by Lorenz‡, but only when n is half an odd integer. The immediate object of Lorenz was to obtain some expansions necessary for his investigation of the scattering of light by a glass sphere, in which, as in most problems of this type, only Bessel functions expressible in finite form are required. His results were first approximations

* Communicated by the Author. Read before the British Association, Dublin, 1908.

† Poisson, *Journal de l'Ecole*, 1823; Stokes, Camb. Phil. Trans. 1856; Hankel, *Math. Ann.* i. 1869; Lipschitz, Crelle, 1859; and others.

‡ *Œuvres Scientifiques*, vol. i. p. 435 *et seq.*

only, and include three cases: (1) z and n very large, but $z-n$ not of low order in comparison; (2) z and n very large, and $n-z$ not of low order; and (3) z and n nearly equal. The limits of validity were left very doubtful in each case, and especially in (3), there were many points in the investigation which cannot bear examination. In a paper by the author*, the defect in this case was indicated, and expansions deduced when n and z do not differ by an amount of higher order than $z^{\frac{1}{3}}$, whether n be greater or less than z. These results are general, and hold for all large real values of n. They were subsequently† applied to the calculation of a table for the function $J_n(z)$ in this case, based upon Airy's‡ tabulation of a type of integral occurring in physical optics.

The Bessel functions of nearly equal argument and order may be reduced to an approximate dependence on this and an associated integral, and thence also to Bessel functions of small argument and fractional order $\frac{1}{3}$, whose tabulation is readily effected. In another paper §, the special case of restricted order of Lorenz has been further investigated when the order is less than the argument, and a type of expansion obtained which can be used to a degree of accuracy determined only by its convergence.

The consideration of corresponding expansions for the remaining cases of large real argument or order is the object of this paper. A scheme is developed which will furnish the approximate values of the functions in all cases in which one or both of the magnitudes n and z is large, and both are real. The order is not restricted in any other way. Some interesting analytical results appear in the course of the work, and a general theory is indicated, applicable to all solutions of differential equations of the second order which can be expressed in series whose general term is known.

The Associated Equation of the Third Order.

If (y_1, y_2) are two independent solutions of a differential equation of the second order with invariant I, so that

$$\left(\frac{d^2}{dz^2}+\mathrm{I}\right)(y_1, y_2)=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Phil. Mag. Aug. 1908.

† Phil. Mag. July 1909.

‡ Airy, Camb. Phil. Trans. vi. p. 379; viii. p. 595.

§ Phil. Mag. Dec. 1907.

we shall give the name "asymptotic substitution" to any one of the following pairs of equations :—

$$\begin{array}{lll} (1)\ y_1 = R^{\frac{1}{2}} \sin \rho & y_2 = R^{\frac{1}{2}} \cos \rho & \\ (2)\ y_1 = S^{\frac{1}{2}} \sinh \sigma & y_2 = S^{\frac{1}{2}} \cosh \sigma & \\ (3)\ y_1 = T^{\frac{1}{2}} e^t & y_2 = T^{\frac{1}{2}} e^{-t}, & \quad . \quad . \quad . \quad (2) \end{array}$$

where $(R\, S\, T\, \rho\, \sigma\, t)$ are functions of z.

It appears at once from (1) that if dashes denote differentiations with respect to z,

$$y_2 y_1{}' - y_1 y_2{}' = C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where C depends on the two solutions chosen.

Thus by the first asymptotic summation,

$$R^{\frac{1}{2}} \cos \rho \left(\frac{R' \sin \rho}{2R^{\frac{1}{2}}} + R^{\frac{1}{2}} \rho' \cos \rho \right) - R^{\frac{1}{2}} \sin \rho \left(\frac{R' \cos \rho}{2R^{\frac{1}{2}}} - R^{\frac{1}{2}} \rho' \sin \rho \right) = C$$

or

$$\frac{d\rho}{dz} = \frac{C}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In a similar manner it may be proved that

$$\frac{d\sigma}{dz} = \frac{C}{S}, \qquad \frac{dt}{dz} = \frac{C}{2T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Asymptotic expansions for y_1 and y_2 of any type may, therefore, be obtained when R, S, and T have been found.

But writing, in the equation

$$\frac{d^2 y}{dz^2} + Iy = 0,$$

$$y = u^{\frac{1}{2}} e^v, \qquad \frac{dv}{dz} = \frac{C}{u},$$

then on reduction

$$uu'' - \tfrac{1}{2} u'^2 + 2Iu^2 = C^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The possible values of u are $(\iota\rho, \sigma, t)$, corresponding to the values (R, S, T) of u, and making $\frac{dv}{dz} = (\iota,\ 1,\ \tfrac{1}{2}) \frac{C}{u}$ respectively.

A solution of (6) is therefore S. Moreover, R and T satisfy similar equations with ιC and $\tfrac{1}{2}C$ written for C. But C disappears on differentiation, and the equation becomes linear, yielding

$$u''' + 4Iu' + 2uI' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and (R, S, T) are three independent solutions of this equation, which will be referred to as the associated equation of the third order. It will be recognized as the equation whose solution is a quadratic function of any two solutions of (1).

Definition of the Bessel Functions.

We shall choose, when n is not an integer,

$$y_1 = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_n(z) \quad \ldots \quad (8)$$

$$y_2 = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} \operatorname{cosec} n\pi \, (J_{-n}(z) - \cos n\pi \, J_n(z)) \quad \ldots \quad (9)$$

where, in accordance with the usual notation,

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)}\left(1 - \frac{z^2}{2^2 . \overline{n+1}} + \frac{z^4}{2^2 . 4^2 . \overline{n+1} . \overline{n+2}} \cdots\right), (10)$$

$$J_{-n}(z) = \frac{\sin n\pi \, 2^n \Gamma(n)}{\pi z^n}\left(1 - \frac{z^2}{2^2 . \overline{1-n}} + \frac{z^4}{2^2 . 4^2 . \overline{1-n} . \overline{2-n}} \cdots\right), \quad \ldots \quad (11)$$

with the ordinary semiconvergent expansions* when z, and not n, is large,

$$\left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_n(z) = U_n(z) \sin\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) + V_n(z) \cos\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) \quad (12)$$

$$\left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_{-n}(z) = U_n(z) \cos\left(z + \frac{n\pi}{2} + \frac{\pi}{4}\right) - V_n(z) \sin\left(z + \frac{n\pi}{2} + \frac{\pi}{4}\right) \quad (13)$$

where

$$U_n(z) = 1 - \frac{4n^2 - 1^2 . 4n^2 - 3^2}{2!\,(8z)^2} + \frac{4n^2 - 1^2 . 4n^2 - 3^2 . 4n^2 - 5^2 . 4n^2 - 7^2}{4!\,(8z)^4} \ldots (14)$$

$$V_n(z) = \frac{4n^2 - 1^2}{1!\,8z} - \frac{4n^2 - 1^2 . 4n^2 - 3^2 . 4n^2 - 5^2}{3!\,(8z)^3} + \ldots \quad (15)$$

These values make

$$y_2 y_1' - y_1 y_2' = 1, \quad \ldots \quad (16)$$

so that, for these standard solutions, $C = 1$, $\frac{d\rho}{dz} = \frac{1}{R}$.

This definition also makes $J_{-n}(z) = (-)^n J_n(z)$ when n is integral. By comparison with Hankel's expansions the formulæ (8, 9) are obviously the most suitable for the expansion of $(y_1 \, y_2)$ in the forms $R^{\frac{1}{2}}(\sin \rho . \cos \rho)$ in general,

* Hankel, *l. c.*

when n ceases to be small. As defined above, $J_n(z)$ admits the integral formula*

$$\pi J_n(z) = \int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta - \sin n\pi \int_0^\infty d\theta\,.\,e^{-n\theta - z\sinh\theta}, \quad (17)$$

z being positive, whilst $\pi J_{-n}(z)$ is represented by the same expression with the sign of n changed.

If n be an integer we select, as standard solutions,

$$y_1 = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_n(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$y_2 = -\left(\frac{z}{2\pi}\right)^{\frac{1}{2}} Y_n(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $Y_n(z)$ is Hankel's second solution of Bessel's equation, defined by

$$Y_n(z) = \left(\frac{\partial J_n(z)}{\partial n} - (-)^n \frac{\partial J_{-n}(z)}{\partial n}\right)_{n=\text{integer}}. \quad . \quad . \quad (20)$$

By proceeding to the limit when n is integral in the former case, it is at once obvious that this substitution is the natural continuation of the first. Thus it is again true that

$$y_2 y_1' - y_1 y_2' = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and, therefore, when n is integral, the expansions deduced for $(J_{-n}(z) - \cos n\pi\, J_n(z))\operatorname{cosec} n\pi$ will remain valid for $-\frac{1}{\pi}Y_n(z)$. This is an obvious property of Hankel's expansions when n is small. When z is small, $Y_n(z)$ may be written

$$Y_n(z) = 2J_n(z)\left\{\gamma + \log\frac{z}{2}\right\} - \left(\frac{z}{2}\right)^{-n}\left(\frac{n-1\,!}{0\,!} + \frac{n-2\,!}{1\,!}\left(\frac{z}{2}\right)^2 + \dots\right)$$
$$- \left(\frac{z}{2}\right)^n \frac{S_n}{n\,!} + \left(\frac{z}{2}\right)^{n+2} \frac{S_1 + S_{n+1}}{n+1\,!\,1\,!} - \dots, \quad . \quad (22)$$

where

$$\gamma = -\cdot 577\dots, \qquad S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n}$$

and negative factorials are to be taken as zero.

The asymptotic expansion when z is large is†

$$\left(\frac{z}{2\pi}\right)^{\frac{1}{2}} Y_n(z) = -U_n(z)\cos\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) + V_n(z)\sin\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right), (23)$$

where U_n, V_n are as before.

* *e. g. vide* Whittaker, Modern Analysis, p. 281.

† Hankel, *Math. Ann.* i. p. 494.

Solutions of the Associated Equation. Values of R, ρ.

If y denote y_1 or y_2 in the substitutions for Bessel functions,

$$(yz^{\frac{1}{2}})'' + \left(1 - \frac{n^2 - \frac{1}{4}}{z^2}\right) yz^{\frac{1}{2}} = 0.$$

Thus
$$\mathrm{I} = 1 - \frac{n^2 - \frac{1}{4}}{z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the associated equation becomes

$$z^3 u''' + (4z^3 - 4n^2 z + z)u' + (4n^2 - 1)u = 0. \quad . \quad (25)$$

Writing
$$u = \Sigma_r \, a_r z^r,$$

the relation between successive coefficients becomes

$$r+1 \,.\, r+2-2n \,.\, r+2+2n \,.\, a_{r+2} = -4ra_r,$$

with an indicial equation

$$s-1 \,.\, s-1+2n \,.\, s-1-2n = 0.$$

The following series solutions therefore exist,

$$u_1 = 1 + \tfrac{1}{2} \,.\, \frac{4n^2 - 1^2}{4z^2} + \frac{1\,.\,3}{2\,.\,4} \,.\, \frac{4n^2 - 1^2 \,.\, 4n^2 - 3^2}{(4z^2)^2} + \dots \quad . \quad . \quad . \quad . \quad (26)$$

$$u_2 = z + \tfrac{1}{2} \frac{z^3}{n^2 - 1^2} + \frac{1\,.\,3}{2\,.\,4} \frac{z^5}{n^2 - 1^2 \,.\, n^2 - 2^2} + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$u_3 = z^{2n+1}\left(1 - \frac{2n+1}{n+1} \frac{z^2}{2\,.\,2n+1} + \frac{2n+1\,.\,2n+3}{n+1\,.\,n+2} \frac{z^4}{2\,.\,4\,.\,2n+1\,.\,2n+2} \dots\right), \quad . \quad . \quad . \quad (28)$$

where if n be an integer, u_2 must be multiplied by the evanescent factor of the denominators, thereby ceasing to be distinct from u_3. For positive real values of n, u_2 and u_3 are convergent for all finite values of z, but u_1 is ultimately divergent except when $2n$ is an odd integer, in which case it terminates.

Proceeding to an examination of u_1, it is seen that when z is infinite in comparison with n, $u_1 = 1$.

But comparing (8, 9) and (12, 23), in this case, $y_1^2 + y_2^2 = 1$, or $\mathrm{R} = 1$ from the first asymptotic substitution. Thus $u_1 = \mathrm{R}$ when z is very great, and being always a linear combination of R, S, and T, which are of similar magnitudes, it must always be R either identically or in an asymptotic sense, not necessarily that of Poincaré.

When n and z are both great, a first approximation to R is obviously

$$R=1+\tfrac{1}{2}\cdot\frac{n^2}{z^2}+\frac{1\cdot 3}{2\cdot 4}\frac{n^4}{z^4}+\dots,$$

or

$$R=\frac{z}{\sqrt{z^2-n^2}} \qquad z>n,$$

which leads to one of Lorenz's expansions, when n is half an odd integer. But the error involved would be very doubtful.

Now writing as in usual notation for Bessel functions of imaginary argument,

$$K_0(\lambda)=\int_0^\infty e^{-\lambda\cosh\phi}\,d\phi, \quad . \quad . \quad . \quad . \quad (29)$$

then it may be shown * that when the series terminates

$$u_1=R=\frac{4z}{\pi}\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt, \qquad (30)$$

or, as a reversible double integral,

$$R=\frac{4z}{\pi}\int_0^\infty\int_0^\infty e^{-2z\sinh t\cosh\psi}\cosh 2nt\,dt\,d\psi. \qquad (31)$$

But this expression remains finite and determinate when n is not half an odd integer, and it may, moreover, be proved by direct substitution that it is still a solution of the same differential equation. Accordingly, it is still the value of R, as it also takes the correct value when $z=\infty$. Thus for all real values of n and z,

$$R=\frac{4z}{\pi}\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt, \quad . \quad . \quad (32)$$

and when $2n$ is not an odd integer, the series u_1, though divergent ultimately, may be used for the computation of the integral.

Expressing the value of R in terms of the Bessel functions, we deduce, when n is not an integer,

$$J_n^2(z)+J_{-n}^2(z)-2J_n(z)J_{-n}(z)\cos n\pi=\frac{8}{\pi^2}\sin^2 n\pi\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt, \quad . \quad . \quad . \quad . \quad (33)$$

and when n is an integer,

$$Y_n^2(z)+\pi^2J_n^2(z)=8\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt. \qquad (34)$$

* Phil. Mag. Dec. 1907.

Some of the special cases in which $2n$ is an odd integer are very interesting, as the integral can then be expressed in terms of trigonometric functions.

Since by (4, 16)

$$\frac{d\rho}{dz}=\frac{1}{\mathrm{R}},$$

and when z is very great in comparison with n, the usual expansions yield

$$\rho=z-\frac{n\pi}{2}+\frac{\pi}{4},$$

it follows that in general

$$\rho=z-\frac{n\pi}{2}+\frac{\pi}{4}-\int_z^\infty\left(\frac{1}{\mathrm{R}}-1\right)dz. \quad . \quad . \quad (35)$$

Second Solutions. Values of T, t *when* n *is not integral.*

The second series solution, when n is not integral, is of the absolutely convergent form

$$u_2=z+\tfrac{1}{2}\frac{z^3}{n^2-1^2}+\frac{1.3}{2.4}\frac{z^5}{n^2-1^2.\,n^2-3^2}+\dots.$$

Now if

$$v_m=\int_0^\pi \sin 2nx\sin^{2m}x\,dx,$$

where n is not, and m is an integer, then it is readily shown that

$$v_m=\frac{-2m.\,2m-1}{4.\,n^2-m^2}v_{m-1},$$

$$=\frac{(-)^m\,2m\,!}{2^{2m}n^2-m^2.\,n^2-m-1^2.\,\dots\,n^2-1^2}\int_0^\pi \sin 2nx\,dx,$$

whence

$$u_2=\frac{2nz}{1-\cos 2n\pi}\int_0^\pi\left(1-\frac{(2z\sin x)^2}{2^2}+\frac{(2z\sin x)^4}{2^24^2}\dots\right)\sin 2nx\,dx$$

$$=\frac{2nz}{1-\cos 2n\pi}\int_0^\pi \sin 2nx\,.\,\mathrm{J}_0(2z\sin x)\,dx. \quad (36)$$

It remains to identify u_2. Now making a substitution (a modification of the third asymptotic substitution),

$$y_1=\left(\frac{\pi z}{2}\right)^{\frac{1}{2}}\mathrm{J}_n(z),\quad y_3=\left(\frac{\pi z}{2}\right)^{\frac{1}{2}}\operatorname{cosec} n\pi.\mathrm{J}_{-n}(z),\quad y_1y_3=\mathrm{T}_1.\ (37)$$

Then, by direct multiplication, an ascending series is obtained whose leading terms are

$$y_1 y_3 = \frac{z}{2n} + \frac{1}{2}\frac{z^3}{2n \, . \, n^2 - 1^2} \dots,$$

and which, moreover, must satisfy the associated equation. Thus by comparison of series

$$u_2 = 2n\mathrm{T}_1,$$

and

$$\mathrm{T}_1 = \frac{z}{1 - \cos 2n\pi} \int_0^{\pi} \sin 2nx \, \mathrm{J}_0(2z \sin x)\, dx. \quad . \quad (38)$$

This result was known to Lorenz for the case in which $2n$ is an odd integer. The substitution (36) is more convenient than (9) when n is non-integral and greater than z. Thus for all real values of z and n, the latter not being an integer,

$$\mathrm{J}_n(z)\mathrm{J}_{-n}(z) = \frac{1}{\pi \sin n\pi} \int_0^{\pi} \sin 2nx \, . \, \mathrm{J}_0(2z \sin x)\, dx. \quad (39)$$

When n is an integer, evaluating the form then presented, by an obviously legitimate process,

$$(-)^n \mathrm{J}_n^2(z) = \frac{1}{\pi} \frac{\partial}{\partial n} \int_0^{\pi} \sin 2nx \, . \, \mathrm{J}_0(2z \sin x)\, dx \div \frac{\partial}{\partial n} \sin n\pi$$

$$= \frac{(-)^n}{\pi^2} \int_0^{\pi} 2x \cos 2nx \, \mathrm{J}_0(2z \sin x)\, dx$$

$$= \frac{(-)^n}{\pi^2} \; \frac{\pi}{2} \int_0^{\pi} 2 \cos 2nx \, \mathrm{J}_0(2z \sin x)\, dx,$$

or $$\mathrm{J}_n^2(z) = \frac{1}{\pi} \int_0^{\pi} \sin 2nx \, . \, \mathrm{J}_0(2z \sin x)\, dx, \quad n = \text{integer}. \; (40)$$

This follows otherwise from a result given by Neumann *. The determination of T (T_1 being infinite) when n is integral is somewhat difficult, for u_2 and u_3 cease to be distinct. For this determination a more direct method is useful †.

It was shown that if t be defined by the third asymptotic substitution,

$$\frac{\partial t}{\partial z} = \frac{1}{2\mathrm{T}}.$$

* *Cf.* Gray and Matthews, p. 28.

† *Cf.* the formula for $\mathrm{J}_n(z)\mathrm{Y}_n(z)$, *infra.*

Now near $z=0$, $e^{2t}=\frac{y_1}{y_2}$ becomes zero like $\frac{z^{2n}\pi}{2^{2n}\Gamma(n)\Gamma(n+1)}$, or near $z=0$,

$$t=n\log\tfrac{1}{2}z+\frac{1}{2}\log\frac{\pi}{n\Gamma^2(n)},$$

and therefore in general

$$t=\frac{1}{2}\log\frac{\pi}{n\Gamma^2(n)}-n\log 2+n\log z+\int_0^z\left(\frac{1}{2\mathrm{T}}-\frac{n}{z}\right)dz, \quad (41)$$

this being the only possible function satisfying all the conditions. In a similar way, if t_1 corresponds to T_1 as t to T,

$$t_1=\frac{1}{2}\log\frac{\pi}{n\Gamma^2(n)}-n\log 2+n\log z+\int_0^z\left(\frac{1}{2\mathrm{T}_1}-\frac{n}{z}\right)dz. \quad (42)$$

These two relations cease to be distinct when n is very great in comparison with z, for it will appear that T_1 and T only differ by an amount which is exponentially evanescent when n tends to infinity.

The third series satisfying the associated equation is

$$u_3=z^{2n+1}\left(1-\frac{2n+1}{n+1}\,\frac{z^2}{2\,.\,2n+1}+\frac{2n+1\,.\,2n+3}{n+1\,.\,n+2}\,\frac{z^4}{2\,.\,4\,.\,2n+1\,.\,2\,.\,n+2}-\dots\right)$$

$$=z^{2n+1}\int_0^\pi\left(\sin^{2n}\theta-\frac{z^2\sin^{2n+2}\theta}{1\,!\,2n+1}+\frac{z^4\sin^{2n+4}\theta}{2\,!\,2n+1\,.\,2n+2}-\dots\right)d\theta\div\int_0^\pi\sin^{2n}\theta\,d\theta$$

$$=\frac{\Gamma(n+1)\,\Gamma(2n+1)}{\pi^{\frac{1}{2}}\Gamma(n+\frac{1}{2})}\,z\int_0^\pi \mathrm{J}_{2n}(2z\sin\theta)\,d\theta. \quad (43)$$

But

$$y_1{}^2=\frac{\pi z}{2}\mathrm{J}_n^2(z)=\frac{\pi\,.\,z^{2n+1}}{2^{2n+1}\Gamma^2(n+1)}\left(1-\frac{z^2}{2\,.\,n+1}+\dots\right),$$

and therefore by comparison, and by the differential equation,

$$y_1{}^2=\frac{\pi u_3}{2^{2n+1}\Gamma^2(n+1)}=\tfrac{1}{2}z\int_0^\pi \mathrm{J}_{2n}(2z\sin\theta)\,d\theta,$$

by the use of a well-known property of gamma functions.

Here $2n$ must be greater than -1, in order that the integral may be finite. Thus for all real values of z, and values of n greater than $-\frac{1}{2}$,

$$\pi\mathrm{J}_n^2(z)=\int_0^\pi \mathrm{J}_{2n}(2z\sin\theta)\,d\theta, \quad (44)$$

which is a known result for integral values of n.

The value of S, in the second asymptotic substitution, may now be expressed, but this substitution is of small importance, as it is identical with the third in all useful cases.

Again, by (44), if $n<\frac{1}{2}$,

$$\pi J_{-n}^2(z)=\int_0^{\pi} J_{-2n}(2z\sin\theta)\,d\theta. \quad . \quad . \quad . \quad (45)$$

Thus, if the argument of the functions be constantly z when not expressed,

$$2\pi J_n\frac{\partial J_n}{\partial n}=\int_0^{\pi}\frac{\partial}{\partial n}J_{2n}(2z\sin\theta)\,d\theta$$

$$2\pi J_{-n}\frac{\partial J_{-n}}{\partial n}=\int_0^{\pi}\frac{\partial}{\partial n}J_{-2n}(2z\sin\theta)\,d\theta.$$

Thus

$$2\pi J_n\left(\frac{\partial J_n}{\partial n}-\frac{J_{-n}}{J_n}\frac{\partial J_{-n}}{\partial n}\right)=\int_0^{\pi}\left\{\frac{\partial}{\partial n}J_{2n}(2z\sin\theta)-\frac{\partial}{\partial n}J_{-2n}(2z\sin\theta)\right\}d\theta,$$

or, when n is made integral, its only possible value being zero,

$$\pi J_0(z)Y_0(z)=\int_0^{\pi}Y_0(2z\sin\theta)\,d\theta. \quad . \quad . \quad (46)$$

A more general result is proved in the next section.

The Formula for $J_n(z)\,Y_n(z)$.

An expression for the product of two Bessel functions of different types, when n is integral, has been given by Neumann *. But it is somewhat unsuitable for our purpose, and an alternative is now developed which can, however, be formally identified with that of Neumann.

By (43), the argument being z unless otherwise specified,

$$2\pi J_n\frac{\partial J_n}{\partial n}=\int_0^{\pi}\frac{\partial}{\partial n}J_{2n}(2z\sin\theta)\,d\theta.$$

But

$$\pi J_{2n}(w)=\int_0^{\pi}\cos(w\sin\phi-2n\phi)\,d\phi$$
$$-\sin 2n\pi\int_0^{\infty}d\phi\, e^{-2n\phi-w\sinh\phi}.$$

* *Bessels'che Functionen*, p. 65.

Thus when n is an integer,

$$\pi\frac{\partial}{\partial n}J_{2n}(w)=2\int_0^{\pi}\phi\sin(w\sin\phi-2n\phi)\,d\phi$$
$$-2\pi\int_0^{\infty}d\phi\,e^{-2n\phi-w\sinh\phi},$$

so that on reduction

$$2J_n\frac{\partial J_n}{\partial n}=\tfrac{1}{2}\int_0^{\pi}\cos 2n\phi\,H_0(2z\sin\phi)\,d\phi-\frac{2}{\pi}\int_0^{\pi}\phi\sin 2n\phi\,J_0(2z\sin\phi)\,d\phi$$
$$-\frac{2}{\pi}\int_0^{\infty}\int_0^{\pi}d\theta\,d\phi\,e^{-2n\phi-2z\sin\theta\sinh\phi}, \quad . \quad . \quad . \quad (47)$$

where $H_0(w)$ is Struve's function * defined by

$$H_0(w)=\frac{2}{\pi}\int_0^{\pi}\sin(w\sin\theta)\,d\theta. \quad . \quad . \quad . \quad (48)$$

Again,

$$\pi J_n J_{-n}=\int_0^{\pi}J_0(2z\sin x)\frac{\sin 2nx}{\sin n\pi}dx,$$

so that when n is integral,

$$\pi\left(J_n\frac{\partial J_{-n}}{\partial n}+(-)^n J_n\frac{\partial J_n}{\partial n}\right)=\int_0^{\pi}J_0(2z\sin x\,.\frac{\partial}{\partial n}\frac{\sin 2nx}{\sin n\pi}dx$$
$$=\int_0^{\pi}J_0(2z\sin x)\frac{(2x\cos 2nx\sin n\pi-\pi\cos n\pi\sin 2nx)}{\sin^2 n\pi}dx.$$

The integrand takes the form $\frac{0}{0}$ when n is integral. Therefore evaluating in the usual way, which is obviously legitimate, we obtain

$$(-)^n J_n\frac{\partial J_{-n}}{\partial n}+J_n\frac{\partial J}{\partial n}=\frac{1}{2\pi^2}\int_0^{\pi}(\pi^2-4x^2)\sin 2nx\,J_0(2z\sin x)\,dx.$$

By subtraction with (47), and with the help of the results, true for integral values of n,

$$\int^{\pi}\sin 2nx\,J_0(2z\sin x)\,dx=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\int_0^{\pi}x^2\sin 2nx\,J_0(2z\sin x)\,dx=\pi\int_0^{\pi}x\sin 2nx\,J_0(2z\sin x)\,dx, \quad . \quad . \quad (50)$$

it appears that

$$J_n(z)Y_n(z)=\tfrac{1}{2}\int_0^{\pi}\cos 2nx\,H_0(2z\sin x)dx-\frac{2}{\pi}\int_0^{\infty}d\phi\int_0^{\pi}d\theta\,e^{-2n\phi-2z\sin\theta\sinh\phi}$$
$$. \quad . \quad . \quad (51)$$

* *Cf.* Struve, Wied. *Ann.* Bd. xvi. 1882, p. 1008; Lord Rayleigh, 'Theory of Sound,' § 302.

The value of T when n is integral may now be deduced, and thence t by (41).

Expansions of the First Type.

Returning to the value of R given in (31), it is first to be noticed that the evaluation (asymptotic) of the integral, given in a previous paper *, was in no way dependent upon the restriction of $2n$ to an odd integer. Accordingly, this evaluation may be used in the general case. The same applies to the subsequent treatment of ρ as given by (35) of the present paper.

Thus quoting the values of R and ρ previously obtained, we obtain the following asymptotic expansions when n is less than z, and $z-n$ is not very small:—

When n is not an integer

$$J_n(z)=\left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\sin\rho$$

$$J_{-n}(z)-\cos n\pi\, J_n(z)=\left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\cos\rho\sin n\pi,$$

and when n is an integer,

$$J_n(z)=\left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\sin\rho$$

$$Y_n(z)=-\left(\frac{2\pi R}{z}\right)^{\frac{1}{2}}\cos\rho,$$

where if $n=z\sin\alpha$, defining an angle α,

$$R=\sec\alpha+\frac{\lambda_2}{z^2}\sec^3\alpha+\frac{\lambda_4}{z^4}\sec^4\alpha+\ldots,\quad . \quad . \quad . \quad (52)$$

where

$$\lambda_2=-\frac{1}{2^3},\quad \lambda_4=\frac{27-96n^2}{2^7},\quad \lambda_6=\frac{4640n^2-1125-640n^4}{2^{10}},$$

and

$$4\,.\,s+3\,.\,\lambda_{s+3}+(s+2)^3\lambda_{s+1}+2n^2s\,.\,s+1\,.\,s+2\,.\,\lambda_{s-1}$$
$$+n^4s\,.\,s^2-4\,.\,\lambda_{s-3}=0,\quad . \quad . \quad (53)$$

and every third term of R, beginning with the second, is two orders (in z or n) smaller than those before.

* Phil. Mag. Dec. 1907.

Moreover, if identically

$$1+\mu_2x^2+\mu_4x^4+\ \dots\ =(1+\lambda_2x^2+\lambda_4x^4+\ \dots\ )^{-1}. \quad (54)$$

Then

$$\rho=\tfrac{1}{4}\pi+z(\cos\alpha-\overline{\tfrac{\pi}{2}-\alpha}\sin\alpha)$$
$$-\frac{1}{n}\left\{\frac{\mu_4}{n^2}\tan\alpha-\frac{\mu_6}{n^4}(\tan\alpha-\tfrac{1}{3}\tan^3\alpha)\right.$$
$$\left.+\frac{\mu_8}{n^6}(\tan\alpha-\tfrac{1}{3}\tan^3\alpha+\tfrac{1}{5}\tan^5\alpha)-\dots\right\}, \quad (55)$$

and the second, fifth, &c. terms in the large brackets are each two orders smaller than the preceding.

An identical relation

$$\mu_2-\frac{\mu_4}{n^2}+\frac{\mu_6}{n^6}-\dots=0, \quad . \quad . \quad . \quad . \quad (56)$$

proved later *, has been used in the reduction.

R may be arranged more conveniently for some purposes in the form

$$\mathrm{R}=z\left\{1+\frac{z}{3\,!}\frac{\delta_1\delta_2^2}{2^2}+\frac{z}{5\,!}\frac{\delta_1\delta_2^4}{2^4}+\frac{10z^2}{6\,!}\frac{\delta_1\delta_2^4}{2^4}+\frac{z}{7\,!}\frac{\delta_1\delta_2^6}{2^6}\right.$$
$$\left.+\frac{56z^2}{8\,!}\frac{\delta_1^2\delta_2^6}{2^6}+\frac{280z^3}{9\,!}\frac{\delta_1^3\delta_2^6}{2^6}+\dots\right\}\frac{1}{\sqrt{z^2-n^2}}, \quad (57)$$

where $\delta_1=\partial/\partial z, \quad \delta_2=\partial/\partial n,$

being here given to an order z^{-6} when z and n are of the same order. We shall refer to this system of expansions subsequently as (A).

The Remainder in the Expansion of R.

From the previous paper the remainder after r terms in the expansion of R is

$$\frac{2z}{\pi}\Sigma\int_0^\infty\frac{d\psi}{\lambda^r}\int_0^\infty\left\{\left(\frac{1}{v'}\frac{d}{dt}\right)^r\frac{1}{v'}\right\}v'\,e^{-\lambda v}\,dt, \quad . \quad (58)$$

where

$$v=\sinh t\mp\mu t,\ v'=\cosh t\mp\mu,\ \lambda=2z\cosh\psi,\ \lambda\mu=2n,$$

and Σ denotes an addition of the two values corresponding to the ambiguity.

Now it is well known that the integral

$$\mathrm{I}=\int_0^\infty f(t)\,e^{-\lambda\mathrm{F}(t)}\,dt, \quad . \quad . \quad . \quad . \quad (59)$$

* *Cf.* (67) *infra.*

where λ is large, and f, F are uniform in the range, is represented to an order λ^{-1} by

$$I=\int_0 f(0)\, e^{-\lambda F(t)}\, dt, \quad . \quad . \quad . \quad . \quad (60)$$

or by

$$I=\int_0^\infty t f'(0)\, e^{-\lambda F(t)}\, dt, \quad . \quad . \quad . \quad . \quad (61)$$

according as $f(0)$ is not or is zero. Moreover, $F(t)$ may be expanded, and the first non-vanishing power of t alone retained, if $F(t)$ does not contain λ. Also $F(0)=0$, and F' must not vanish in the ranges except perhaps at $t=0$.

In the present case $\mu=\dfrac{n}{z\cosh\psi}$, and n is less than z, so that $F'=v'$ cannot vanish. The other conditions are obviously satisfied.

Now when $t=0$, $v=0$, and $\left(\dfrac{1}{v'}\dfrac{d}{dt}\right)^r\dfrac{1}{v'}$ is zero when r is odd, and when r is even, it is given by $\dfrac{\pi}{2z}\lambda^{r+1}u_{r+1}$, where u_{r+1} represents the term in R next after the final one retained (in the *second* expression for R) with $\dfrac{1}{\lambda(1\mp\mu)}$ substituted for $\dfrac{\pi}{\sqrt{z^2-n^2}}$, and the summation and integration prefixed. Thus the remainder after r terms only differs by an order λ^{-1} from

$$\Sigma\int_0^\infty d\psi\, u_{r+1}\int_0^\infty v'\, e^{-\lambda v}\, dt, \quad \text{or} \quad \Sigma\int_0^\infty \lambda u_{r+1}\frac{d\psi}{\lambda},$$

when r is odd. When r is even its order is less.

If the rth term of R be therefore denoted by $\pi U_r(z^2-n^2)^{-\frac{1}{2}}$, where U_r is a certain operation, the error involved in stopping at the rth term is of magnitude

$$\frac{1}{\pi}U_{r+1}\int_0^\infty d\psi\left(\frac{1}{z\cosh\psi-n}+\frac{1}{z\cosh\psi+n}\right),$$

or

$$\frac{1}{\pi}U_{r+1}\frac{\pi}{\sqrt{z^2-n^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and is therefore of the same magnitude as the term next after the last retained. The expansion of R is thus asymptotic in the proper sense. That of ρ will also be asymptotic, but less convergent.

Expansions when n *is greater than* z.

When n is non-integral, in the notation of (36),

$$T_1 = \frac{z}{1-\cos 2n\pi}\int_0^\pi \sin 2nx \,.\, J_0(2z \sin x)dx$$

$$= \frac{z}{2\pi(1-\cos 2n\pi)}\int_0^\pi (I_1+I_2)d\phi, \quad . \quad . \quad . \quad (62)$$

where

$$I_1,\ I_2 = \int_0^\pi \sin 2n(x \pm \mu \sin x)dx, \quad . \quad . \quad . \quad (63)$$

$$\mu = \frac{z \sin \phi}{n} > 1$$

Now $\frac{\partial}{\partial x}(x \pm \mu \sin x)$ is never zero, or a very small quantity for any possible values of x and μ in the double range of integration. Therefore the integration of I_1 and I_2 may be effected by the method of the last section, and in so far as the leading terms are concerned,

$$I_1,\ I_2 = \int_0^\pi \frac{1}{1 \pm \mu}(1 \pm \mu \cos x) \sin 2n(x \pm \mu \sin x)dx$$

$$= \frac{1}{1 \pm \mu}\int_0^\pi \sin 2nt\, dt = \frac{1-\cos 2n\pi}{2n \pm 2z \sin \phi}.$$

Thus

$$T_1 = \frac{z}{4\pi}\int_0^\pi d\phi\left(\frac{1}{n+z\sin\phi} + \frac{1}{n-z\sin\phi}\right)$$

$$\doteq \frac{1}{2}\cdot\frac{z}{(n^2-z^2)^{\frac{1}{2}}},$$

on reduction.

The leading term of $\frac{2T_1}{z}$ is therefore $(n^2-z^2)^{-\frac{1}{2}}$. Let its expansion as a series be of the form

$$\frac{2T_1}{z} = \frac{1}{x} + \frac{\nu_2}{x^3} + \frac{\nu_4}{x^5} + \dots, \quad . \quad . \quad . \quad . \quad (64)$$

where $x = (n^2-z^2)^{\frac{1}{2}}$.

It is a solution of the same differential equation as $\frac{R}{z}$ which had the value when $n < z$,

$$\frac{R}{z} = \frac{1}{(z^2-n^2)^{\frac{1}{2}}} + \frac{\lambda_2}{(z^2-n^2)^{\frac{3}{2}}} + \dots,$$

Thus when $n > z$, the real form

$$\frac{1}{x} - \frac{\lambda_2}{x^3} + \frac{\lambda_4}{x^5} \dots, \quad (x^2 = n^2 - z^2)$$

is a solution of the differential equation for $\frac{T_1}{z}$. By comparison, it represents $\frac{2T_1}{z}$ if $\nu_{2s} = (-)^s \lambda_{2s}$; and therefore if $x = (n^2 - z^2)^{\frac{1}{2}}$, and the coefficients λ have the same values as in expansion A,

$$2T_1 = \frac{z}{x} - \lambda_2 \frac{z}{x^3} + \lambda_4 \frac{z}{x^5} - \dots ; \quad . \quad . \quad . \quad (65)$$

and if the coefficients μ are also given by expansion A,

$$\frac{1}{T_1} = \frac{2x}{z}\left(1 - \frac{\mu_2}{x^2} + \frac{\mu_4}{x^4} - \dots\right). \quad . \quad . \quad . \quad (66)$$

We may now prove the existence of the identical relation (55). For

$$\left(\frac{z}{2T_1}\right)_{z=0} = 2n\left(1 - \frac{\mu_2}{n^2} + \frac{\mu_4}{n^4} - \dots\right).$$

But by the integral (37),

$$\left(\frac{z}{2T_1}\right) = 2n;$$

and therefore

$$\mu_2 - \frac{\mu_4}{n^2} + \frac{\mu_6}{n^4} \dots = 0. \quad . \quad . \quad . \quad .$$

This is readily verified to any order by direct substitution of the values of μ calculated from (53).

Now, with the use of the relation $\left(\frac{z}{2T_1}\right)_0 = 2n$, it follows from (41) that

$$t_1 = \tfrac{1}{2} \log \frac{\pi}{n\Gamma^2(n)} - n \log 2 + \left(\frac{z}{4T_1}\right)_0 \log z + \int_0^z dz \left(\frac{1}{2T_1} - \frac{1}{z}\left(\frac{z}{4T_1}\right)_0\right)$$

or

$$t_1 + n \log 2 - n \log z - \tfrac{1}{2} \log \frac{\pi}{n\Gamma^2(n)} = \int_0^z dz \left(\frac{1}{2T_1} - \frac{n}{z}\right)$$

$$= n \int_\beta^\infty \tanh \beta (\tanh \beta - 1) d\beta - \frac{1}{n} \int_\beta^\infty d\beta \left(\mu_2 - \frac{\mu_4}{n^2} \cosh^2 \beta + \dots\right), \quad (68)$$

where $z = n \operatorname{sech} \beta$, $x = n \tanh \beta$.

But

$$\int_{\beta}^{\infty} \tanh\beta(\tanh\beta-1)d\beta = -\log\frac{e^{\beta}}{\cosh\beta} + \tanh\beta + \log 2 - 1;$$

and thus

$$t_1 - n\log z - \tfrac{1}{2}\log\frac{\pi}{n\Gamma^2(n)} + n + n\log\cdot\frac{e^{\beta}}{\cosh\beta} - n\tanh\beta$$

$$= -\frac{1}{n}\int_{\beta}^{\infty} d\beta\left(\mu_2 - \frac{\mu_4}{n^2}\coth^2\beta + \frac{\mu_6}{n^4}\coth^4\beta - \ldots\right)$$

$$= \frac{1}{n}\int_{\beta}^{\infty} d\beta\left(\frac{\mu_4}{n^2}\operatorname{cosech}^2\beta - \frac{\mu_6}{n^4}\operatorname{cosech}^2\beta\coth^2\beta + \ldots\right)$$

by the use of the identical relation, and finally

$$= \left(\frac{\mu_4}{n^2} - \frac{\mu_6}{3n^4} + \frac{\mu_8}{5n^6} - \ldots\right) - \left(\frac{\mu_4}{n^2}\coth\beta - \frac{\mu_6}{3n^4}\coth^3\beta\ldots\right).$$

But by Stirling's series, n being large,

$$n - n\log n - \tfrac{1}{2}\log\frac{\pi}{n\Gamma^2(n)} = \tfrac{1}{2}\log 2 + \frac{B_1}{1.2.n} - \frac{B_2}{3.4\,n^3} + \frac{B_3}{5.6.n^5} - \ldots \quad (6$$

where the B's are Bernoullian numbers *.

And finally

$$t_1 = n(\tanh\beta - \beta) - \tfrac{1}{2}\log 2 - \frac{1}{n^3}\left(\mu_4 - \frac{1}{3}\frac{\mu_6}{n^2} - \ldots\right)$$

$$+ \frac{1}{n^3}\left(\mu_4\coth\beta - \frac{\mu_6}{3n^2}\coth^3\beta\ldots\right) + \frac{B_1}{n}\frac{1\,!}{2\,!} - \frac{B_2}{n^3}\frac{2\,!}{4\,!} + \ldots. \quad (70)$$

It appears at once that $J_n(z)$ is ultimately rapidly evanescent when n greatly exceeds z. Accordingly, T_1 and T become identical, as also t_1 and t.

Thus when n is not integral, and the coefficients λ, μ in the expansion are identical with those of expansion A, then if

$$n > z = z\cos\beta,$$

we have

$$J_n(z) = \left(\frac{2T}{\pi z}\right)^{\frac{1}{2}} e^{t}$$

$$J_{-n}(z) - \cos n\pi J_n(z) = \sin n\pi\,.\left(\frac{2T}{\pi z}\right)^{\frac{1}{2}} e^{-t}$$

where

$$2T\sinh\beta = 1 - \frac{\lambda_2}{n^2}\coth^2\beta + \frac{\lambda_4}{n^4}\coth^4\beta - \ldots; \quad . \quad (71)$$

* *E. g. vide* Whittaker, 'Modern Analysis,' p. 194.

and the second, fifth, ... terms are each two orders in n smaller than those preceding, and

$$t=n(\tanh\beta-\beta)-\tfrac{1}{2}\log_e 2-\frac{1}{n^3}\left(\mu_4-\frac{1}{3}\frac{\mu_6}{n^2}+\frac{1}{5}\frac{\mu_8}{n^4}\ldots\right)$$

$$+\frac{1}{n^3}\left(\mu_4\coth\beta-\frac{\mu_6}{3n^2}\coth^3\beta+\ldots\right)+\frac{B_1}{n}\frac{1!}{2!}-\frac{B_2}{n^3}\frac{2!}{4!}+\frac{B_3}{n^5}\frac{4!}{6!}-\ldots, \quad (72)$$

where 1, $\mu_2 n^{-2}$, $\mu_4 n^{-4}$, decrease in order by n^{-2} at each third member of the series.

This will be called expansion B.

This value of $J_n(z)$ continues to hold when n is an integer, and it may be expected that $-\frac{1}{\pi}Y_n(z)$ will then replace $\{J_{-n}(z)-\cos n\pi J_n(z)\}\operatorname{cosec} n\pi$; but the formal proof is necessary owing to the use of T_1 in place of T. This proof is given in the next section.

Expansions when n *is integral and greater than* z.

In order to verify the result last suggested, concerning the expression for $Y_n(z)$, it is only necessary to prove that if

$$J_n(z)=\left(\frac{2T}{\pi z}\right)^{\frac{1}{2}}e^t,$$

$$Y_n(z)=-\left(\frac{2T\pi}{z}\right.\ e^{-t}.$$

Then the value of T is, to a first order, given by

$$2T=\frac{z}{(n^2-z^2)^{\frac{1}{2}}}. \quad (73)$$

For if this be true, the subsequent proof follows the lines above. Now by (50)

$$J_n(z)Y_n(z)=\tfrac{1}{2}\int_0^\pi\cos 2nx\,.\,H_0(2z\sin x)dx-\frac{2}{\pi}\int_0^\infty d\phi\int_0^\pi d\theta\, e^{-2n\phi-2z\sin\theta\sinh\phi}$$

$$=\tfrac{1}{2}I_1-\frac{2}{\pi}I_2\text{ (say)}. \quad (74)$$

The first approximation to I_1, found in the usual manner, is, when $n>z$, n is large, and not too close to z in value,

$$I_1=\frac{1}{\pi}\int_0^\pi\left(\frac{1}{n-z\sin\phi}-\frac{1}{n+z\sin\phi}\right)d\phi$$

$$=-\frac{4z}{\pi}\int_0^1\frac{dt}{n^2-z^2+z^2t^2}$$

$$=-\frac{4}{\pi\sqrt{n^2-z^2}}\,.\sin^{-1}\frac{z}{n}. \quad (75)$$

To the same order,

$$I_2 = -\int_0^\pi d\theta \cdot \Big[(2n+2z\sin\theta\cosh\phi)^{-1}e^{-2n\phi-2z\sin\theta\sinh\phi}\Big]_0^\infty$$

$$= \tfrac{1}{2}\int_0^\pi \frac{d\theta}{n+z\sin\theta}$$

$$= \frac{1}{\sqrt{n^2-z^2}}\left\{\frac{\pi}{2} - \sin^{-1}\frac{z}{n}\right\}, \text{ on reduction}, \quad . \quad . \quad . \quad (76)$$

Thus

$$J_n(z)Y_n(z) = \tfrac{1}{2}I_1 - \frac{2}{\pi}I_2 = -\frac{1}{\sqrt{n^2-z^2}}, \quad . \quad . \quad (77)$$

giving the proper value for T.

Finally, in expansion B, when n is integral, we write

$$J_n(z) = \left(\frac{2T}{\pi z}\right)^{\frac{1}{2}} e^t, \quad Y_n(z) = -\left(\frac{2T\pi}{z}\right)^{\frac{1}{2}} e^{-t}; \quad . \quad . \quad (78)$$

and the functions (T, t) are those previously given.

The limits of accuracy of expansion B will be the same as those for A, with n and z interchanged.

The transition between different forms of expansion.

It will now be shown that there is no range of large values of n or z for which expansions are yet to be determined, when n and z are real, and that each expansion passes naturally into the other, without the necessity of an intermediate expansion. We shall first define expansion C as that of a previous paper, where $n-z$ is not large in comparison with $z^{\frac{1}{3}}$, viz.

$$J_n(z) = \frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}} f_1(\rho),$$

$$J_{-n}(z) = \frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}}\{f_1(\rho)\cos n\pi + \overline{f_2(\rho)+f_3(\rho)}\sin n\pi\}, \ n \text{ not integral.}$$

$$Y_n(z) = -\left(\frac{6}{z}\right)^{\frac{1}{3}}\{f_2(\rho)+f_3(\rho)\}, \ n \text{ integral}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

where

$$\rho \quad = (n-z)\left(\frac{6}{z}\right)^{\frac{1}{3}}$$

$$f_1(\rho) = \int_0^\infty \cos(w + \rho w)\,dw,$$

$$f_2(\rho) = \int_0^\infty \sin(w^3 + \rho w)\,dw,$$

$$f_3(\rho) = \int_0^\infty dw\, e^{-w^3+\rho w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Now Stokes * has shown that when σ is not small, an asymptotic value of

$$f_1(-\sigma) - \iota f_2(-\sigma) = \int_0^\infty dw\, e^{-\iota w^3 + \iota\sigma w},$$

is given by

$$\frac{\pi^{\frac{1}{2}}}{3^{\frac{1}{4}}\sigma^{\frac{1}{4}}}\, e^{2\iota\left(\frac{\sigma}{3}\right)^{\frac{3}{2}} - \frac{1}{4}2\pi};$$

so that

$$f_1(-\sigma) = \frac{\pi^{\frac{1}{2}}}{3^{\frac{1}{4}}\sigma^{\frac{1}{4}}}\cos\left(2\left(\frac{\sigma}{3}\right)^{\frac{3}{2}} - \frac{1}{4}\pi\right),$$

$$f_2(-\sigma) = -\frac{\pi^{\frac{1}{2}}}{3^{\frac{1}{4}}\sigma^{\frac{1}{4}}}\sin\left(2\left(\frac{\sigma}{3}\right)^{\frac{3}{2}} - \frac{1}{4}\pi\right), \quad . \quad . \quad . \quad (81)$$

where σ is negative.

Moreover, when ρ is positive, and not too small,

$$\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\int_0^\infty \cos\frac{\pi}{2}(w^3 + \rho w)\,dw = \frac{\pi^{\frac{1}{2}}}{2\,.\,3^{\frac{1}{4}}\rho^{\frac{1}{4}}}\, e^{-\frac{2}{3}\sqrt{\frac{\rho^3}{3}}}. \quad . \quad (82)$$

The values of $J_n(z)$ and $Y_n(z)$ of expansion C, when these formulæ are valid, become, on reduction, identical with

$$J_n(z) = \left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\sin\rho,$$

$$Y_n(z) = -\left(\frac{2\pi R}{z}\right)^{\frac{1}{2}}\cos\rho, \quad . \quad . \quad . \quad . \quad (83)$$

* Camb. Phil. Trans. ix., Math. and Phys. Papers, ii. p. 329 *et seq.*

provided that

$$R = \left(\frac{z}{2 \,.\, z - n}\right)^{\frac{1}{2}},$$

$$\rho = \tfrac{1}{4}\pi + \tfrac{2}{3}(z-n)^{\frac{3}{2}}\left(\frac{2}{z}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (84)$$

when n is less than z.

The corresponding values from expansion A are, retaining the leading terms

$$R = \frac{z}{(z^2 - n^2)^{\frac{1}{2}}},$$

$$\rho = (z^2 - n^2)^{\frac{1}{2}} + n \sin^{-1}\frac{n}{z} - \tfrac{1}{2}\, n\pi + \tfrac{1}{4}\pi, \quad . \quad . \qquad (85)$$

But when n and z are nearly equal, if $\sin^{-1}\frac{n}{z} = \frac{\pi}{2} - \epsilon$.

Then

$$\cos\epsilon = \frac{n}{z}, \quad \sin\epsilon = \frac{1}{z}(z-n)^{\frac{1}{2}}(z+n)^{\frac{1}{2}};$$

and

$$\rho = z(\sin\epsilon - \epsilon\cos\epsilon) + \tfrac{1}{4}\pi$$

$$= \tfrac{1}{4}\pi + \tfrac{2}{3}(z-n)^{\frac{3}{2}}\left(\frac{2}{z}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (86)$$

on reduction, which is the value furnished by expansion C.

Accordingly there exists a region in which either A or C may be used. Similarly, such a region exists for C and B.

Finally, the scheme of expansions A, B, C is complete for a real argument. For the case of purely imaginary argument, only one set of expansions is necessary. These may be expressed in terms of the same coefficients (λ, μ) as A and B*.

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* *Cf.* British Association Report, Dublin, 1908.