FUNCTIONS OF LIMITING MATRICES

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1. Interesting problems arise in the theory of matrices when two or more of the roots are equal. This case has been discussed by Frobenius,* Sylvester,† Buchheim,‡ and Taber,§ by different methods. Sylvester||, gave the formula for any function of a matrix with unequal roots, and suggested that the case of equal roots might be treated by passing to the limit. Sylvester's suggestion has to be modified before it can be put into practice, as the purely symbolic method leads to difficulties when we come to consider the non-scalar fractional powers of a scalar.¶, §

It appears that several lines of investigation can be coordinated by reverting to Grassmann's treatment of a matrix as an open product.** The limiting process can then be carried out in full generality, and thus we have convenient explicit formulæ in terms of the scalar coefficients of the degenerate matrix.

If all the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix of order n are distinct there are n distinct (generalised) axes $u_1, u_2, \ldots, u_n$. For simplicity we shall denote Grassmann's external multiplication by simple juxtaposition as in the latter part of A1, avoiding the square brackets of A2. Write

$$u_1' = (-)^{n-1}\frac{u_2u_3\ldots u_n}{u_1u_2\ldots u_n}, \quad u_2' = (-)^{n-2}\frac{u_1u_3\ldots u_n}{u_1u_2\ldots u_n}, \quad \ldots, \quad u_n' = \frac{u_1u_2\ldots u_{n-1}}{u_1u_2\ldots u_n}. \tag{1}$$

* G. Frobenius, *Jour. f. Math. (Crelle)*, Vol. 84 (1878), p. 1.

† J. J. Sylvester, *Johns Hopkins Univ. Circulars*, Vol. 3 (1884), p. 9 (*Math. Papers*, Vol. 4, p. 133); *Amer. Jour. Math.*, Vol. 6 (1884), p. 270 (*Math. Papers*, Vol. 4, p. 208).

‡ A. Buchheim, *Proceedings*, Vol. 16 (1884), p. 63; *Phil. Mag.* [5], Vol. 22 (1886), p. 173.

§ H. Taber, *Amer. Jour. Math.* Vol. 12 (1890), p. 337.

|| J. J. Sylvester, *Comptes Rendus*, Vol. 94 (1882), p. 55 (*Math. Papers*, Vol. 3, p. 562).

¶ J. J. Sylvester, *Phil. Mag.* [5], Vol. 16 (1883), p. 267 (*Math. Papers*, Vol. 4, p. 110).

** H. Grassmann, *Die lineale Ausdehnungslehre*, p. 266, 1844 [referred to as A1; *Ges. Werke*, Vol. 1 (1), p. 284]; *Die Ausdehnungslehre*, p. 245, 1862 [referred to as A2; *Ges. Werke*, Vol. 1 (2), p. 243].

Then the matrix is expressed as an open product in the form

$$\Phi = \lambda_1 u_1 . u_1' + \lambda_2 u_2 . u_2' + \ldots + \lambda_n u_n . u_n'.$$

The notation is a combination of Grassmann's first notation with Gibbs' dyadic notation as modified by Heaviside. Thus

$$\Phi x = \lambda_1 u_1 . u_1' x + \lambda_2 u_2 . u_2' x + \ldots + \lambda_n u_n . u_n' x,$$

where $u_1' x$ is the external product

$$(-)^{n-1} \frac{u_2 \ldots u_n x}{u_1 u_2 \ldots u_n}$$

and may be described without impropriety as the scalar product of u_1' and x.* Following Gibbs, u_r' is said to be the extensive quantity reciprocal to u_r.

2. Let $s+1$ axes coalesce with u_r. In addition, let u_t be a typical axis which remains distinct from u_r while the corresponding root tends to λ_r, and u_m a typical axis distinct from u_r with a root λ_m distinct from λ_r. We have therefore four sets of quantities to be considered in the first place, represented by the scheme

$$\begin{array}{c|ccc|c|c} \lambda_r & \lambda_r+\epsilon_1 & \ldots & \lambda_r+\epsilon_s & \lambda_r+\epsilon_t & \lambda_m, \\ u_r & u_r+x_1 & \ldots & u_r+x_s & u_t & u_m, \end{array}$$

where the ϵ's are ultimately indefinitely small scalars and the x's ultimately indefinitely small extensive quantities. For the present they are finite. Write P for the complete external product

$$u_r (u_r+x_1) \ldots (u_r+x_s) \Pi u_t \Pi u_m,$$

and PEu for the product omitting any factor u, with such a sign that PEu/P is the extensive quantity reciprocal to u in the n-ad P. Write Q similarly for the complete external product

$$u_r x_1 \ldots x_s \Pi u_t \Pi u_m.$$

Then

$$P = Q, \quad PE(u_r+x_\sigma) = QEx_\sigma, \quad PEu_t = QEu_t, \quad PEu_m = QEu_m,$$

and

$$PEu_r = QEu_r - \sum_{\sigma=1}^{s} QEx_\sigma,$$

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* See A2, Abschnitt 1, Kap. 4: the scalar product is used in this sense in A1, pp. 268 *et seq.* The quantities u' are practically Grassmann's complementary quantities.

a case of a general theorem of Grassmann's to be used later. Hence

$$\Phi = \lambda_r u_r . \frac{PEu_r}{P} + \sum_{\sigma=1}^{s} (\lambda_r+\epsilon_\sigma)(u_r+x_\sigma) . \frac{PE(u_r+x_\sigma)}{P}$$

$$+\sum_t (\lambda_r+\epsilon_t) u_t . \frac{PEu_t}{P} + \sum_m \lambda_m u_m . \frac{PEu_m}{P}$$

$$= \lambda_r \left[u_r . \frac{QEu_r}{Q} + \sum_{\sigma=1}^{s} x_\sigma . \frac{QEx_\sigma}{Q} + \sum_t u_t . \frac{QEu_t}{Q} \right] + \sum_m \lambda_m u_m . \frac{QEu_m}{Q}$$

$$+ \sum_{\sigma=1}^{s} \epsilon_\sigma (u_r+x_\sigma) . \frac{QEx_\sigma}{Q} + \sum_t \epsilon_t u_t . \frac{QEu_t}{Q} .$$

Since $1 = \Sigma u . QEu/Q$ in any n-ad Q, we have more simply

$$\Phi = \lambda_r + \sum_m (\lambda_m - \lambda_r) u_m . \frac{QEu_m}{Q} + \sum_{\sigma=1}^{s} \epsilon_\sigma (u_r+x_\sigma) . \frac{QEx_\sigma}{Q} + \sum_t \epsilon_t u_t . \frac{QEu_t}{Q}. \quad (2)$$

The problem before us is to find the limiting form of Φ when the scalars ϵ and the extensive quantities x tend to zero. Take a set of extensive quantities $u_{r+1}, u_{r+2}, \ldots, u_{r+s}$ arbitrarily, but fixed. Let R be the complete external product

$$u_r u_{r+1} \ldots u_{r+s} \Pi u_t \Pi u_m,$$

and let $\quad x_\sigma = (\sigma 0) u_r + (\sigma 1) u_{r+1} + \ldots + (\sigma s) u_{r+s} + \Sigma (\sigma t) u_t + \Sigma (\sigma m) u_m. \quad (3)$

Thus

$$Q = u_r \{ (10) u_r + (11) u_{r+1} + \ldots + (1s) u_{r+s} + \Sigma (1t) u_t + \Sigma (1m) u_m \}$$

$$\{ (20) u_r + (21) u_{r+1} + \ldots + (2s) u_{r+s} + \Sigma (2t) u_t + \Sigma (2m) u_m \}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\{ (s0) u_r + (s1) u_{r+1} + \ldots + (ss) u_{r+s} + \Sigma (st) u_t + \Sigma (sm) u_m \} \, \Pi u_t \Pi u_m.$$

Write Δ for the determinant

$$\begin{vmatrix} (11) & (12) & \ldots & (1s) \\ (21) & (22) & \ldots & (2s) \\ \ldots & \ldots & & \ldots \\ (s1) & (s2) & \ldots & (ss) \end{vmatrix}$$

Then leaving QEx_σ undetermined for the moment,

$$Q = \Delta R, \quad QEu_r = \Delta REu_r - \sum_{\sigma=1}^{s} (\sigma 0) \, QEx_\sigma,$$

$$QEu_t = \Delta REu_t - \sum_{\sigma=1}^{s} (\sigma t) QEx_\sigma, \qquad QEu_m = \Delta REu_m - \sum_{\sigma=1}^{s} (\sigma m) QEx_\sigma.$$

Write $u' = REu/R$ for the quantity reciprocal to any factor u of the n-ad R, in accordance with (1). Then equation (2) gives

$$\Phi = \Phi_1 + \Phi_2,$$

where

$$\Phi_1 = \lambda_r \left[u_r \,.\, u_r' + \sum_{\rho=1}^{s} u_{r+\rho} \,.\, u_{r+\rho}' \right] + \sum_t (\lambda_r + \epsilon_t)\, u_t \,.\, u_t' + \sum_m \lambda_m u_m \,.\, u_m', \quad (4)$$

$$\Phi_2 = \sum_{\sigma=1}^{s} \left[\epsilon_\sigma (1+\sigma 0)\, u_r + \sum_{\rho=1}^{s} \epsilon_\sigma (\sigma\rho) u_{r+\rho} + \sum_t (\epsilon_\sigma - \epsilon_t)(\sigma t)\, u_t \right.$$
$$\left. + \sum_m (\epsilon_\sigma - \lambda_m + \lambda_r)(\sigma m)\, u_m \right] . \frac{QEx_\sigma}{Q}. \quad (5)$$

No new terms arise in Φ_1 in the limit. As regards Φ_2, we have*

$$\frac{QEx_\sigma}{Q} = \sum_{a=1}^{s} (\sigma a)'\, u_{r+a}',$$

where $(pq)'$ is the minor of (pq) in Δ, divided by Δ. The coefficient of $u_r \,.\, u_{r+a}'$ $(a = 1 \text{ to } s)$ in Φ_2 is

$$\mu_a = \sum_{\sigma=1}^{s} \epsilon_\sigma (1+\sigma 0)(\sigma a)'. \quad (6)$$

The coefficient of $u_{r+\rho} \,.\, u_{r+a}'$ $(\rho,\ a = 1 \text{ to } s)$ is

$$\nu_{\rho a} = \sum_{\sigma=1}^{s} \epsilon_\sigma (\sigma\rho)(\sigma a)'. \quad (7)$$

The coefficient of $u_t \,.\, u_{r+a}'$ $(a = 1 \text{ to } s)$ is

$$\eta_{ta} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \epsilon_t)(\sigma t)(\sigma a)'. \quad (8)$$

The coefficient of $u_m \,.\, u_{r+a}'$ $(a = 1 \text{ to } s)$ is

$$\zeta_{ma} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \lambda_m + \lambda_r)(\sigma m)(\sigma a)'. \quad (9)$$

Since the vanishing of the quantities (σq) tends in general to make the inverse set $(\sigma q)'$ infinite, we must consider the possibility of all the quantities μ_a, $\nu_{\rho a}$, η_{ta}, ζ_{ma} tending to finite limits. Using the same letters for the limits we have therefore the semi-canonical form of the limiting

* A2, pp. 38, 39.

open product, namely*

$$\Phi = \lambda_r \left[u_r . u'_r + \sum_{\rho=1}^{s} u_{r+\rho} . u'_{r+\rho} + \sum_t u_t . u'_t \right] + \sum_m \lambda_m u_m . u'_m$$

$$+ \sum_{\alpha=1}^{s} \left[\mu_\alpha u_r + \sum_{\rho=1}^{s} \nu_{\rho\alpha} u_{r+\rho} + \sum_t \eta_{t\alpha} u_t + \sum_m \zeta_{m\alpha} u_m \right] . u'_{r+\alpha}. \quad (10)$$

3. To find any function $f(\Phi)$ of an open product Φ we replace each root λ by $f(\lambda)$, leaving the axes unaltered. Thus

$$f(\Phi) = f(\lambda_r) \left[u_r . u'_r + \sum_{\rho=1}^{s} u_{r+\rho} . u'_{r+\rho} + \sum_t u_t . u'_t \right] + \sum_m f(\lambda_m) u_m . u'_m$$

$$+ \sum_{\alpha=1}^{s} \left[\mu'_\alpha u_r + \sum_{\rho=1}^{s} \nu'_{\rho\alpha} u_{r+\rho} + \sum_t \eta'_{t\alpha} u_t + \sum_m \zeta'_{m\alpha} u_m \right] . u'_{r+\alpha}, \quad (11)$$

where
$$\mu'_\alpha = \sum_{\sigma=1}^{s} \epsilon'_\sigma (1+\sigma 0)(\sigma\alpha)', \quad (12)$$

$$\nu'_{\rho\alpha} = \sum_{\sigma=1}^{s} \epsilon'_\sigma (\sigma\rho)(\sigma\alpha)', \quad (13)$$

$$\eta'_{t\alpha} = \sum_{\sigma=1}^{s} (\epsilon'_\sigma - \epsilon'_t)(\sigma t)(\sigma\alpha)', \quad (14)$$

$$\zeta'_{m\alpha} = \sum_{\sigma=1}^{s} \{\epsilon'_\sigma - f(\lambda_m) + f(\lambda_r)\} (\sigma m)(\sigma\alpha)', \quad (15)$$

and
$$\epsilon'_k = f(\lambda_r + \epsilon_k) - f(\lambda_r). \quad (16)$$

So far approximations have only been made in the first lines of (10) and (11), where no limiting problem arises. We have now to calculate the limiting values of μ'_α, $\nu'_{\rho\alpha}$, $\eta'_{t\alpha}$, $\zeta'_{m\alpha}$ in terms of those of μ_α, $\nu_{\rho\alpha}$, $\eta_{t\alpha}$, $\zeta_{m\alpha}$. We first eliminate the scalars (σk), that is the modes of evanescence of the x's, leaving only the ϵ's. From (6) and (7) we have

$$\sum_{\beta=1}^{s} \mu_\beta \nu_{\beta\alpha} = \sum_{\beta=1}^{s} \sum_{\sigma=1}^{s} \sum_{\tau=1}^{s} \epsilon_\sigma (1+\sigma 0)(\sigma\beta)' \epsilon_\tau (\tau\beta)(\tau\alpha)'.$$

The summation with respect to β gives zero if $\sigma \neq \tau$ and unity if $\sigma = \tau$. Hence

$$\sum_{\beta=1}^{s} \mu_\beta \nu_{\beta\alpha} = \sum_{\sigma=1}^{s} \epsilon_\sigma^2 (1+\sigma 0)(\sigma\alpha)'.$$

* The degenerate forms of open products of the third order have been found by different methods by J. W. Gibbs, *Scientific Papers*, Vol. 2, p. 71, and F. L. Hitchcock, *Proc. Roy. Soc. Edinburgh*, Vol. 35 (1915), p. 171.

The result can obviously be generalised. Write

$$\mu_{pa} = \Sigma \mu_\beta \nu_{\beta\gamma} \nu_{\gamma\delta} \ldots \nu_{\lambda a} \quad (p \leqslant s),$$

where $\beta, \gamma, \ldots, \lambda$ are $p-1$ of the first s positive integers, and summation is over all possible values of each, equality included. Then we can prove by induction that

$$\mu_{pa} = \sum_{\sigma=1}^{s} \epsilon_\sigma^p (1+\sigma 0)(\sigma a)'.$$

Equation (6) is included if we put $\mu_{1a} = \mu_a$. From the first s of these equations we can calculate the s coefficients

$$(1+10)(1a)',\ (1+20)(2a)',\ \ldots,\ (1+s0)(sa)'$$

and express μ_a' in terms of $\epsilon_1, \epsilon_2, \ldots, \epsilon_s, \mu_a, \mu_{2a}, \ldots, \mu_{sa}$. Write Δ for the determinant

$$\begin{vmatrix} \epsilon_1 & \epsilon_2 & \ldots & \epsilon_s \\ \epsilon_1^2 & \epsilon_2^2 & \ldots & \epsilon_s^2 \\ \ldots & \ldots & & \ldots \\ \epsilon_1^s & \epsilon_2^s & \ldots & \epsilon_s^s \end{vmatrix}$$

and $\Delta_{\sigma p}$ for the minor of ϵ_σ^p in Δ. Write further

$$f_p = \sum_{\sigma=1}^{s} \frac{\epsilon_\sigma' \Delta_{\sigma p}}{\Delta}.$$

Then we have without approximation

$$\mu_a' = f_1 \mu_a + f_2 \mu_{2a} + \ldots + f_s \mu_{sa}.$$

If $f(\lambda)$ is holomorphic in the neighbourhood of λ_r, the quantity f_p tends to $f^{(p)}(\lambda_r)/p!$ as the ϵ's tend to zero. Hence

$$\mu_a' = \mu_a f'(\lambda_r) + \frac{\mu_{2a}}{2!} f''(\lambda_r) + \ldots + \frac{\mu_{sa}}{s!} f^{(s)}(\lambda_r). \tag{17}$$

The theory for $\nu_{\rho a}$ is similar. Writing

$$\nu_{p\rho a} = \Sigma \nu_{\rho\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \ldots \nu_{\lambda a} \quad (p \leqslant s)$$

we have
$$\nu_{\rho a}' = \nu_{\rho a} f'(\lambda_r) + \frac{\nu_{2\rho a}}{2!} f''(\lambda_r) + \ldots + \frac{\nu_{s\rho a}}{s!} f^{(s)}(\lambda_r). \tag{18}$$

As regards η_{ta}, we have the derived formula

$$\eta_{pta} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \epsilon_t) \epsilon_\sigma^{p-1} (\sigma t)(\sigma a)',$$

where $\eta_{1t\alpha} = \eta_{t\alpha}$ and

$$\eta_{pt\alpha} = \Sigma \eta_{t\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \dots \nu_{\lambda\alpha} \quad (p \leqslant s).$$

Writing Δ for the determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \epsilon_1 & \epsilon_2 & \dots & \epsilon_s \\ \dots & \dots & \dots & \\ \epsilon_1^{s-1} & \epsilon_2^{s-1} & \dots & \epsilon_s^{s-1} \end{vmatrix}$$

and $\Delta_{\sigma p}$ for the minor of ϵ_σ^p, we find as before

$$\eta'_{t\alpha} = \phi_1 \eta_{t\alpha} + \phi_2 \eta_{2t\alpha} + \dots + \phi_s \eta_{st\alpha},$$

where

$$\phi_p = \sum_{\sigma=1}^{s} \frac{\epsilon'_\sigma - \epsilon'_t}{\epsilon_\sigma - \epsilon_t} \frac{\Delta_{\sigma(p-1)}}{\Delta}.$$

It is clear from the form of equations (6), (8), (12) and (14) that if ϕ_p tends to a definite limit when ϵ_t, as well as $\epsilon_1, \epsilon_2, \dots, \epsilon_s$, tends to zero, that limit is $f^{(p)}(\lambda_r)/p!$. This will be the case if, for every finite value of ϵ_t, ϕ_p tends to a definite value as $\epsilon_1, \epsilon_2, \dots, \epsilon_s$ tend to zero. Let ϵ_t therefore be finite, and write $\lambda_t = \lambda_r + \epsilon_t$. Then

$$\phi_p = \sum_{\sigma=1}^{s} \frac{f(\lambda_r + \epsilon_\sigma) - f(\lambda_t)}{\lambda_r + \epsilon_\sigma - \lambda_t} \frac{\Delta_{\sigma(p-1)}}{\Delta}.$$

This expression, however, has the definite limit

$$\frac{1}{(p-1)!} \frac{\partial^{p-1}}{\partial \lambda_r^{p-1}} \left\{ \frac{f(\lambda_r) - f(\lambda_t)}{\lambda_r - \lambda_t} \right\},$$

which proves the theorem. Hence we have finally

$$\eta'_{t\alpha} = \eta_{t\alpha} f'(\lambda_r) + \frac{\eta_{2t\alpha}}{2!} f''(\lambda_r) + \dots + \frac{\eta_{st\alpha}}{s!} f^{(s)}(\lambda_r). \tag{19}$$

The theory for $\zeta_{m\alpha}$ is implicitly contained in the above, and we have

$$\zeta'_{m\alpha} = \zeta_{m\alpha} \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} + \frac{\zeta_{2m\alpha}}{1!} \frac{\partial}{\partial \lambda_r} \left\{ \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} \right\} + \dots$$

$$+ \frac{\zeta_{sm\alpha}}{(s-1)!} \frac{\partial^{s-1}}{\partial \lambda_r^{s-1}} \left\{ \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} \right\}, \tag{20}$$

where

$$\zeta_{pm\alpha} = \Sigma \zeta_{m\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \dots \nu_{\lambda\alpha} \quad (p \leqslant s).$$

The enunciation of the general theorem for any number of sets of coincident axes is cumbrous, but the theorem itself is easily understood.

Corresponding to each set of roots such as λ_r we have terms of the type

$$\lambda_r \left[u_r . u_r' + \sum_{\rho=1}^{s} u_{r+\rho} . u_{r+\rho}' + \sum_t u_t . u_t'\right]$$

$$+ \sum_{a=1}^{s} \left[\mu_a u_r + \sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho} + \sum_t \eta_{ta} u_t + \sum_m \zeta_{ma} u_m\right] . u_{r+a}',$$

where no account is taken of coincidences outside the r set, except that u_r', ... are calculated from a complete external product in which only one axis of a coincident set is retained and the factors made up to the full number by arbitrary extensive quantities. The rules for $f(\Phi)$, or rather the part of $f(\Phi)$ belonging to r, are as above.

4. The quantity $\nu_{p\rho a}$ may be regarded as obtained by repeated application of the formula

$$\nu_{(p+q)\rho a} = \sum_{\beta=1}^{s} \nu_{p\rho\beta}\, \nu_{q\beta a} \quad (p+q \leqslant s), \tag{21}$$

and then the other quantities are given by

$$\mu_{pa} = \sum_{\beta=1}^{s} \mu_\beta \nu_{(p-1)\beta a}, \quad \eta_{pta} = \sum_{\beta=1}^{s} \eta_{t\beta}\, \nu_{(p-1)\beta a}, \quad \zeta_{pm\beta} = \sum_{\beta=1}^{s} \zeta_{m\beta}\, \nu_{(p-1)\beta a}. \tag{22}$$

Equation (21) is of the form of matricular multiplication, as it should be, for considering the special open product

$$\Psi = \sum_{a=1}^{s} \left[\sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho}\right] . u_{r+a}', \tag{23}$$

we have

$$\Psi^p = \sum_{a=1}^{s} \left[\sum_{\rho=1}^{s} \nu_{p\rho a} u_{r+\rho}\right] . u_{r+a}' \quad (p \leqslant s). \tag{24}$$

The possibility of identical relations between the coefficients in (10) has been left open in passing to the limit, and it remains to apply the test that $s+t+1$ of the roots are equal to λ_r. We have

$$\Phi - \lambda = (\lambda_r - \lambda) u_r . u_r' + \sum_{a=1}^{s} v_{ra} . u_{r+a}' + (\lambda_r - \lambda) \sum_t u_t . u_t' + \sum_m (\lambda_m - \lambda)\, u_m . u_m',$$

where

$$v_{ra} = \mu_a u_r + (\lambda_r - \lambda)\, u_{r+a} + \sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho} + \sum_t \eta_{ta} u_t + \sum_m \zeta_{ma} u_m.$$

The external product of the antecedents of $\Phi - \lambda$, whose vanishing determines the roots, is obviously independent of the quantities μ, η, ζ. Considering Ψ as a special form of (10) we see that it must have all its roots zero; and these are all the necessary restrictions on the generality of (10). Since Ψ is an open product of order s, Ψ^s vanishes identically, and

$$\nu_{s\rho\alpha} \equiv 0. \tag{25}$$

It follows that the series (18) must stop at $\nu_{(s-1)\rho\alpha} f^{(s-1)}(\lambda_r)/(s-1)!$ at most.

Not more than s of the relations (25) are, of course, independent. Other forms may be obtained by considering that the quantities ν' satisfy equations of the same form as the quantities ν. Thus from the relation

$$\sum_{\alpha=1}^{s} \nu_{\alpha\alpha} = 0$$

we derive
$$\sum_{\alpha=1}^{s} \nu_{\alpha\alpha} = \sum_{\alpha=1}^{s} \nu_{2\alpha\alpha} = \dots = \sum_{\alpha=1}^{s} \nu_{(s-1)\alpha\alpha} = 0. \tag{26}$$

To find the identical equation of lowest degree satisfied by the open product Φ we proceed as follows. Let Φ_r be a function of Φ which does not contain any of the extensive quantities u_r, $u_{r+\alpha}$, u_t (which we call the r set) as antecedents. Let Φ_v be another function similarly related to a second simple or multiple root. Then $\Phi_r\Phi_v$ does not contain the r set as antecedents, nor $\Phi_r\Phi_v$ the v set. But since Φ_r and Φ_v are functions of Φ, $\Phi_r\Phi_v = \Phi_v\Phi_r$. Hence $\Phi_r\Phi_v$ contains no member of either the r set or the v set as antecedent. Proceeding to the end we see that the produc $\Pi\Phi_r$ extended over all the roots is identically zero. Hence we have to find the function Φ_r of lowest degree having the required property with respect to u_r, $u_{r+\alpha}$, u_t, and we know from the general theory that this is a power of $\Phi-\lambda_r$.

Put $f(\Phi) = (\Phi-\lambda_r)^q$. Then

$$f^{(p)}(\lambda_r)/p! = 0 \quad \text{if} \quad p \neq q, \quad = 1 \quad \text{if} \quad p = q.$$

Thus $\mu'_\alpha = \nu'_{\rho\alpha} = \eta'_{t\alpha} = 0$ if $q > s$, and

$$\mu'_\alpha = \mu_{q\alpha}, \quad \nu'_{\rho\alpha} = \nu_{q\rho\alpha}, \quad \eta'_{t\alpha} = \eta_{qt\alpha} \quad \text{if} \quad q \leqslant s.$$

Hence $\Phi_r = (\Phi-\lambda_r)^q$, where q is the least number for which

$$\mu_{q\alpha} = \nu_{q\rho\alpha} = \eta_{qt\alpha} = 0, \tag{27}$$

provided that any such number less than s exists, failing which we have $q = s$. It follows that coalescence of the t type *always* causes reduction of the degree of the identical equation, by an amount equal to the number of terms involved, and further reduction may take place. From (21) and (22), if equations (27) are satisfied for any value of q they are satisfied for all higher values up to s, and we have also

$$\zeta_{(q+1)m\beta} = \dots = \zeta_{sm\beta} = 0 \quad (q < s). \tag{28}$$

Hence we have what seems to be the essential point of Buchheim's second paper,* that if $(\Phi-\lambda_r)^q$ is the highest power of $\Phi-\lambda_r$ in the identical equation, differential coefficients up to the order $q-1$ occur in the corresponding part of $f(\Phi)$.

5. If $f(z)$ is q-valued and Φ a matrix of order n, $f(\Phi)$ is in general n^q-valued,† but functions of special matrices may contain arbitrary constants, or be non-existent. One cause of indeterminacy is that Φ may admit a transformation of axes (as for example the matrix unity), the multiformity of $f(z)$ causing the constants of transformation to appear in the result.‡ The other cause is more closely connected with our present subject. The q-th root of a degenerate matrix is found in practice by assuming a form of the greatest admissible generality as to roots and axes, and comparing its q-th power with the given matrix.§ That arbitrary constants may enter into the solution is clear from the preceding formulæ. Let $f(\Phi)$ in equation (11) be a q-th root of Φ, and let $\lambda_r = 0$. Then from § 4, if $q > s$, the original matrix must satisfy the conditions

$$\mu_a = \nu_{\rho a} = \eta_{ta} = 0,$$

and then there are not equations enough to determine the assumed constants μ'_a, $\nu'_{\rho a}$, η'_{ta}, ζ'_{ma}. Theoretically, there is no need to assume a tentative standpoint, since the indeterminate solutions can be found by a direct limiting process. This point seems of interest, as it leads us to something approaching a general theory of functions of open products.

Starting from the finite polynomial, which is interpreted directly as the sum of a sequence of intelligible operations, we ascend to the Taylor or Laurent series with the roots λ_1, λ_2, ... in the belt of convergence.‖ But there is no need to stop at this. The domain can, in general, be extended by quasi-analytical continuation in powers of $\Phi-\lambda$, where λ is some complex quantity, and thus a larger class of products, namely those whose roots lie within the extended domain, brought within the scope of our formulæ. If we calculate $f(\Phi)$ and then let λ_r move up to a singularity of $f(z)$, there may be either a single limiting form, or one with arbitrary constants, or no finite limit. A branch-point without infinity gives rise to the first two, any infinity of $f(z)$ to the last. We know beforehand that a pole of $f(z)$ will have this effect, since $\Phi-\lambda_r$ cannot be inverted.

* Buchheim, *loc. cit.*

† Taber, *loc. cit.*

‡ C. J. Joly, *Manual of Quaternions*, p. 99.

§ F. L. Hitchcock, *Proc. Roy. Soc. Edinburgh*, Vol. 37 (1917), p. 350.

‖ E. Weyr, *Bull. des Sciences Math.* [2], Vol. 11 (1887), p. 205.

Indeterminacy can occur even in an open product of the third order with a double root: thus $\lambda_3 u_3 . u_3' + \zeta u_3 . u_2'$ has the square root

$$\lambda_3^{\frac{1}{2}} u_3 . u_3' + \mu' u_1 . u_2' + \zeta \lambda_3^{-\frac{1}{2}} u_3 . u_2'.$$

To illustrate the general theory consider the q-th power of the open product

$$\Phi = \lambda (u_1 . u_1' + u_2 . u_2') + \lambda_3 u_3 . u_3' + (\mu u_1 + \zeta u_3) . u_2',$$

where q is real and commensurable and z^q uniformised by a radial cut, so that if $q = m/n$, $f(z)$ is the m-th power of one branch of $z^{1/n}$. From (17) and (20),

$$\Phi^q = \lambda^q (u_1 . u_1' + u_2 . u_2') + \lambda_3^q u_3 . u_3' + \left[q\mu\lambda^{q-1} u_1 + \zeta \frac{\lambda_3^q - \lambda^q}{\lambda_3 - \lambda} u_3 \right] . u_2'.$$

If $q > 1$ there is a determinate limit as $\lambda \to 0$, namely

$$\lambda_3^q u_3 . u_3' + \zeta \lambda_3^{q-1} u_3 . u_2'.$$

If $0 < q < 1$ there is a finite limit if μ tends to zero in the order λ^{1-q}, giving the indeterminate result $\lambda_3^q u_3 . u_3' + \mu' u_1 . u_2' + \zeta \lambda_3^{q-1} u_3 . u_2'$. Finally if q is negative no compensation of coefficients can give a finite limit, illustrating what has been said about the effect of a pole.