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tables are due to the fact that the substances in question are chemically related; and perhaps it is scarcely justifiable to generalize from such a limited number; and it may well be that a wide range of elements would show other peculiarities.

I desire to express my hearty appreciation of the efficient service rendered in this work by Mr. C. R. Mann, and especially to recognize the patience and skill shown in the tedious and delicate process of preparation of the vacuum-tubes, to which in great measure the success of the investigation is due.

XXXIX. *On Discontinuities connected with the Propagation of Wave-motion along a Periodically Loaded String.* By CHARLES GODFREY, B.A., Scholar of Trinity College, Isaac Newton Student in the University of Cambridge*.

1. **T**HE system described below shows rather remarkable discontinuous properties. The work was suggested by a passage in Sir George Stokes' Read lecture, and formed part of an essay written in December 1896.

A heavy string of density ρ under tension T extends from $-\infty$ to $+\infty$. From $-\infty$ to 0 it is free from loads; from 0 to $+\infty$ it is loaded at equal intervals l with equal particles of mass M . To avoid ambiguity we will suppose that the motion of each mass is retarded by a small viscous force; this will finally be neglected. We will investigate the steady vibration of the system when simple transverse waves are travelling along the string from $-\infty$. These impinge on the system of masses; a reflected wave is generated which travels back along the string; furthermore, the masses are agitated in a certain manner.

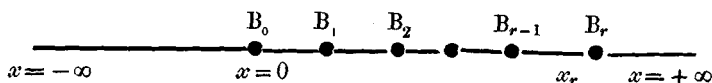
2. We will denote by ξ the lateral displacement of a point on the string between $x = -\infty$ and $x = 0$. The velocity of propagation along the string is $\sqrt{\frac{T}{\rho}} = v$. For a motion whose frequency is given by e^{int} we have

$$\xi = Ae^{in(t - \frac{x}{v})} + Be^{in(t + \frac{x}{v})}. \quad \dots \quad (i.)$$

Let the displacement of the mass B_r at time t be denoted by y_r . For a point in the r th string $B_{r-1}B_r$ let the displacement be ξ_r ; and let the distance of such a point from B_{r-1} be x_r .

* Communicated by R. T. Glazebrook, F.R.S.

Fig. 1.



For the string $B_{r-1}B_r$

$$\xi_r = \left(P_r \cos \frac{nx_r}{v} + Q_r \sin \frac{nx_r}{v} \right) e^{int} \quad . \quad . \quad . \quad (ii.)$$

Now at $x_r = 0$, $\xi_r = y_{r-1}$, and at $x_r = l$, $\xi_r = y_r$;

$$\left. \begin{aligned} \therefore P_r e^{int} &= y_{r-1} \\ Q_r e^{int} &= \frac{y_r - y_{r-1} \cos \frac{nl}{v}}{\sin \frac{nl}{v}} \end{aligned} \right\} . \quad . \quad . \quad . \quad (iii.)$$

The equation of motion of B_r , for $r > 0$, is

$$M\ddot{y}_r + 2k\dot{y}_r = T \left\{ \left(\frac{\partial \xi_{r+1}}{\partial x_{r+1}} \right)_{x_{r+1}=0} - \left(\frac{\partial \xi_r}{\partial x_r} \right)_{x_r=l} \right\} . \quad . \quad (iv.)$$

Substituting from (ii.) and (iii.) and remembering that $y_r \propto e^{int}$,

$$\begin{aligned} y_r(-Mn^2 + 2kin) &= \frac{Tn}{v} \left(Q_{r+1} + P_r \sin \frac{nl}{v} - Q_r \cos \frac{nl}{v} \right) e^{int} \\ &= \frac{Tn}{v \sin \frac{nl}{v}} \left(y_{r+1} - 2y_r \cos \frac{nl}{v} + y_{r-1} \right), \end{aligned}$$

or

$$y_{r+1} + y_{r-1} - 2y_r \left(\cos \frac{nl}{v} + \frac{v \sin \frac{nl}{v}}{2Tn} (2kni - Mn^2) \right) = 0 . \quad . \quad (v.)$$

The equation of motion of B_0 is

$$M\ddot{y}_0 + 2k\dot{y}_0 = T \left\{ \left(\frac{\partial \xi_1}{\partial x_1} \right)_{x_1=0} - \left(\frac{\partial \xi}{\partial x} \right)_{x=0} \right\} . \quad . \quad . \quad (vi.)$$

This gives

$$\begin{aligned} y_0(-Mn^2 + 2kni) &= \left(\frac{Tn}{v} Q_1 + \frac{Tin}{v} (A - B) \right) e^{int} \\ &= \frac{Tn}{v \sin \frac{nl}{v}} \left(y_1 - y_0 \cos \frac{nl}{v} \right) + \frac{Tin}{v} (A - B) e^{int} . \quad . \quad (vii.) \end{aligned}$$

We will now abbreviate by writing

$$\frac{nl}{v} \equiv \psi,$$

$$\frac{Mv^2}{2lT} \equiv \mu;$$

$$\cos \frac{nl}{v} + \frac{v \sin \frac{nl}{v}}{2Tn} (2kni - Mn^2) \equiv \cos(\alpha + i\beta) \equiv \cos \theta, \quad . \quad (\text{viii.})$$

where α and β are real.

$$\therefore \left. \begin{aligned} \cos \alpha \cosh \beta &= \cos \frac{nl}{v} - \frac{Mnv}{2T} \sin \frac{nl}{v} = \cos \psi - \mu \psi \sin \psi, \\ \sin \alpha \sinh \beta &= -\frac{vk \sin \frac{nl}{v}}{T} = -\frac{vk}{T} \sin \psi. \end{aligned} \right\} \quad (\text{ix.})$$

With these substitutions, (v.) will give

$$y_{r+1} - 2y_r \cos \theta + y_{r-1} = 0 \quad . \quad . \quad . \quad (\text{x.})$$

The general solution of this set of equations is

$$\begin{aligned} y_r &= (C e^{ir\theta} + D e^{-ir\theta}) e^{int}, \\ &\equiv (C e^{ir\alpha - r\beta} + D e^{-ir\alpha + r\beta}) e^{int}. \end{aligned}$$

Now (ix.) will not determine the sign of β ; we will always take the positive value. It is then obvious that $D=0$; otherwise the motion would be great for great values of r .

$$\therefore y_r = C e^{i(r\theta + nt)}. \quad . \quad . \quad . \quad (\text{xi.})$$

Again, (vii.) becomes

$$\begin{aligned} B - A &= \frac{e^{-int}}{i \sin \psi} (y_1 + y_0 \cos \psi - 2y_0 \cos \theta) \\ &= \frac{iC}{\sin \psi} (e^{-i\theta} - \cos \psi). \quad . \quad . \quad . \quad (\text{xii.}) \end{aligned}$$

Furthermore, at $x=0$,

$$B + A = C. \quad . \quad . \quad . \quad (\text{xiii.})$$

From (xii.) and (xiii.),

$$\left. \begin{aligned} 2A &= \frac{iC}{\sin \psi} (-e^{-i\theta} + e^{-i\psi}), \\ 2B &= \frac{iC}{\sin \psi} (+e^{-i\theta} - e^{i\psi}). \end{aligned} \right\} \quad . \quad . \quad (\text{xiv.})$$

These are equivalent to

$$\left. \begin{aligned} \frac{A}{C} &= \frac{e^{ir}}{2 \sin \psi} (e^{2\beta} - 2e^{\beta} \cos \alpha - \psi + 1) \\ \frac{B}{C} &= \frac{e^{ir'}}{2 \sin \psi} (e^{2\beta} - 2e^{\beta} \cos \alpha + \psi + 1) \end{aligned} \right\} \quad (\text{xv.})$$

where

$$\tan \tau = \frac{e^{\beta} \cos \alpha - \cos \psi}{e^{\beta} \sin \alpha - \sin \psi}, \quad \tan \tau' = \frac{e^{\beta} \cos \alpha - \cos \psi}{e^{\beta} \sin \alpha + \sin \psi}. \quad (\text{xvi.})$$

3. We are now provided with a complete solution of the motion ; this we will proceed to interpret.

The quantities α and β are determined by equations (ix.) :

$$\left. \begin{aligned} \cos \alpha \cosh \beta &= \cos \psi - \mu \psi \sin \psi \equiv z \text{ say,} \\ \sin \alpha \sinh \beta &= -\frac{vk}{T} \sin \psi. \end{aligned} \right\} \quad (\text{ix.})$$

We shall clearly perceive the drift of the matter if we neglect the friction and put $k=0$. Then either $\sin \alpha$ or $\sinh \beta$ vanishes ; the former or the latter being the case according as $z^2 \geq 1$.

If $z^2 < 1$, $\beta=0$, and a wave-like motion will be propagated through the masses, for

$$y_r = C e^{i(r\alpha + nt)}.$$

If $z^2 > 1$, β is finite, while α is a multiple of π . The equation

$$y_r = C e^{i(r\alpha + nt) - r\beta}$$

will represent an exponential falling off of motion, consecutive masses being either in the same or in opposite phases. In this case the deeper masses will be practically unaffected by the incident wave.

In order to understand how these phenomena depend upon the frequency of the incident wave, we must trace the changes of z for different values of ψ , or $\frac{nl}{v}$.

4. Graph of $z = \cos \psi - \mu \psi \sin \psi$.

This is readily constructed on finding the roots of

$$z=0, \quad z=1, \quad z=-1.$$

For $z=0$ we have $\cot \psi = \mu \psi$. By the usual graphic method, we find that the roots of this equation lie between

$$0 \text{ and } \frac{\pi}{2}, \quad \pi \text{ and } \frac{3\pi}{2}, \quad 2\pi \text{ and } \frac{5\pi}{2}, \text{ \&c.,}$$

approaching closer to the lower limit for the greater values.

For $z=1$ we have

$$\sin \frac{\psi}{2} \left(\sin \frac{\psi}{2} + \mu \psi \cos \frac{\psi}{2} \right) = 0.$$

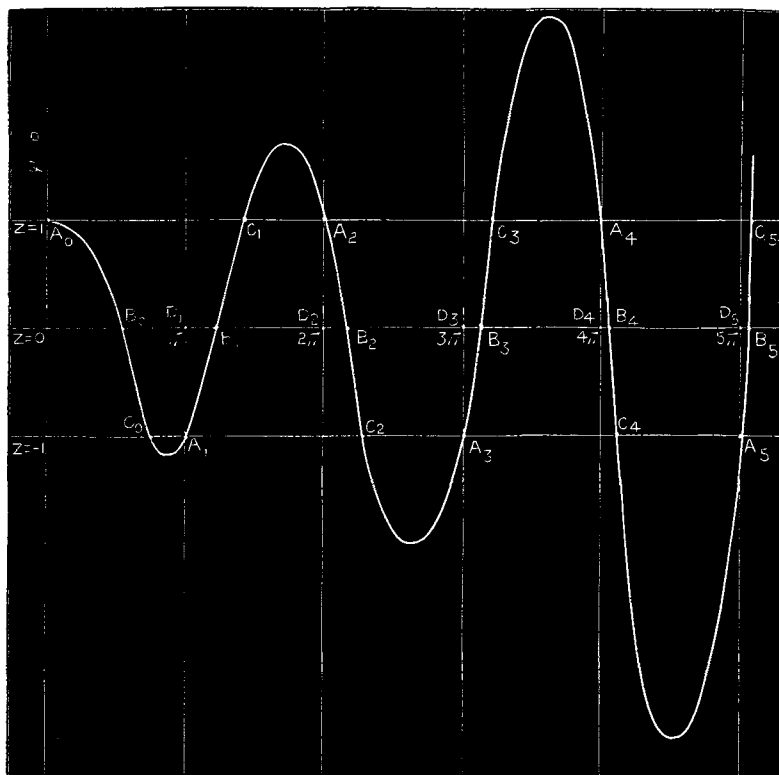
The factor $\sin \frac{\psi}{2}$ gives $\psi = 2s\pi$, where s is integral. The other factor gives roots lying between π and 2π , 3π and 4π , &c. It will be seen that they lie beyond the zeros of the corresponding regions.

For $z=-1$ we have

$$\cos \frac{\psi}{2} \left(\cos \frac{\psi}{2} - \mu \psi \sin \frac{\psi}{2} \right) = 0.$$

The factor $\cos \frac{\psi}{2}$ gives $\psi = (2s+1)\pi$. The other factor gives roots between 0 and π , 2π and 3π , &c.; and again lying beyond the zeros of the corresponding regions.

Fig. 2.



It is also clear that the maxima and minima become more marked as ψ increases. With these data it is easy to see that the general shape of the curve is as above.

From what has been already said, it is obvious that the wave will penetrate the masses if ψ lies within limits corresponding to the portions A_0C_0 , A_1C_1 , A_2C_2 , &c. of the curve. If, on the other hand, ψ belongs to the regions C_0A_1 , C_1A_2 , C_2A_3 , &c., the motion will only enter to a small distance.

5. It is interesting to look at the magnitude of the different amplitudes rather more closely. We shall lose no generality if we suppose that α lies between 0 and -2π . On considering the signs of $\sin \alpha$ and $\cos \alpha$ as given by (ix.), we have the following table:—

Region.	ψ between	$\sin \alpha$.	z and $\cos \alpha$.	α between
A_0B_0	0 and π	—	+	0 and $-\frac{\pi}{2}$.
B_0A_1	0 and π	—	—	$-\frac{\pi}{2}$ and $-\pi$.
A_1B_1	π and 2π	+	—	$-\pi$ and $-\frac{3\pi}{2}$.
B_1A_2	π and 2π	+	+	$-\frac{3\pi}{2}$ and -2π .

Similar limits recur for the other reaches, A_2A_4 , A_4A_6 , . . .

6. Regions for which $z^2 > 1$.

From the above table it appears that α is equal to 0 or $-\pi$ according as ψ lies between $2s\pi$ and $(2s+1)\pi$, or between $(2s-1)\pi$ and $2s\pi$. Now, denoting moduli of complex quantities by straight brackets,

$$\left| \frac{A^2}{C^2} \right| = \left| \frac{B^2}{C^2} \right| = \frac{e^{2\beta} \mp 2e^{\beta} \cos \psi + 1}{4 \sin^2 \psi},$$

the upper or lower sign being taken according as α is 0 or $-\pi$.

This expression is equal to

$$\frac{e^{\beta}}{2} \frac{\cosh \beta \mp \cos \psi}{\sin^2 \psi}.$$

But $\mp \cosh \beta \pm \cos \psi = \mu \psi \sin \psi$;

$$\therefore \left| \frac{A^2}{C^2} \right| = \left| \frac{B^2}{C^2} \right| = \mp \frac{e^{\beta}}{2} \frac{\mu \psi}{\sin \psi}. \quad \text{. . . (xvii.)}$$

For the frequencies which are not propagated we thus

have $|A| = |B|$, *i. e.* total reflexion, and it is clear from (xvi.) that there is reversal of phase. For the upper limit of such a region of frequency, $\psi = s\pi$ and $C = 0$ (xvii.).

7. Region for which $z^2 < 1$.

Here we shall have $\beta = 0$.

$$\left| \frac{A^2}{C^2} \right| = \frac{\sin^2 \frac{\alpha - \psi}{2}}{\sin^2 \psi}, \quad \left| \frac{B^2}{C^2} \right| = \frac{\sin^2 \frac{\alpha + \psi}{2}}{\sin^2 \psi}.$$

But

$$\cos \alpha = \cos \psi - \mu \psi \sin \psi,$$

$$\therefore 2 \sin \frac{\alpha - \psi}{2} \sin \frac{\alpha + \psi}{2} = \mu \psi \sin \psi,$$

$$\therefore \left| \frac{A^2}{C^2} \right| = \frac{\mu^2 \psi^2}{4 \sin^2 \frac{\alpha + \psi}{2}}, \quad \left| \frac{B^2}{C^2} \right| = \frac{\mu^2 \psi^2}{4 \sin^2 \frac{\alpha - \psi}{2}}. \quad (\text{xviii.})$$

Here equation (xvi.) shows that the incident, transmitted, and reflected waves will be in different phases.

For the lower limit of such a region both ψ and α are multiples of π ; further, they will be even or odd together.

It therefore appears that $\sin \frac{\alpha + \psi}{2}$ and $\sin \frac{\alpha - \psi}{2}$ are zero, and

$C = 0$. But for these points $n = \frac{s\pi v}{2}$. If then the incident

wave is of frequency corresponding to any one of the natural nodes of the intervals of string, the masses will be entirely undisturbed.

The case of ψ small is seen to be exceptional: we easily find that

$$\left| \frac{A^2}{C^2} \right| = \frac{\mu^2}{\{1 - \sqrt{1 + \mu}\}^2}, \quad \left| \frac{B^2}{C^2} \right| = \frac{\mu^2}{\{1 + \sqrt{1 + \mu}\}^2}.$$

8. It may perhaps be allowed that the phenomena here discussed have some mathematical analogy with an ideal case in optics. We may think of the incidence of light from the free æther upon a solid of periodic structure. Without pressing the analogy, we will recapitulate our results in optical phraseology; this presentation will have the advantage of brevity.

Light being incident upon a periodic distribution of molecules, the light is analysed by a spectroscope after transmission through a considerable thickness. We shall find narrow bright bands, their lower edges ranged harmonically; each band will

be faint on the lower side and terminate abruptly on the upper. Their width diminishes as we ascend the scale of frequency. Their lower edges correspond to the proper periods of the intermolecular spaces. If we view the light reflected we shall see total reflexion corresponding to the frequencies of the dark bands of the transmission spectrum; for these wave-lengths there will be reversal of phase.

XL. A Numerical Evaluation of the Absolute Scale of Temperature. By R. A. LEHFELDT.*

Introduction.

NUMEROUS attempts have been made to reduce the readings of thermometers to the absolute scale, since that scale was first clearly defined by Thomson and Joule; but though the process of calculation has been varied a good deal, the most essential experimental basis of all the reductions is the same, viz. Thomson and Joule's own experiments on the outflow of gases through a porous plug. It is very remarkable, therefore, that no one, so far as I know, has attempted to repeat or extend those experiments, except in one case studied by E. Natanson, and that notwithstanding the great discrepancies in Thomson and Joule's measurements. For hydrogen, the substance which is probably the best for thermometric purposes, there were twelve experiments carried out at about 7° and five at about 90° ; but these results varied from $+0.9$ to -0.1 ! while the hydrogen was in no case even approximately pure; and although the determination of absolute temperatures depends upon that experiment, it has been left in so unsatisfactory a state for half a century. Of the other data required, some, such as the specific heat and specific volume of the gases used, are known with sufficient accuracy, since they enter only in a small correction term: the most important, however, is the coefficient of pressure in the case of a gas thermometer at constant volume (or the coefficient of expansion in the constant-pressure thermometer). On this point, and this only, a distinct advance has been made since the time of Regnault. This has been accomplished by Chappuis, working at the Bureau International, who has measured the coefficient of pressure between 0° and 100° for hydrogen, nitrogen, and carbon dioxide, with all the care and scrupulousness that modern physical methods can suggest. My object in writing this paper is partly to take advantage of Chappuis's results, and partly to draw attention

* Communicated by the Author.