ON NON-HARMONIC FOURIER SERIES

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1. Following English usage, I have found it convenient to restrict the term Fourier series to those trigonometric series which proceed by cosines and sines of integral values of the variable, or variables, and have the further property that their coefficients are expressible by means of integrals of the well-known kind, involving a function of the independent variable, or variables. I have, for example, not followed the usage adopted by Jordan, and other French writers, of employing the term to characterise analogous series of functions, other than $\cos nx$ and $\sin nx$, such as Bessel functions. It seems to me, however, desirable to retain the term when the very slight change—formally speaking—of substituting $n+k$ for n is made. Though the series so obtained have not, however, as far as I know, been the subject of systematic study, and I have found it necessary to obtain among other things, the expressions for the coefficients as integrals, such series naturally present themselves. Special examples of such series are indeed known.* I myself have been led to the study of these series naturally by the necessity of extending to the case when the order is irrational, properties of series of Bessel functions, which, in the rational case, I have been able to deduce from the theory of what I have called Restricted Fourier Series. Needless to say Series of Bessel Functions are not the only series whose study might be expected to demonstrate the same need.

In the researches in question, these non-harmonic trigonometrical series do not present themselves in what I propose to call the Fourier form, being of a more general type. It is evident, however, that a proper grasp of the whole class of trigonometrical series in question cannot be obtained without investigating the properties of the particular class in question. Indeed the behaviour of a Bessel series† at the further bound

* See below, footnote to § 11.

† I distinguish between a series of Bessel functions $\Sigma A_r J_m(k_r z)$ and the special case, Bessel series, when the coefficients A have the appropriate form.

of the interval in which the expansion is valid, is most conveniently discussed by reference to the behaviour of a non-harmonic Fourier series, restricted in character generally, at an extremity of the interval in which this series is defined.

The paper falls naturally into three parts. The first of these, §§ 2–16, treats of the definition and summation of the series in question.

A non-harmonic trigonometrical $\left.\begin{matrix}\text{cosine}\\ \text{sine}\end{matrix}\right\}$ series

$$\sum_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix} (n+k)\,x$$

is said to have *the Fourier form* if

$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \begin{matrix}\sin\\ \cos\end{matrix} (n+k)(2\pi - t)\,dt,$$

and the function $f(x)$ is then said to be *the associated function* of the series.

At an internal point of the interval $(0, 2\pi)$ these series are proved to have, apart from an additive term which tends uniformly to zero, the same expression for the n-th partial summation

$$s_n = o(1) + \frac{1}{\pi}\int_0^{c} \tfrac{1}{2}\{f(x+t)+f(x-b)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})t\,dt,$$

as the Fourier series of a function equal to $f(x)$ at, and in the neighbourhood of, the point x, and having any convenient values elsewhere. *The upper and lower functions, and the modes of oscillation, are accordingly independent of the form of the associated function, except in an arbitrary small neighbourhood of the point x considered, and are the same as those of this auxiliary Fourier series. In particular, the conditions of convergence, and of uniform or bounded convergence, are the same for the non-harmonic Fourier series and the auxiliary harmonic Fourier series.*

These results are still true at the end-points of the interval $(0, 2\pi)$, provided the function $f(x)$, whose values have hitherto only been supposed known almost everywhere in the interval $(0, 2\pi)$, is supposed "continued" outside the interval in accordance with certain laws, which are different for the cosine and sine series. The law of "continuation" appropriate to the cosine is embodied in the formulæ:

$$\left.\begin{aligned} f(t) &= f(-t), \\ \tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} &= f(t)\cos 2k\pi. \end{aligned}\right\}$$

The law of "continuation" appropriate to the sine series is as follows:

$$\left.\begin{aligned} f(t) &= -f(-t), \\ \tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} &= f(t)\cos 2k\pi. \end{aligned}\right\}$$

These "continuation" formulæ play in our present theory the part of the periodicity in the case of harmonic series.

The proof of these properties depends on certain fundamental lemmas, which, in our theory, take the place of the ordinary normal properties of the sine and cosine. These are

$$\int_0^{2\pi} \begin{matrix}\sin\\ \cos\end{matrix}\,(n+k)(2\pi-u)\,\begin{matrix}\cos\\ \sin\end{matrix}\,(r+k)u\,du = 0 \quad (r \neq n)$$

or

$$= \pm\,\pi \sin 2k\pi \quad (r = n),$$

r and n being integers, positive, negative, or zero. In particular it appears that, if a function is expressible as the sum of either a convergent non-harmonic Fourier cosine series, or the corresponding sine series, at a point x internal to the interval $(0, 2\pi)$, it is expressible by both these series; but that this is not the case at the points 0 and 2π, unless the value of the function at each of these points is zero. At the origin indeed a sine series, of course, converges to zero, and at 2π each term of a cosine series returns to its value at the origin, multiplied by $\cos 2k\pi$.

It follows from the theorem of Riemann-Lebesgue, that the coefficients of a non-harmonic Fourier series tend to zero as we advance along the series. Hence the second integrated series necessarily converges uniformly to a continuous function. We are thus able to show at once that *the necessary and sufficient condition that a non-harmonic* $\left.\begin{matrix}\textit{cosine}\\ \textit{sine}\end{matrix}\right\}$ *series should have the Fourier form is that the second integrated series should converge to a second integral,* and the associated function is then the second differential coefficient of this second integral almost everywhere in* $(0, 2\pi)$: *also, if we prefer to utilise at once both types of series, that we may in this statement replace "second" by "first."*

The first part of the paper terminates with some simple examples of non-harmonic Fourier cosine and sine series, the nature of whose convergence, uniform or bounded, is discussed.

The object of the second part of the paper, §§ 17–25, is mainly to shew that there is no other way of obtaining the development of a given function in such a series as is here contemplated. For this and later

* That is the integral of an integral.

purposes we require the generalisation of Riemann's "Three Theorems on Trigonometrical Series," given in his *Habilitationsschrift.*

Harnack's theorem, too, as to certain conditions under which the coefficients of a trigonometrical series tend to zero is generalised so as to apply to non-harmonic series. In particular we have the result that, *if the points of non-convergence form a set which is at most of the first category in some interval, then the coefficients tend to zero.*

One of the most important consequences of these theorems is the result we were in search of: *no two distinct non-harmonic* $\left.\begin{matrix}\textit{cosine}\\ \textit{sine}\end{matrix}\right\}$ *series of the type* $\sum\limits_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix} (n+k)x$, *with the same* k, *which converge to the same value at each point of* $(0, 2\pi)$ *with the exception at most of a countable set of points* can exist.*

The main object of the third part of the paper (§§ 26 to end), is to apply the considerations which precede to establish rigidly a statement which I made in a previous paper† as to the behaviour of a Bessel series at the point $z = 1$. For this purpose it is convenient to use the theory of restricted non-harmonic Fourier series in a slightly extended form. In the light of the continuation formulæ this theory is found to apply not only when the interval of restriction belongs to the completely open interval $(0, 2\pi)$, but even when one of these points is an included end-point of the interval of restriction.

In consequence we may now state that *the conditions of convergence of a Bessel series at any point of the half-open interval* $(0 < x \leqslant 1)$ *are the same as those of the Fourier series of a function equal to the associated function* $f(z)$ *of the Bessel series in an arbitrary small neighbourhood surrounding the point considered and having any convenient values elsewhere, provided the value assigned to* $f(z)$ *at any point to the right of the point* $z = 2\pi$ *is defined as the same as that at the reflection of this point in* $z = 2\pi$.

2. I begin by stating the following lemmas, which take the place of the familiar normal properties of the sine and cosine in that of Fourier series :—

LEMMA 1.—*If* r *and* n *are different integers, positive, negative, or zero,*

$$\int_0^{2\pi} \sin(n+k)(2\pi-u)\cos(r+k)u\,du = 0 \quad (0 < k < 1),$$

* Or totally imperfect set.

† "On Series of Bessel Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 163–200.

and if $r = n$, *the value of the integral is*

$$\pi \sin 2k\pi.$$

LEMMA 2.—*If* r *and* n *are different integers, positive, negative, or zero,*

$$\int_0^{2\pi} \cos(n+k)(2\pi-u) \sin(r+k)\, u\, du = 0,$$

and, if $r = n$, *the value of the integral is*

$$-\pi \sin 2k\pi.$$

The proofs may be left to the reader.

3. Hence, in place of the ordinary Fourier expansion, we are led to examine the possibility of expanding a function $f(x)$ in a series of the form*

$$\sum_{n=-\infty}^{\infty} a_n \cos(k+n)\, x, \tag{1}$$

where, in view of Lemma 1, the coefficients a_n have what may be called the Non-harmonic Fourier Form,

$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \sin(n+k)(2\pi - t)\, dt, \tag{2}$$

for all integral values of n, positive, negative, or zero, and $f(t)$ is supposed defined and summable only in the closed interval $(0, 2\pi)$, to which we restrict our attention. When the coefficients have this form, in which case $a_n \to 0$, by the theorem of Riemann-Lebesgue, we shall write symbolically

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(k+n)\, x \quad (0 \leqslant x \leqslant 2\pi). \tag{3}$$

We shall then find that the partial summation s_n, that is

$$s_n(x) = \sum_{r=-n}^{n} a_r \cos(k+r)x,$$

apart from an additive term which tends uniformly to zero, has the same form as that of the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small interval enclosing the point x considered, and having any convenient values elsewhere, provided x is internal to the completely open interval $(0, 2\pi)$.

* This symbol is to be understood to mean $\underset{n\to\infty}{\mathrm{Lt}} \sum_{r=-n}^{n} a_r \cos(k+r)\, x$.

At the point 0 the same is still true, provided $f(x)$ be regarded as "continued" beyond the origin on the left, so as to be an even function in the arbitrary small interval in question.

At the point 2π, the formula for s_n only differs from that at 0 by the multiplicative factor $\cos 2k\pi$, and the connection with the Fourier series of $\phi_x(t)$ still holds true, provided $f(t)$ be regarded as "continued" beyond the point 2π on the right, in accordance with the relation

$$\tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} = f(t)\cos 2k\pi, \tag{4}$$

in the arbitrary small interval in question. It will be noticed that the "continuation" defined in the present article is precisely that which is fulfilled of itself when the non-harmonic series converges to $f(x)$ everywhere.

4. To prove these formulæ, we have

$$s_n = \frac{1}{\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\cos (k+r)\, x \sin (k+r)(2\pi-t)\}\, f(t)\, dt$$

$$= \frac{1}{2\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\sin (k+r)(x+t) - \sin (k+r)(x-t)\}\, f(2\pi-t)\, dt \tag{A}$$

$$= \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t) \Big[\sin k(x+t) \Big\{\tfrac{1}{2} + \sum_{r=1}^{n} \cos r(x+t)\Big\}$$
$$- \sin k(x-t) \Big\{\tfrac{1}{2} + \sum_{r=1}^{n} \cos r(x-t)\Big\}\Big]\, dt$$

$$= \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t) [\sin k(x+t) \operatorname{cosec} \tfrac{1}{2}(x+t) \sin (n+\tfrac{1}{2})(x+t)$$
$$- \sin k(x-t) \operatorname{cosec} \tfrac{1}{2}(x-t) \sin (n+\tfrac{1}{2})(x-t)]\, dt. \tag{A'}$$

5. Now suppose first that

$$0 < x < 2\pi,$$

then the integrand is the product of $f(2\pi-t)$ into the sum of two terms, each of the form

$$u(x, t) \sin nt + v(x, t) \cos nt,$$

where, in the first of the two terms, u and v are bounded, except in the neighbourhood of $t = 2\pi - x$, and, in the second, without exception, since $\sin k(x-t) \operatorname{cosec} \tfrac{1}{2}(x-t)$ remains bounded in the neighbourhood of the point $t = x$.

Thus, by the theorem of Riemann-Lebesgue, we may write

$$s_n = o(1) + \frac{1}{2\pi \sin 2k\pi} \int_{2\pi - x - e}^{2\pi - x + e} f(2\pi - t) \sin k(x+t) \operatorname{cosec} \tfrac{1}{2}(x+t) \sin(n+\tfrac{1}{2})(x+t)\, dt$$

$$= o(1) + \frac{1}{2\pi \sin 2k\pi} \int_0^e \{f(x+t) \sin k(2\pi - t) + f(x-t) \sin k(2\pi + t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt$$

$$= o(1) + \frac{1}{2\pi} \int_0^e \{f(x+t) + f(x-t)\} \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt;$$

for, by the theorem of Riemann-Lebesgue, the integral which we have neglected tends to zero, since $\cos 2k\pi \sin kt \operatorname{cosec} \frac{1}{2}t$ remains bounded throughout $(0, e)$.

Now in the last integral we may replace $\cos kt$ by unity, since this integral may be broken up into two parts, differing from the whole only in having unity and $-1 + \cos kt$ respectively in place of $\cos kt$, and the latter part tends to zero by the theorem of Riemann-Lebesgue.

Thus, finally,

$$s_n(x) = o(1) + \frac{1}{2\pi} \int_0^e \{f(x+t) + f(x-t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt, \tag{5}$$

which is the well-known Fourier formula.

This proves the first of our statements (§ 3).

6. Returning to (A′) and putting $x = 0$, we get

$$s_n(0) = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi - t) \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt$$

$$= o(1) + \frac{1}{\pi \sin 2k\pi} \int_{2\pi - e}^{2\pi} f(2\pi - t) \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2}) t\,dt,$$

by the theorem of Riemann-Lebesgue,

$$= o(1) + \frac{1}{\pi \sin 2k\pi} \int_0^e f(t) \sin k(2\pi - t) \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt.$$

Expanding $\sin k(2\pi - t)$ and using again the theorem of Riemann-Lebesgue, we get

$$= o(1) + \frac{1}{\pi} \int_0^e f(t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt$$

$$= o(1) + \frac{1}{\pi} \int_0^e f(t) \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\,dt, \tag{5'}$$

as at the end of the preceding article.

This proves the statement made (§ 3) as to the identity of this formula with that corresponding to the Fourier series of an even function $\phi_0(t)$ equal to $f(t)$ in $(0, e)$.

Thus, *if $f(x)$ is continuous on the right at the origin, the series converges there to $f(0)$.*

7. The statement as to the formula at the point $x = 2\pi$, follows at once. Putting $x = 2\pi$ in formula (A′) it becomes

$$s_n(2\pi) = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi - t) \cos 2k\pi \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})t$$

$$= \cos 2k\pi\, s_n(0)$$

$$= o(1) + \frac{1}{\pi} \int_0^e f(t) \cos 2k\pi \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})t\,dt$$

$$= o(1) + \frac{1}{2\pi} \int_0^e \{f(2\pi - t) + f(2\pi + t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})t\,dt, \quad (5'')$$

supposing $f(t)$ "continued" in accordance with the law formulated in equation (4).

The series therefore converges at 2π if, and only if, it converges at the origin, and the value is $\cos 2k\pi$ times the value at the origin.

8. By the formulæ (5), (5′) and (5″) we have at once the following theorem :—

THEOREM.—*The non-harmonic Fourier cosine series*

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(n+k)x, \quad (3)$$

where
$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \sin(n+k)(2\pi - t)\,dt, \quad (2)$$

converges at any point x of the closed interval $(0, 2\pi)$ under the same conditions, and to the same value, as the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x in question and having any convenient values elsewhere, provided $f(t)$ be supposed "continued" outside the interval in question, so that, in an arbitrary small neighbourhood of the origin it is an even function, and in an arbitrary small neighbourhood of the point 2π,

$$\tfrac{1}{2}\{f(2\pi - t) + f(2\pi + t)\} = f(t) \cos 2k\pi. \quad (4)$$

We remark that it follows that the non-harmonic Fourier series cannot represent the function $f(t)$ at the point 2π unless

$$f(2\pi) = f(+0)\cos 2k\pi, \tag{6}$$

nor at the origin unless $f(t)$ is continuous on the right.

9. Remembering that the theorem of Riemann-Lebesgue asserts, not merely that a certain integral tends to zero, but that it does so uniformly, we may combine the formulæ (5), (5′), and (5″) in the formula

$$s_n(x) - \sigma_n(x) = \eta_n(x), \tag{7}$$

where $\sigma_n(t)$ denotes the n-th partial summation of the Fourier series of the auxiliary function $\phi_x(t)$, and $\eta_n(x)$ tends uniformly to zero.

Now by the definition of the auxiliary function $\phi_x(t)$ as equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x and having any convenient values elsewhere, we see that, taking $|h| < e$, we have for all these values of h, choosing the auxiliary functions conveniently,

$$\phi_{x+h}(t) \equiv \phi_x(t),$$

t lying in the arbitrary small neighbourhood $(x-e,\ x+e)$ already utilised in constructing $\phi_x(t)$. Hence we may write

$$s_n(x+h) - \sigma_n(x+h) = \eta_n(x+h),$$

where $\eta_n(x+h)$ tends uniformly to zero. Therefore

$$\{s_m(x) - s_n(x+h)\} - \{\sigma_m(x) - \sigma_n(x+h)\} = \eta_m(x) - \eta_n(x+h). \tag{8}$$

This shews that if one of the series converges uniformly, or boundedly, at the point x, on the right, or on the left, so does the other.

If the series do not converge, we see, by (7) and (8), that they have the same upper and lower functions, and that, if one of them oscillates boundedly on the right or on the left, so does the other, while if one of them oscillates uniformly (in the first mode) above, or below, on the right, or on the left, so does the other.

In particular if $\qquad f(2\pi) = \cos 2k\pi \,.\, f(0),$

and $f(x)$ is an integral in the closed interval $(0,\ 2\pi)$, *the non-harmonic Fourier cosine series of (x) converges uniformly to $f(x)$ in the closed interval* $(0,\ 2\pi)$.

10. This last result enables us to prove that *the necessary and sufficient condition that a non-harmonic cosine series should have the Fourier form*

is that the second integrated series should converge to a second integral, that is to the integral of an integral, in the closed interval (0, 2π), *and the associated function of the original series is then almost everywhere equal to the second differential coefficient of that repeated integral.**

Indeed writing
$$\int_0^x f(x)\,dx = F(x),$$
$$a+\int_0^x F(x)\,dx = G(x),$$
we shall have
$$G(2\pi) = \cos 2k\pi \,.\, G(0),$$
if a has the value
$$a = -\frac{1}{1-\cos 2k\pi}\int_0^{2\pi} F(x)\,dx.$$

Now let
$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(n+k)x,$$
$$G(x) \sim \sum_{n=-\infty}^{\infty} A_n \cos(n+k)x\,;$$
then the latter series converges uniformly to $G(x)$ in the closed interval (0, 2π).

But
$$-\pi \sin 2k\pi (k+n)^{-2} a_n$$
$$= -\int_0^{2\pi} (k+n)^{-2} f(x) \sin(k+n)(2\pi-x)\,dx$$
$$= -\int_0^{2\pi} (k+n)^{-2} \sin(k+n)(2\pi-x)\,dF(x)$$
$$= -\Big[(k+n)^{-2} \sin(k+n)(2\pi-x)\,F(x)\Big]_0^{2\pi}$$
$$-\int_0^{2\pi} (k+n)^{-1} F(x) \cos(k+n)(2\pi-x)\,dx$$
$$= -\int_0^{2\pi} (k+n)^{-1} \cos(k+n)(2\pi-x)\,dG(x)$$
$$= -\Big[(k+n)^{-1} G(x) \cos(k+n)(2\pi-x)\Big]_0^{2\pi} + \int_0^{2\pi} G(x) \sin(k+n)(2\pi-x)\,dx$$
$$= -(k-n)^{-1}\{G(2\pi)-\cos 2k\pi G(0)\} + A_n$$
$$= A_n.$$

* It will be seen below that in this statement we may replace the second by the first.

The relation between the series

$$\sum_{n=-\infty}^{\infty} a_n \cos(n+k)x \quad \text{and} \quad \sum_{n=-\infty}^{\infty} A_n \cos(n+k)x$$

is therefore that the second is the second integrated series of the first.

Thus the second integrated series of the non-harmonic Fourier cosine series of $f(x)$ converges uniformly to a second integral of $f(x)$, and conversely, the second differentiated series of the non-harmonic Fourier cosine series of a second integral of $f(x)$ is the non-harmonic Fourier cosine series of $f(x)$.

The result stated at the beginning of this article is equivalent to this last double statement and is therefore proved.

11. *Example.*—Put $f(t) = 1$.

We get for all integral values of n, positive, negative, and zero,

$$a_n \pi \sin 2k\pi = (1-\cos 2k\pi)/(k+n),$$

that is,

$$a_n \pi \cot k\pi = 1/(k+n).$$

The "continuation" of the function gives us

$$f(-t) = 1, \quad f(2\pi+t) = 2\cos 2k\pi - 1,$$

where $0 < t < e$.

Thus at each point the auxiliary Fourier series converges, and it converges uniformly except at $x = 2\pi$, where it still converges boundedly. The sum is 1 except at $x = 2\pi$, where it is $\cos 2k\pi$.

Hence, by our general theory the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{k+n} \cos(k+n)x$$

converges uniformly to $\pi \cot k\pi$ at each point of $(0, 2\pi)$, except at the point 2π, where it converges boundedly to $\pi \cot k\pi \cos 2k\pi$.

Thus* in $0 \leqslant x < 2\pi$ we have, uniformly,

$$\pi \cot k\pi = \sum_{n=-\infty}^{\infty} \frac{1}{k+n} \cos(k+n)x$$

$$= \frac{\cos kx}{k} + 2\sum_{n=1}^{\infty} \frac{k \cos kx \cos nx + n \sin kx \sin nx}{k^2-n^2}. \tag{9}$$

* This result has already been obtained (for $0 < x < 2\pi$) by a direct method. See Bromwich, *Theory of Infinite Series*, § 90, p. 231, top, where references are supplied.

Putting $x = 0$ in this equation, we get the well-known formula

$$\pi \cot k\pi = \sum_{n=-\infty}^{\infty} \frac{1}{k+n}. \tag{10}$$

We get the same formula, with each side multiplied by $\cos 2k\pi$, when $x = 2\pi$.

When $x = \pi$, we get the known cosecant formula.

That the convergence is bounded is evident from the series itself, as well as that it is uniform, except at $x = 2\pi$, for

$$2k \cos kx \sum_{n=1}^{\infty} \frac{\cos nx}{k^2-n^2} \quad \text{and} \quad 2k \sin kx \sum_{n=1}^{\infty} \frac{\sin nx}{k^2-n^2}$$

converge uniformly for all values of x, while

$$-2 \sin kx \sum_{n=1}^{\infty} \frac{(k-n) \sin nx}{k^2-n^2} = 2 \sin kx \sum_{n=1}^{\infty} \frac{\sin nx}{k+n}.$$

Now, by Abel's lemma, the latter series converges uniformly for all values of x in $(0, 2\pi)$ excluding the end-points 0 and 2π. Also the series is known to converge boundedly in $(0 \leqslant x \leqslant 2\pi)$, and therefore, when multiplied by $\sin kx$, converges at the origin uniformly to zero. Thus our series, being the sum of these three series, converges boundedly, and, except at $x = 2\pi$, uniformly.

12. Again, in virtue of Lemma 2, we are led to examine the possibility of expanding a function $f(x)$ in a series of the form

$$\sum_{n=-\infty}^{\infty} b_n \sin(k+n)\, x, \tag{11}$$

where, in view of Lemma 2, the coefficients have what may be called the non-harmonic Fourier form,

$$b_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \cos (n+k)(2\pi - t)\, dt, \tag{12}$$

for all integral values of n, positive, negative, and zero: $f(t)$ being supposed defined and summable only in the closed interval $(0, 2\pi)$, to which we restrict our attention.

We again write symbolically

$$f(x) \sim \sum_{n=-\infty}^{\infty} b_n \sin(k+n)\, x, \tag{13}$$

where, by the theorem of Riemann-Lebesgue, $b_n \to 0$ $(n \to \infty)$.

We then find the same result as before, in so far as the internal points of the interval $(0, 2\pi)$ are concerned. We have, however, in order to extend the result to the end-points 0 and 2π to use a different mode of "continuation" from that applicable to the non-harmonic Fourier cosine series. This is again that which is naturally fulfilled when the series converges everywhere to $f(x)$. Indeed the partial summation s_n of the sine series (11) has the same form (A) as before, except that the two terms inside the bracket are now united by the sign + instead of −,

$$s_n = \frac{1}{2\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\sin(k+r)(x+t) + \sin(k+r)(x-t)\} f(2\pi - t)\, dt. \quad \text{(B)}$$

This only affects the term which, at the next stage, disappears by the theorem of Riemann-Lebesgue. We get, in fact, corresponding to the formula (A′),

$$s_n = \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi - t) \left[\sin k(x+t) \operatorname{cosec} \tfrac{1}{2}(x+t) \sin(n+\tfrac{1}{2})(x+t) \right.$$
$$\left. + \sin k(x-t) \operatorname{cosec} \tfrac{1}{2}(x-t) \sin(n+\tfrac{1}{2})(x-t)\right] dt, \quad \text{(B}'\text{)}$$

which leads as before to the formula (5).

13. At the origin $s_n = 0$, always, which implies the law of "continuation" by which $f(t)$ is an odd function in an arbitrary small neighbourhood enclosing the origin.

14. At $x = 2\pi$, we have, using (A′), after changing the − into +,

$$s_n(2\pi) = \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi - t) \{\sin k(2\pi + t) + \sin k(2\pi - t)\}$$
$$\operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(2\pi - t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} f(2\pi - t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$+ \frac{1}{\pi} \int_{2\pi - e}^{2\pi} f(2\pi - t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} \{f(2\pi - t) + f(t) \cos k(2\pi - t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} \{f(2\pi - t) + f(t) \cos 2k\pi\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

using the theorem of Riemann-Lebesgue in the third and fifth of these equations.

Thus if we "continue" our function $f(t)$ beyond the right-hand end-point 2π, according to the law

$$\tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} = f(t)\cos 2k\pi, \tag{14}$$

we shall have

$$s_n(2\pi) = o(1)+\frac{1}{2\pi}\int_0^e \{f(2\pi-t)+f(2\pi+t)\}\operatorname{cosec}\tfrac{1}{2}t\sin(n+\tfrac{1}{2})t\,dt, \tag{5''}$$

which is the Fourier formula.

15. Hence we have the following theorem:—

THEOREM.—*The non-harmonic Fourier sine series*

$$f(x) \sim \sum_{n=-\infty}^{\infty} b_n \sin(n+k)x, \tag{13}$$

where
$$b_n = \frac{1}{\pi\sin 2k\pi}\int_0^{2\pi} f(t)\cos(n+k)(2\pi-t)\,dt, \tag{12}$$

converges at any point x of the closed interval $(0, 2\pi)$ under the same conditions, and to the same value, as the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x in question and having any convenient values elsewhere, provided $f(t)$ be supposed "continued" outside the interval in question, so that, in an arbitrary small neighbourhood of the origin it is an odd function, and in an arbitrary small neighbourhood of the point 2π,

$$\tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} = f(t)\cos 2k\pi. \tag{14}$$

We remark that it follows that the non-harmonic Fourier series *cannot represent the function $f(t)$ at the origin unless*

$$f(0) = 0, \tag{15}$$

and it cannot represent the function $f(t)$ at the point 2π, unless

$$f(2\pi) = f(2\pi-0)+f(+0)\cos 2k\pi. \tag{16}$$

If therefore continuous on the right at the origin, it must be continuous on the left at 2π.

The discussion of the uniform, or bounded, convergence or oscillation, on the right, or on the left, given in § 9, evidently applies equally to the sine series, the appropriate law of "continuation" being supposed utilised.

In particular, if $f(0) = 0$, *and* $f(x)$ *is an integral in the closed interval* $(0, 2\pi)$, *in which case* (16) *holds, then the non-harmonic Fourier sine series of* $f(x)$ *converges uniformly to* $f(x)$ *in the closed interval* $(0, 2\pi)$.

Thus, as an application of this result, we see that if

$$F(x) = \int_0^x f(x)\,dx \sim \sum_{n=-\infty}^{\infty} b_n \sin(k+n)x,$$

where

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(k+n)x,$$

the non-harmonic Fourier sine series of $F(x)$ converges uniformly in $(0, 2\pi)$ to $F(x)$. But

$$\begin{aligned}\pi \sin 2k\pi (k+n)^{-1} a_n &= \int_0^{2\pi} (k+n)^{-1} \sin(k+n)(2\pi - x) f(x)\,dx \\ &= \int_0^{2\pi} (k+n)^{-1} \sin(k+n)(2\pi - x)\,dF(x) \\ &= \int_0^{2\pi} F(x) \cos(k+n)(2\pi - x)\,dx = b_n.\end{aligned}$$

Thus we see that *the first integrated series of a non-harmonic Fourier cosine series converges uniformly in the closed interval* $(0, 2\pi)$ *to the integral of the associated function of the cosine series; and, conversely, the first derived series of the non-harmonic Fourier sine series of an integral is a non-harmonic Fourier cosine series, whose associated function has that integral for integral.*

Similarly, using the result proved in § 9, that the non-harmonic cosine series of an integral which satisfies the proper "continuation" law, converges uniformly to that integral in the closed interval $(0, 2\pi)$, we see that in the preceding statement in *italics* we may write *cosine* for *sine*.

We may state these results otherwise by saying: *The necessary and sufficient condition that a non-harmonic sine or cosine series should have the Fourier form is that the first integrated series should converge towards an integral in the closed interval* $(0, 2\pi)$.

We have seen that a similar condition may be stated in the case of the cosine series, and may evidently also be given for the sine series, in which the second integrated series takes the place of the first in this statement, the integral (first integral) being replaced by a second integral. This form of the condition has the advantage of only comparing sine series with sine series, and cosine series with cosine series.

16. As an example, suppose that

$$f(x) = 1.$$

We get for all integral values of n, positive, negative, and zero,

$$b_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \cos (k+n)(2\pi - t)\, dt = \frac{1}{\pi(k+n)}.$$

The "continuation" according to the law appropriate to the sine series, gives us $f(-t) = 1$, and $f(2\pi+t) = 2\cos 2k\pi + 1$. Thus at the origin and at the point 2π the auxiliary Fourier series, which is everywhere convergent, does not represent the function. The convergence is uniform except at the end-points, where it is bounded. Hence, for $0 < x < 2\pi$, we have*

$$\pi = \sum_{n=-\infty}^{\infty} \frac{\sin (k+n)\, x}{k+n} = \frac{\sin kx}{k} + \sum_{n=1}^{\infty} \frac{k \sin kn \cos nx - n \cos kn \sin nx}{k^2 - n^2}, \tag{17}$$

the convergence being uniform, while at the origin the series converges boundedly to zero, and, at the point 2π, it converges boundedly to $\pi\{1+\cos 2k\pi\} = 2\pi \cos^2 k\pi$, agreeing with (10).

17. We next proceed to generalise Riemann's three theorems on trigonometrical series.† Riemann's reasoning applies with comparatively slight alterations to non-harmonic trigonometrical series, and in order to save space I have not written out the proofs at length.

RIEMANN'S FIRST THEOREM (*generalised*).—

If the series $\sum_{n=-\infty}^{\infty} A_n^{(k)}$, *where*

$$A_n^{(k)} = a_n \cos (n+k)\, x + b_n \sin (n+k) x,$$

converges at the point x *to, say,* $f(x)$, *and its second integrated series converges at* x *and in its neighbourhood to a function* $G(x)$, *then*

$$\{G(x+a) - 2G(x) + G(x-a)\}/a^2 \to f(x) \quad (a \to 0).$$

* See footnote to § 11.

† B. Riemann, *Habilitationsschrift*, § 8, *Ges. Werke*, pp. 246–249.

We have

$$\{G(x+2a)-2G(x)+G(x-2a)\}/4a^2$$

$$=-\sum_{n=-\infty}^{\infty}(k+n)^{-2}a_n\{\cos(k+n)(x+2a)-2(k+n)\cos x+\cos(k+n)(x-2a)\}$$

$$=\sum_{n=-\infty}^{\infty}A_n\left\{\frac{\sin(k+n)a}{(k+n)a}\right\}^2. \tag{18}$$

From this point on the proof is, in all essentials, identical with Riemann's.*

COR.—*Under the same circumstances if* α *and* β *approach zero so that the ratio of each to the other does not tend towards zero*

$$\frac{G(x+\alpha+\beta)-G(x+\alpha-\beta)-G(x-\alpha+\beta)+G(x-\alpha-\beta)}{4\alpha\beta}\to f(x).$$

This is the form in which Riemann enunciates his theorem, but we use it in the simpler form, from which as he points out, it immediately follows.

18. RIEMANN'S SECOND THEOREM (*generalised*).—

If the second integrated series of the series

$$\sum_{n=-\infty}^{\infty}A_n\equiv\sum_{n=-\infty}^{\infty}\{a_n\cos(n+k)x+b_n\sin(n+k)x\},$$

converges at x *and in its neighbourhood to a function* $G(x)$, *then*

$$\{G(x+\alpha)-2G(x)+G(x-\alpha)\}/\alpha\to 0 \quad (\alpha\to 0).$$

The proof of this theorem also is practically identical with Riemann's.

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* Similarly we may shew that *when the series* $\sum_{n=-\infty}^{\infty}A_n$ *does not converge, all the limits of*

$$\{G(x+h)-2G(x)+G(x-\alpha)\}/\alpha^2,$$

when $\alpha\to 0$, *lie between the two bounds*

$$\tfrac{1}{2}(U+L)\pm\tfrac{1}{2}(U-L)\left(1+\frac{2}{\pi^2}\right),$$

where U *and* L *are the values of the upper and lower functions of* $\sum_{-\infty}^{\infty}A_n$ *at the point* x, *provided only the second integrated series converge and* $G(x)$ *denote its sum.*

This form of the result is due to my wife, being a closer pair of bounds than that given in the case of harmonic series by Hassenfelder, "Zur Theorie der trigonometrischen Reihe" (1900), *Jahresbericht des königlichen Gymnasiums zu Strassburg* (Teubner), as an emendation of a similar result due to Du Bois Reymond. See a forthcoming paper by Grace Chisholm Young, also Hobson's *Theory of Functions of a Real Variable.*

19. The extended form of Riemann's Third Theorem, I have already given in the recent communication to the Royal Society already cited.[*] The enunciation is as follows:—

RIEMANN'S THIRD THEOREM (*generalised*).—

If in (b, c) *the function* $\lambda(x)$ *is the integral of an integral, and if* $\lambda(x)$ *and* $\lambda'(x)$ *are both zero at* b *and* c, *then*

$$I \equiv \mu^2 \int_b^c \{G(x+u)+G(x-u)\}\, \lambda(u) \begin{matrix}\cos\\ \sin\end{matrix}\, \mu u\, du \to 0 \quad (\mu \to \infty),$$

where

$$G = -\sum_{n=-\infty}^{\infty} (n+k)^{-2}\{a_n \cos(n+k)x + b_n \sin(n+k)x\} = -\sum_{n=-\infty}^{\infty} (n+k)^{-2} A_n^{(k)},$$

where $(0 \leqslant k < 1)$, *provided only* (i) *the series converge uniformly, and* (ii) $A_n^{(k)} \to 0$ $(n \to \infty)$.

The proof, which also is closely modelled on Riemann's will not be introduced here.

20. Harnack has proved the following result:—

If (a, b) *be any interval, and* $r_1, r_2, \ldots$ *any sequence of continually increasing positive integers, we can find a sub-sequence* $n_1, n_2, \ldots$ *and a point* x, *internal to* (a, b), *such that, from and after a certain index, the values*

$$n_i x,$$

all differ from an odd multiple of $\frac{1}{2}\pi$ *by less than an assigned (arbitrary) small positive number* ζ.[†]

Hence it immediately follows that if at every internal point x of (a, b), we have

$$|\lambda_{n_i} \sin n_i x| < \delta,$$

for all indices i, we must have for all indices i from and after a certain integer,

$$|\lambda_{n_i}| < \delta \sec \zeta,$$

and therefore

$$\operatorname*{Lt}_{i \to \infty} |\lambda_{n_i}| \leqslant \delta \sec \zeta \leqslant \delta,$$

since ζ is arbitrary, and may be chosen as small as we please.

From this result Harnack easily deduces a theorem for harmonic

[*] "On Non-harmonic Trigonometrical Series", 1918, § 10.

[†] The proof may be consulted in Hobson's *Theory of Functions of a Real Variable*, p. 747.

trigonometric series, equally true for non-harmonic series. His proof is equally valid in the more general case. The theorem is as follows :—

HARNACK'S THEOREM (*generalised*).—

If $\sum_{-n}^{n} A_n^{(k)}(x)$ *is a non-harmonic or harmonic trigonometric series, where*

$$A_n^{(k)} = a_n \cos(n+k)\,x + b_n \sin(n+k)\,x \quad (0 \leqslant k < 1),$$

and the series is such that in (0, 2π) *there is a sub-interval in which the upper and lower functions at each point* x *differ by less than* δ, *then all the limits of the coefficients* a_n *and* b_n, *when* $n \to \infty$, *are numerically less than or equal to* δ.

COR. 2.—*If the points of non-convergence form a set which is at most of the first category in some interval, then the coefficients tend towards zero.*

For the points at which the excess of the upper function over the lower function is greater than δ form, as is known, an inner limiting set, and therefore form in every interval in which they are dense everywhere a set of the second category, and cannot therefore be a sub-set of a set of the first category; they must be nowhere dense therefore in an interval in which the points of non-uniform convergence form a set of the first category. Thus the conditions of the corollary hold good, since this is true for all values of δ.

21. To prove the uniqueness of the expansion in a non-harmonic Fourier cosine or sine series, we state and prove the theorem in a form which is practically the most general possible, though, by employing a less simple method the words "countable set of points" might be replaced by "totally imperfect set," that is a set containing no perfect subset. We give the proof for the cosine series, but it will be seen that we may in it substitute the sine for the cosine. The theorem is as follows :—

THEOREM.—*No two distinct non-harmonic* $\begin{cases} \textit{cosine} \\ \textit{sine} \end{cases}$ *series of the type* $\sum_{n=-\infty}^{\infty} a_n \cos(n+k)\,x$, *with the same* k, *which converge to the same value at each point of* (0, 2π), *with the exception at most of a countable set of points, can exist.*

Suppose, if possible, that there were two such series, and let their difference be denoted by $\sum_{n=-\infty}^{\infty} a_n \cos(n+k)x$: then this latter series must converge to zero, except at a countable set of points.

Since a countable set is a set of the first category, it follows from the Cor. 2 of § 20, that the coefficient a_n tends to zero.

If we then write

$$G(x) = -\sum_{n=-\infty}^{\infty} (n+k)^{-2} a_n \cos(n+k)x,$$

we see that $G(x)$ is a continuous function, satisfying the two conditions that

$$\text{(i)} \quad \{G(x+t)+G(x-t)-2G(x)\} = 0,$$

except at the countable set of points; and further, without exception

$$\text{(ii)} \quad \{G(x+t)+G(x-t)-2G(x)\}\, t \to 0 \quad (t \to 0).$$

Hence by the extension of Schwartz's theorem,* $G(x)$ is a linear function of x, and we may write

$$Ax+B = -\sum_{n=\infty}^{\infty} (n+k)^{-2} a_n \cos(n+k)x,$$

the convergence being uniform.

Multiply by $\sin(n+k)(2\pi-x)$ and integrate from 0 to 2π, and we get, since term-by-term integration is allowable,

$$\int_0^{2\pi} (Ax+B)\sin(n+k)(2\pi-x)\,dx = -(n+k)^{-2} a_n \pi \sin 2k\pi,$$

whence

$$\int_0^{2\pi} (Ax+B)\,d\cos(n+k)(2\pi-x) = (n+k)^{-1} a_n \pi \sin 2k\pi,$$

that is, integrating by parts,

$$\Big[(Ax+B)\cos(n+k)(2\pi-x)\Big]_0^{2\pi} - \int_0^{2\pi} A\cos(n+k)(2\pi-x)\,dx$$
$$= (n+k)^{-1} a_n \pi \sin 2k\pi,$$

that is,

$$(2\pi A+\beta) - B\cos 2k\pi + (n+k)^{-1} A \sin 2k\pi = (n+k)^{-1} a_n \pi \sin 2k\pi.$$

* Hobson's *Theory of Functions of a Real Variable*, p. 211.

Now let $n \to \infty$, then, since $A \sin 2k\pi$ is finite, and $a_n \to 0$, we get

$$(2\pi A + B) - B \cos 2k\pi = 0.$$

Therefore also, by the preceding equation,

$$A \sin 2k\pi = a_n \pi \sin 2k\pi.$$

Now again letting $n \to \infty$, we get

$$A \sin 2k\pi = 0;$$

and therefore $$A = 0.$$

Hence also from the former equation

$$B = 0.$$

Thus, by the above expression for $(n+k)^{-1} a_n \pi \sin 2k\pi$, we have, for all values of n,

$$a_n = 0,$$

which proves the theorem.

22. The considerations above exposed lead naturally, as in the case of harmonic series,* from the theory of non-harmonic Fourier series to that of restricted non-harmonic Fourier series. The definition is as follows:

The p-th derived series of a non-harmonic Fourier cosine or sine series of $F(x)$ is said to be a non-harmonic restricted Fourier cosine or sine series of the p-th class, and to be restricted to one or more intervals (α, β), if, throughout each completely open interval in (α, β), $F(x)$ is a p-th integral. The p-th differential coefficient of $F(x)$ is then said to be the associated function of the restricted Fourier series in any of these intervals of restriction.

If, in addition, the coefficients of the restricted Fourier series tend to zero as we advance along the series, it is said to be "ordinary," and is denoted briefly as a non-harmonic R. F. series.

We have already seen that a non-harmonic Fourier sine or cosine series satisfies both these conditions in the closed interval $(0, 2\pi)$.

One of the fundamental properties of non-harmonic restricted Fourier

* W. H. Young, "On the Convergence of the Derived Series of Fourier Series" (1916), *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1919), pp. 195–236; "On the Ordinary Convergence of Restricted Fourier Series" (1917), *Proc. Roy. Soc.*, (A), Vol. 93, pp. 276–292; "On Restricted Fourier Series and the Convergence of Power Series" (1917), *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1919), pp. 353–366.

series* is embodied in the following theorems :—

THEOREM.—*The upper and lower functions at any point internal to the interval of restriction are independent of the nature of the associated function, except in an arbitrary small neighbourhood surrounding the point, provided the individual terms of the series tend to zero at the point as we advance along the series ; indeed the part of the expression for the n-th partial summation depending on the remaining values of the function tends uniformly to zero.*

THEOREM.—*In the case when the typical coefficient itself converges to zero so that the series is "ordinary", we can go still further and assert that the series behaves at any internal point of its interval of restriction precisely like the Fourier series of a function equal to the associated function of the series in the neighbourhood of the point and having any convenient values elsewhere.*

These theorems remain true at the end-points, provided the associated function satisfy at the point in question the appropriate "continuation" formula applying to the point.

23. For the proof of these theorems we require a modified form of the expressions (A′) and (B′). These expressions involve only the values of $f(x)$ in the interval $(0, 2\pi)$. We shall now suppose that $f(x)$ is "continued" according to the appropriate law. The formulæ (A″) and (A‴) obtained are identical for the sine and cosine series, only varying in these "continuation" formulæ, and in the meaning of the symbol $A_n^{(k)}$. Taking, for example, the formula (A′) with the corresponding "continuation" formulæ

$$f(t) = f(-t),$$

$$\tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} = \cos 2k\pi \, . \, f(t),$$

we may transform (A′) as follows :

Writing in the first term of the bracket

$$x+t = 2\pi - u,$$

and in the second term $\qquad x-t = u-2\pi,$

* These theorems were given by me in the communication to the Royal Society quoted above. The proof is reproduced here, slightly adjusted so as to agree with the present point of view, and so as to include the end-points.

we get for (A′), omitting the factor $1/(2\pi \sin 2k\pi)$,

$$\int_{-x}^{2\pi-x} f(x+u) \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du$$
$$-\int_{x}^{2\pi+x} f(u-x) \sin k(u-2\pi) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du.$$

Adding and subtracting the integrals requisite to change the limits of integration in both these terms to $(0, 2\pi)$, and then making the obvious changes of the independent variable in the superfluous integrals, we get

$$\int_0^{2\pi} \{f(x+u)+f(u-x)\} \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du$$
$$+\int_0^{x} [f(x-u) \sin k(2\pi+u) - f(u-x) \sin k(2\pi-u)$$
$$- \{f(x-u+2\pi)+f(2\pi+u-x)\} \sin ku] \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2}) u\, du.$$

Since f satisfies the "continuation" formulæ appropriate to a cosine series, the superfluous integral will be zero, and we get

$$s_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2}) u\, du \qquad \text{(A'')}$$
$$= \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du$$
$$+ \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \cos nu\, du.$$

The last of these integrals, being one half the sum of the n-th and the $-n$)-th coefficients in the non-harmonic Fourier cosine series of $\tfrac{1}{2}\{f(x+u)+f(x-u)\}$ is, as will be shewn in the next article, equal to $\tfrac{1}{2}\{A_n^{(k)}+A_{-n}^{(k)}\}$. Thus the partial summation s_n of our series is given by

$$s_n = \sum_{r=-n}^{n} A_r^{(k)} = \sum_{r=-n}^{n} a_r \cos(r+k)\, x$$
$$= \tfrac{1}{2}\{A_n^{(k)}+A_{-n}^{(k)}\} + \frac{1}{\pi \sin k\pi} \int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\}$$
$$\times \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du, \qquad \text{(A''')}$$

in which formula $\{f(x+u)+f(x-u)\}$ must be supposed to obey the "continuation" laws appropriate to the cosine series, since $\sum_{r=-\infty}^{\infty} A_r^{(k)}$ is a cosine series. If this series is a sine series the same formula holds, but the continuation is that appropriate in this case (15) and (16).

24. Indeed, supposing $f(t) = f(-t)$, we have

$$\int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\} \sin(n+k)(2\pi-u)\,du$$

$$= \int_x^{2\pi+x} \tfrac{1}{2}f(t)\sin(n+k)(2\pi-t+x)\,dt + \int_{-x}^{2\pi-x} \tfrac{1}{2}f(t)\sin(n+k)(2\pi-t-x)\,dt$$

$$= \int_0^{2\pi} f(t)\sin(n+k)(2\pi-t)\cos(n+k)x\,dt$$

$$-\int_0^x \tfrac{1}{2}\{f(t)\sin(n+k)(2\pi-t+x)-f(t)\sin(n+k)(2\pi+t-x) -f(2\pi+t)\sin(n+k)(x-t)+f(2\pi-t)\sin(n+k)(t-x)\}\,dt$$

$$= \cos(n+k)x\int_0^{2\pi} f(t)\sin(n+k)(2\pi-t)\,dt, \qquad (19)$$

supposing the "continuation" formula (4) to hold.

If on the other hand the "continuation" formulæ (15) and (16) holds, we get, similarly,

$$\int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\} \sin(n+k)(2\pi-u)\,du = \sin(n+k)x\int_0^{2\pi} f(t)\cos(n+k)(2\pi-t)\,dt. \qquad (20)$$

Thus supposing known the non-harmonic Fourier $\left\{\begin{matrix}\text{cosine}\\ \text{sine}\end{matrix}\right.$ series of $f(x)$, say

$$f(x) \sim \sum_{n=-\infty}^{\infty} A_n^{(k)} = \sum_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix}(n+k)x,$$

we have $$\tfrac{1}{2}\{f(x+u)+f(x-u)\} \sim \sum_{n=-\infty}^{\infty} A_n^{(k)}\cos(n+k)u. \qquad (21)$$

25. We proceed now to the proof of the theorems enunciated in § 22. Writing for the second integrated series,

$$G(x) \sim -\sum_{n=-\infty}^{\infty}(n+k)^{-2}A_n^{(k)} = -\sum_{n=-\infty}^{\infty}(n+k)^{-2}a_n \begin{matrix}\sin\\ \cos\end{matrix}(n+k)x,$$

we have seen that, by (A‴), and § 25,

$$-\sum_{r=-n}^{n}(r+k)^{-2}A_r^{(k)} = -\tfrac{1}{2}(n+k)^{-2}\{A_n^{(k)}+A_{-n}^{(k)}\}$$
$$+\frac{1}{2\pi\sin 2k\pi}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin k(2\pi-u)\cot\tfrac{1}{2}u\sin nu\,du.$$

Thus the upper and lower functions of the non-harmonic restricted Fourier series in question, are the upper and lower limits, when $n \to \infty$, of the second differential coefficient of this expression. Since $A_n^{(k)}+A_{-n}^{(k)}$ tends by hypothesis to zero, we have only to consider the upper and lower limits of

$$I \equiv \frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\} \sin k\,(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du.$$

In treating this expression, the first step is to introduce in place of

$$\sin k(2\pi-u) \cot \tfrac{1}{2}u,$$

a function $\phi(u)$, defined equal to it in $(e < u \leqslant 2\pi)$, and having at the origin the value $-2k$, while its differential coefficient is zero there. In $(0, e)$ we suppose $\phi(u)$ to be the integral of an integral, with the same value and the same left and right hand differential coefficients at the point e, and to be otherwise subject to no conditions. The function $[\phi(u)+2k]$ then satisfies the conditions imposed on $\lambda(u)$ in the extension of Riemann's Third Theorem (§ 19).

We thus have

$$\frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\}\{\phi(u)+2k\} \sin nu\, du \to 0 \quad (n \to \infty).$$

Hence we may replace (I) in our investigation by

$$\frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\}\{\sin k(2\pi-u) \cot \tfrac{1}{2}u - \phi(u) - 2k\} \sin nu\, du$$

$$= \frac{d^2}{dx^2}\int_0^{e} \{G(x+u)+G(x-u)\}\{\sin k(2\pi-u) \cot \tfrac{1}{2}u - \phi(u)\} \sin nu\, du$$

$$-2k\frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\} \sin nu\, du.$$

We see then we only need to prove that the last term tends to zero.

For this purpose we introduce a new function $\psi(u)$, defined as equal to

$$1-\cos k(2\pi-u) \quad (e \leqslant u \leqslant 2\pi),$$

and having at the origin zero for its value and for that of its differential coefficient. This function when suitably filled in from 0 to e, satisfies the conditions imposed on $\lambda(u)$ in the extended form of Riemann's Third Theorem. Therefore, omitting for the moment the factor $-2k$, we may replace the last term in question by

$$\frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\} [1-\psi(u)] \sin nu\,du$$

$$= \frac{d^2}{dx^2}\int_0^{e} \{G(x+u)+G(x-u)\} \{1-\psi(u)-\cos k(2\pi-u)\} \sin nu\,du$$
$$+ \frac{d^2}{dx^2}\int_0^{2\pi} \{G(x+u)+G(x-u)\} \cos k(2\pi-u) \sin nu\,du.$$

The last term is equal to

$$\frac{d^2}{dx^2}\int_0^{2\pi} \tfrac{1}{2} \{G(x+u)+G(x-u)\} \{\sin(k+n)(2\pi-u)-\sin(k-n)(2\pi-u)\}\,du$$
$$= \pi \sin 2k\pi \{A_n^{(k)}-A_{-n}^{k}\} \to 0.$$

Thus, finally, writing now $\phi(u)$ instead of

$$\phi(u)+2k\{1-\psi(u)-\cos k(2\pi-u)\},$$

we see that the partial summation s_n of our non-harmonic restricted Fourier series is given by the formula

$$s_n = o(1)+\frac{d^2}{dx^2}\int_0^{e} \tfrac{1}{2}\{G(x+u)+G(x-u)\}$$
$$\times \{\sin k(2\pi-u)\cot \tfrac{1}{2}u-\phi(u)\} \sin nu\,du/\pi \sin 2k\pi, \quad (22)$$

where $\phi(u)$ is the integral of an integral, with the value $-2k\cos 2k\pi$ at the origin, and $\sin k(2\pi-e)\cot\frac{1}{2}e$ at the point e. If the point x is internal to the interval $(0, 2\pi)$, we can, in this formula, choose e so small that no "continuation" is necessary. But if $x=0$, we have to regard G as an even function, and, if $x=2\pi$, we have to assume that the "continuation" formula used is that appropriate to a $\left\{\begin{array}{l}\text{cosine}\\ \text{sine}\end{array}\right.$ series.

Since our upper and lower functions are the upper and lower limits of s_n, this proves the first theorem of § 22.

25. Now if in the interval $(x-e, x+e)$ the function G is the integral of the integral of f, we may in the expression (22) perform the differentiation under the integral sign, writing f in place of G. By the theorem of Riemann-Lebesgue, the part of the integral involving the arbitrary function ϕ then tends to zero. Thus we are left with

$$s_n = o(1)+\frac{1}{\pi}\int_0^{e}\{f(x+u)+f(x-u)\}\frac{\sin k(2\pi-u)}{\sin 2k\pi}\cot\tfrac{1}{2}u \sin nu\,du.$$

Now, when $u \to 0$,

$$\left\{ \frac{\sin k(2\pi u)}{\sin 2k\pi} - 1 \right\} \cot \tfrac{1}{2}u = \frac{[\sin k(2\pi - u) - \sin 2k\pi] \cos \frac{1}{2}u}{\sin 2k\pi \sin \frac{1}{2}u} \to - \frac{k \cot 2k\pi}{\frac{1}{2}},$$

and is therefore bounded in the neighbourhood of the origin.

By the theorem of Riemann-Lebesgue, therefore, we may replace our expression by

$$s_n = o(1) + \int_0^e \{f(x+u) + f(x-u)\} \cot \tfrac{1}{2}u \sin nu \, du,$$

which is the Fourier expression.

This completes the proof of the second of the theorems of § 22.

26. We come now to the application of the results of the present paper to the theory of Bessel series. In particular we are now able to obtain the conditions of convergence at the end-point $z = 1$ of the interval of restriction of the Bessel series.

The asymptotic approximation

$$\sqrt{(\tfrac{1}{2}\pi x)}\, J_m(x) = \cos\psi + \frac{m^2 - \frac{1}{4}}{2x} \sin\psi - \frac{(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{2!\,(2x)^2} \left[\cos\psi + \theta \frac{m^2 - \frac{16}{4}}{3\,(2x)}\right],$$

where $|\theta| < 1$ and $\psi = -x + \frac{1}{4}(2m+1)\pi$, holds up to *and including* the point $x = 1$.

Consequently we may regard as valid up to *and including* $z = 1$, the formal restrictions (21) and (29) obtained in my paper on "Series of Bessel Functions", viz., firstly, the relation (21),

$$\sum_{r=0}^{\infty} A_r J_m(k_r z) \equiv z^{-\frac{1}{2}} \{S_1 + z^{-1} S_2 + z^{-2} S_3\},$$

where S_3, and the first integrated series of S_2, and the second integrated series of S_1, are convergent series whose sums are integrals; while the second integrated series of the Bessel series on the left of this formal identity is a uniformly convergent series whose sum is the integral of an integral; and, secondly, the relation (29),

$$S_1 \equiv T + \beta y T_1 + T_2, \quad y = \pi(1-z),$$

where T_2 and the first integrated series of T_1 converge uniformly to integrals, while

$$T = \sum_{r=0}^{\infty} (-)^{r+s} A_r \sqrt{\left(\frac{2}{\pi k_r}\right)} \cos(k+r)\, y.$$

We recall also the fact that, apart from an irrelevant factor, S_2 and T_1 are the first integrated series of S_1 and T respectively.

Now, by hypothesis $$A_r k_r^{-\frac{1}{2}} \to 0 \quad (r \to \infty),$$

and, integrating twice with respect to $y = \pi(1-z)$, the series

$$S_1 - \beta y T_1 - T_2,$$

we get the integral of an integral. Thus *T is a non-harmonic restricted Fourier series in* $(0 \leqslant y < 1)$, and therefore in this interval T_1, the first integrated series of T, converges to an integral, the convergence being uniform in every closed interval inside $(0 \leqslant y < 1)$. Similarly now we see that S_2 converges uniformly to an integral.

Thus we get from the given relations

$$\sum_{r=0}^{\infty} A_r J_m(k_r z) = z^{-\frac{1}{2}} \{T + \phi(z)\},$$

where $\phi(z)$ is an integral in $(0 < z \leqslant 1)$.

Hence, denoting the function associated with the Bessel series by $f(z)$, and that associated with the series T by $g(y)$, where $y = \pi(1-z)$, we get, after multiplying by $z^{\frac{1}{2}}$ and integrating twice with respect to $z = 1 - y/\pi$,

$$\int dz \int z^{\frac{1}{2}} f(z)\, dz = \int dz \int g\, \{\pi(1-z)\}\, dz + \int dz \int \phi(z)\, dz,$$

whence, differentiating twice, and choosing the function g suitably at the set of content zero at which the differentiation does not work,

$$g\{\pi(1-z)\} = z^{\frac{1}{2}} f(z) - \phi(z).$$

Thus, by our theory, the non-harmonic restricted Fourier series T behaves at the point $\pi(1-z)$ like the Fourier series of a function which in the neighbourhood of the point considered has the form

$$z^{\frac{1}{2}} f(z) - \phi(z),$$

provided the point z is *internal* to the interval (0, 1); and at the point $z = 1$ [which is the same as $\pi(1-z) = 0$], the function has this form, and on the right is reflected in the point $z = 1$. This will be satisfied if $z^{\frac{1}{2}} f(z)$ and $\phi(z)$ separately have this "continuation" property. Now $\phi(z)$ is an integral up to and including the point $z = 1$, and is therefore, when "continued" by reflection in this point, still an integral in the whole neighbourhood considered. Therefore its Fourier series converges uniformly, and, consequently, the term $\phi(z)$ may be suppressed in the expression for our auxiliary function. Also the Fourier series of $z^{\frac{1}{2}} f(z)$

behaves in $(0 < z < 1)$ like the Fourier series of $f(z)$, and the Fourier series of the function which in the neighbourhood of $z = 1$ is equal to $z^{\frac{1}{2}} f(z)$ on the left and is reflected on the right, behaves like that of the function equal to $f(z)$ on the left and reflected on the right, for in each case the factor removed represents a bounded function.

Thus, finally, observing that, in virtue of the relations found, the Bessel series behaves in respect of convergence, and so forth, according to the behaviour of the series T, we see that *the Bessel series converges, oscillates, or diverges, under precisely the same conditions and in precisely the same manner (uniform, bounded, &c.) as the Fourier series of a function which in a neighbourhood enclosing the point z considered is equal to $f(z)$, if $(0 < z < 1)$, and, if $z = 1$, has this form on the left of the point and is reflected in the point.*