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XXIII. *On a Proposition relating to the Theory of Equations.* By JAMES COCKLE, *M.A., of Trinity College, Cambridge; of the Middle Temple, Special Pleader**.

1. LET x be the root of the general equation of the nth degree, and

$$y = \Lambda' x^{\lambda'} + \Lambda'' x^{\lambda''} + \Lambda''' x^{\lambda'''} + \Lambda^{\text{iv}} x^{\lambda^{\text{iv}}}; \quad . \quad . \quad . \quad (\text{a.})$$

also let ${}_m\text{Y}$ be composed of symmetric functions of, and be homogeneous and of the mth degree with respect to y; then, if $n > 2$, ${}_2\text{Y}$ may be reduced to the form

$$(a'_1 \Lambda' + a''_1 \Lambda'' + b')^2 + (a''_2 \Lambda'' + b'')^2, \quad . \quad . \quad . \quad (\text{b.})$$

where b' and b'' are not both zero.

2. For, let

$$\Lambda''' x_n^{\lambda'''} + \Lambda^{\text{iv}} x_n^{\lambda^{\text{iv}}} = l' x_n^{\lambda'} + l'' x_n^{\lambda''}, \quad . \quad . \quad . \quad (\text{c.})$$

then if $\quad y_r = (\Lambda' + l') x_r^{\lambda'} + (\Lambda'' + l'') x_r^{\lambda''} + \text{l}_r, \quad . \quad . \quad (\text{d.})$

$$\text{l}_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{e.})$$

Now ${}_2\text{Y}$ is to be reduced, by means of (d.), to the form (b.), independently of Λ, or, what is the same thing, of $\Lambda + l$; but†

$${}_2\text{Y} = (\text{b.}) + [\text{l}_1 \ldots \text{l}_{n-1}]^2, \quad . \quad . \quad . \quad . \quad . \quad (\text{f.})$$

$[\quad]^m$ denoting a homogeneous function of the enclosed quantities of the mth degree. And, if $n - 1 > 1$,

$$[\text{l}_1 \ldots \text{l}_{n-1}]^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{g.})$$

may be satisfied without making the l's zero.

3. Following a notation similar to that used in my last paper‡, let (p, q) represent the coefficient of $\Lambda^{(p)} \Lambda^{(q)}$ in the development of

$$t p_2 - s p_1^2 = {}_2\text{Y} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{h.})$$

p_2 and p_1 being respectively the coefficients of the third and second terms of the transformed equation in y; then, if (h.) be reducible to the form (b.), we have

$$\ldots + \ldots + b' \pm \sqrt{-1} \, . \, b'' = 0; \quad . \quad . \quad . \quad (\text{i.})$$

and both the values of the above expression can only vanish when $b' = 0 = b''$. Substitute for b' and b'', equate each expression to zero, and eliminate $\frac{\Lambda'''}{\Lambda^{\text{iv}}}$ between the two; then we have

$$(1, 3)(2, 4) - (1, 4)(2, 3) = 0, \quad . \quad . \quad . \quad . \quad (\text{j.})$$

where, for instance,

$$(1, 3) = t \Sigma (x_1^{\lambda'} x_2^{\lambda'''}) - 2 s \Sigma (x_1^{\lambda'}) \, . \, \Sigma (x_1^{\lambda'''}); \quad . \quad . \quad (\text{k.})$$

* Communicated by the Author.

† For the process, see par. 3 of the place which I have before cited, at the first line of p. 126 of vol. xxvii. of the Phil. Mag. S. 3.

‡ Phil. Mag. S. 3. vol. xxvii. p. 292.

so that, on developing, we shall have on writing $\lambda'.\lambda''$ for $\Sigma(x^{\lambda'}).\Sigma(x^{\lambda''})$, &c.,

$$0=(t-2s)\times\{\lambda'.\lambda^{\text{iv}}.(\lambda''+\lambda''')+\lambda''.\lambda'''.(\lambda'+\lambda^{\text{iv}})$$
$$-\lambda'.\lambda'''.(\lambda''+\lambda^{\text{iv}})-\lambda''.\lambda^{\text{iv}}.(\lambda'+\lambda''')\}$$
$$+t\{(\lambda'+\lambda''').(\lambda''+\lambda^{\text{iv}})-(\lambda'+\lambda^{\text{iv}}).(\lambda''+\lambda''')\}. \quad . \quad (\text{l.})$$

Let $t=2n$, and $s=n-1$, then, if $n<3$, the last equation is identically true, but not in any other case. The method of the two first paragraphs, consequently, detects every case of failure; the last-mentioned instance of which is connected with the fact that, implicitly at least, every expression of the form (a.) contains in its right-hand side a term free from x which, with the above values of t and s, vanishes from ${}_2\text{Y}$. These values are those which occur in exterminating the 2nd, 3rd, and rth terms of an equation.

4. If, in the case of $n=2$, $t=4$, and $s=1$, we reject in (g.) the solution $l_1=0$, we are conducted to

$$(x_2^{\lambda'}-x_1^{\lambda'})^2(x_2^{\lambda''}-x_1^{\lambda''})=0, \quad . \quad . \quad . \quad . \quad (\text{m.})$$

having multiplied by the coefficient of Λ'^2 before commencing our operations. This agrees with what we have inferred from (l.).

5. It seems to follow from this, that biquadratics can be reduced to a binomial, and equations of the fifth degree to a trinomial form, by an expression for y consisting of four terms, determinable by one linear*, one quadratic, and one cubic equation.

6. At p. 384 of the 26th vol. of this work, I have only alluded to the equation (3.), which, for cubics, conducts to the reducing equation

$$\xi^2+\xi\Sigma\left(\frac{\phi'_1}{\phi_1}\right)+\frac{\phi'_1\phi'_2}{\phi_1\phi_2}=0; \quad . \quad . \quad . \quad . \quad (3.)'$$

and to a similar one for biquadratics; but if we discuss the equation $\quad \phi\{(\Lambda x^{\lambda}+\text{M}x^{\mu})^{-1}\}=0, \quad . \quad . \quad . \quad . \quad . \quad (3.)''$
it will be found that, though in appearance more complicated, it is in reality simpler than the former, inasmuch as the case of $\lambda=0$ is not excluded; and if $\lambda=0$ and $\mu=1$, we have the form actually taken by the reducing equation in my solution of a perfect cubic at p. 248 of vol. ii. of the Cambridge Mathematical Journal.

Devereux Court, Temple Bar, December 29, 1845. JAMES COCKLE, Jun.

* The 'base' equations are linear, as will be seen on referring to my definition at note † of p. 126 of this (27th) vol. If the roots of the trinomial equation of the fifth degree are given by the expression $b\psi(a)$, then ψ is contained in the solutions of the functional equation $\psi^2(a)-a=0$.