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XXVII. *On the Free Vibrations of Systems affected with Small Rotatory Terms.* By LORD RAYLEIGH, O.M., F.R.S.*

BY a suitable choice of coordinates the expressions for the kinetic and potential energies of the system may be reduced to the forms

$$T = \frac{1}{2} a_1 \dot{\phi}_1^2 + \frac{1}{2} a_2 \dot{\phi}_2^2 + \dots, \quad \dots \quad (1)$$

$$V = \frac{1}{2} c_1 \phi_1^2 + \frac{1}{2} c_2 \phi_2^2 + \dots \quad \dots \quad (2)$$

If there be no dissipative forces, the equations of free vibration are

$$\left. \begin{aligned} a_1 \ddot{\phi}_1 + c_1 \phi_1 + \beta_{12} \ddot{\phi}_2 + \beta_{13} \ddot{\phi}_3 + \dots &= 0, \\ a_2 \ddot{\phi}_2 + c_2 \phi_2 + \beta_{21} \ddot{\phi}_1 + \beta_{23} \ddot{\phi}_3 + \dots &= 0, \\ \dots &\dots \end{aligned} \right\} \quad \dots \quad (3)$$

where $\beta_{rs} = -\beta_{sr}$; and under the restriction contemplated all the quantities β are *small*.

If in equations (3) we suppose that the whole motion is proportional to $e^{i\sigma t}$,

$$\left. \begin{aligned} (c_1 - \sigma^2 a_1) \phi_1 + i\sigma \beta_{12} \phi_2 + i\sigma \beta_{13} \phi_3 + \dots &= 0 \\ (c_2 - \sigma^2 a_2) \phi_2 + i\sigma \beta_{21} \phi_1 + i\sigma \beta_{23} \phi_3 + \dots &= 0 \\ \dots &\dots \end{aligned} \right\}; \quad \dots \quad (4)$$

and it is known that whatever may be the magnitudes of the β 's, the values of the σ 's are real. The *frequencies* are equal to $\sigma/2\pi$.

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If there were no rotatory terms, the above system of equations would be satisfied by supposing one coordinate ϕ_r to vary suitably, while the remaining coordinates vanish. In the actual case there will be *in general* a corresponding solution in which the value of any other coordinate ϕ_s will be small relatively to ϕ_r .

Hence if we omit the terms of the *second* order in β , the *r*th equation becomes

$$(c_r - \sigma_r^2 a_r) \phi_r = 0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which we see that σ_r is approximately the same as if there were no rotatory terms.

From the *s*th equation we obtain

$$(c_s - \sigma_r^2 a_s) \phi_s + i \sigma_r \beta_{sr} \phi_r = 0,$$

terms of the second order being omitted; whence

$$\phi_s : \phi_r = - \frac{i \sigma_r \beta_{sr}}{c_s - \sigma_r^2 a_s} = \frac{i \sigma_r \beta_{sr}}{a_s (\sigma_r^2 - \sigma_s^2)}, \quad . \quad . \quad . \quad (6)$$

where on the right the values of σ_r , σ_s from the first approximation (5) may be used. This equation determines the altered type of vibration; and we see that the coordinates ϕ_s are in the same phase, but that this phase differs by a quarter period from the phase of ϕ_r .

We have seen that when the rotatory terms are small, the value of σ_r may be calculated approximately without allowance for the change of type; but by means of (6) we may obtain a still closer approximation, in which the squares of the β 's are retained. The *r*th equation (4) gives

$$a_r \sigma_r^2 = c_r + \sum \frac{\sigma_r^2 \beta_{rs}^2}{a_s (\sigma_r^2 - \sigma_s^2)}. \quad . \quad . \quad . \quad (7)$$

Since the squares of the σ 's are positive, as well as a_r , a_s , c_r , we recognize that the effect of β_{rs} is to increase σ_r^2 if σ_r^2 be already greater than σ_s^2 , and to diminish it if it be already the smaller. Under the influence of the β 's the σ 's may be considered to *repel* one another. If the smallest value of σ_r be finite, it will be lowered by the action of the rotatory terms*.

The vigour of the repulsion increases as the difference between σ_r and σ_s diminishes. If σ_r and σ_s are equal, the formulæ (6), (7) break down, unless indeed $\beta_{rs} = 0$. It is

* This conclusion was given in Phil. Mag. v. p. 138 (1903), but without some reservations presently to be discussed. Similar reservations are called for in 'Theory of Sound,' §§ 90, 102.

clear that the original assumption that ϕ_r is small relatively to ϕ_r fails in this case, and the reason is not far to seek. When two normal modes have exactly the same frequency, they may be combined in any proportions without alteration of frequency, and the combination is as much entitled to be considered normal as its constituents. But the smallest alteration in the system will in general render the normal modes determinate; and there is no reason why the modes thus determined should not differ finitely from those originally chosen.

A simple example is afforded by a circular membrane vibrating so that one diameter is nodal. When all is symmetrical, any diameter may be chosen to be nodal; but if a small excentric load be attached, the nodal diameter must either itself pass through the load or be perpendicular to the diameter that does so ('Theory of Sound,' § 208). Under the influence of the load the two originally coincident frequencies separate.

In considering the modifications required when equal frequencies occur, it may suffice to limit ourselves to the case where *two* normal modes only have originally the same frequency, and we will suppose that these are the first and second. Accordingly, the coincidence being supposed to be exact,

$$c_1/a_1 = c_2/a_2 = \sigma_0^2. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The relation between ϕ_1 and ϕ_2 and the altered frequencies are to be obtained from the first two equations of (3), in which the terms in ϕ_3 , ϕ_4 , &c. are at first neglected as being of the second order of small quantities. Thus

$$\left. \begin{aligned} (c_1 - \sigma^2 a_1) \phi_1 + i \sigma \beta_{12} \phi_2 &= 0 \\ (c_2 - \sigma^2 a_2) \phi_2 - i \sigma \beta_{12} \phi_1 &= 0 \end{aligned} \right\}, \quad . \quad . \quad . \quad . \quad (9)$$

in which the two admissible values of σ^2 are given by

$$(c_1 - a_1 \sigma^2)(c_2 - a_2 \sigma^2) - \sigma^2 \beta_{12}^2 = 0. \quad . \quad . \quad . \quad (10)$$

If one of the factors of the first term, *e. g.* the second, be finite, β_{12}^2 may be neglected and a value of σ^2 is found by equating the first factor to zero; but in the present case both factors are small together. On writing σ_0 for σ in the small term, (10) becomes

$$(\sigma^2 - \sigma_0^2)^2 = \sigma_0^2 \beta_{12}^2 / a_1 a_2, \quad . \quad . \quad . \quad . \quad (11)$$

so that

$$\sigma^2 - \sigma_0^2 = \pm \sigma_0 \beta_{12} / \sqrt{(a_1 a_2)}, \quad . \quad . \quad . \quad (12)$$

or

$$\sigma = \sigma_0 \pm \frac{1}{2} \beta_{12} / \sqrt{(a_1 a_2)}. \quad . \quad . \quad . \quad (13)$$

The disturbance of the frequency from its original value is now of the *first* order in β_{12} , and one frequency is raised and the other depressed by the same amount.

As regards the ratios in which ϕ_1, ϕ_2 enter into the new normal modes, we have from (9)

$$\frac{\phi_1}{\phi_2} = \frac{a_2(\sigma_0^2 - \sigma^2)}{i\sigma_0\beta_{12}} = \pm i\sqrt{(a_2/a_1)}. \quad . \quad . \quad . \quad (14)$$

From (14) we see that in the new normal vibrations the two original coordinates are combined so as to be in quadrature with one another, and in such proportion that the energies of the constituent motions are equal.

The value of any other coordinate ϕ_s accompanying ϕ_1 and ϕ_2 in vibration σ is obtained from the *s*th equation (4). Thus, squares of β 's being neglected,

$$(c_s - \sigma^2 a_s)\phi_s + i\sigma\beta_{s1}\phi_1 + i\sigma\beta_{s2}\phi_2 = 0, \quad . \quad . \quad . \quad (15)$$

in which, if we please, we may substitute for ϕ_2 in terms of ϕ_1 from (14).

For the second approximation to σ we get from (15) and the two first equations (4)

$$\left\{ c_1 - \sigma^2 a_1 - \sum \frac{\sigma^2 \beta_{1s}^2}{c_s - \sigma^2 a_s} \right\} \phi_1 + \left\{ i\sigma\beta_{12} + \sum \frac{\sigma^2 \beta_{1s}\beta_{s2}}{c_s - \sigma^2 a_s} \right\} \phi_2 = 0,$$

$$\left\{ c_2 - \sigma^2 a_2 - \sum \frac{\sigma^2 \beta_{2s}^2}{c_s - \sigma^2 a_s} \right\} \phi_2 + \left\{ i\sigma\beta_{21} + \sum \frac{\sigma^2 \beta_{2s}\beta_{s1}}{c_s - \sigma^2 a_s} \right\} \phi_1 = 0,$$

in which the summation extends to all values of *s* other than 1 and 2. In the coefficients of the second terms it is to be observed that $\beta_{12} = -\beta_{21}$, and that $\beta_{1s}\beta_{s2} = \beta_{2s}\beta_{s1}$; so that the determinant of the equations becomes

$$\left\{ c_1 - \sigma^2 a_1 - \sum \frac{\sigma^2 \beta_{1s}^2}{c_s - \sigma^2 a_s} \right\} \left\{ c_2 - \sigma^2 a_2 - \sum \frac{\sigma^2 \beta_{2s}^2}{c_s - \sigma^2 a_s} \right\} - \sigma^2 \beta_{12}^2 = 0, \quad . \quad . \quad . \quad (16)$$

terms of the fourth order in β being omitted. In (16) $c_1 - \sigma^2 a_1$, $c_2 - \sigma^2 a_2$ are each of the order β . Correct to the third order we obtain with the use of (12)

$$(\sigma^2 - \sigma_0^2)^2 - \sigma_0^2 \frac{\beta_{12}^2}{a_1 a_2} \mp \frac{\sigma_0 \beta_{12}^3}{(a_1 a_2)^{\frac{3}{2}}} \pm \frac{\sigma_0^3 \beta_{12}}{(a_1 a_2)^{\frac{3}{2}}} \sum \frac{a_1 \beta_{2s}^2 + a_2 \beta_{1s}^2}{c_s - \sigma_0^2 a_s} = 0. \quad (17)$$

whence

$$\sigma^2 - \sigma_0^2 = \pm \sigma_0 \frac{\beta_{12}}{\sqrt{(a_1 a_2)}} + \frac{\frac{1}{2} \beta_{12}^2}{a_1 a_2} - \frac{\frac{1}{2} \sigma_0^2}{a_1 a_2} \sum \frac{a_1 \beta_{2s}^2 + a_2 \beta_{1s}^2}{c_s - \sigma_0^2 a_s}. \quad . \quad (18)$$

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In (18) β_{12} is supposed to be of not higher order of small quantities than β_{1s}, β_{2s} . For example, we are not at liberty to put $\beta_{12}=0$.

In the above we have considered the modification introduced by the β 's into a vibration which when undisturbed is one of two with equal frequencies. If the type of vibration under consideration be one of those whose frequency is not repeated, the original formulæ (6), (7) undergo no essential modification.

In the following paper some of the principles of the present are applied to a hydrodynamical example.

XXVIII. *On the Vibrations of a Rectangular Sheet of Rotating Liquid.* By LORD RAYLEIGH, O.M., F.R.S.*

THE problem of the free vibrations of a rotating sheet of gravitating liquid of small uniform depth has been solved in the case where the boundary is circular†. When the boundary is rectangular, the difficulty of a complete solution is much greater; but I have thought that it would be of interest to obtain a partial solution, applicable when the angular velocity of rotation is *small*.

If ζ be the elevation, u, v the component velocities of the relative motion at any point, the equations of free vibration, when these quantities are proportional to $e^{i\sigma t}$, are

$$\left. \begin{aligned} i\sigma u - 2nv &= -g \, d\zeta/dx, \\ i\sigma v + 2nu &= -g \, d\zeta/dy, \end{aligned} \right\} \quad . \quad . \quad . \quad (1)$$

and

$$\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + \frac{\sigma^2 - 4n^2}{gh} \zeta = 0, \quad . \quad . \quad . \quad (2)$$

in which n denotes the angular velocity of rotation, h the depth of the water (as rotating), and g the acceleration of gravity. The boundary walls will be supposed to be situated at $x = \pm \frac{1}{2}\pi, y = \pm y_1$.

When n is evanescent, one of the principal vibrations is represented by

$$u = \cos x, \quad v = 0; \quad . \quad . \quad . \quad (3)$$

and ζ is proportional to $\sin x$, so that

$$\sigma^2 = gh. \quad . \quad . \quad . \quad (4)$$

This determines the frequency when $n=0$. And since by

* Communicated by the Author.

† Kelvin, Phil. Mag. Aug. 1880; Lamb, 'Hydrodynamics,' §§ 200, 202, 203.