



# XLVIII. On the problems of the calculus of variations

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XLVIII. *On the Problems of the Calculus of Variations.*  
*By HUGH KER CANKRIEN, Esq. M.A. Trin. Coll. Cambridge.\**

THERE are few students who do not find difficulty in the study of the calculus of variations. This does not arise so much from their want of familiarity with the notation made use of, since this is nearly the same as that of the differential calculus, but rather from their not having a distinct view presented to them at the outset, of the necessity for, and the object of, the several operations performed. It will hardly be denied that this is a fault of the method of Lagrange, when applied only to the solution of the more simple problems. His investigation contains the solution of the difficult as well as the easier problems, and consequently a great part of the process is unnecessary for the solution of the latter. For these reasons it is here attempted, with the assistance to be derived from M. Poisson's solution of the problem of the brachystochron, contained in the first volume of his treatise on Mechanics, to give first a solution of the easier problems, and then to show how far this solution is applicable to the more difficult, and in what way the solution of these may be completed.

The object proposed is to investigate the relation which the variables involved in a proposed function must have to one another, in order that a definite value of this given function shall be a maximum or minimum. The most common form in which this function is proposed, is the integral taken between limits of an expression containing the variables themselves and the differential coefficients of one of them considered as a function of the other. The limits are some-

\* Communicated by the Author.

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times

times invariable, and sometimes variable: the problem of the brachystochron is an instance of one or other of these two classes of problems, accordingly as we investigate the line of quickest descent from one given point to another given point, or from one given curve to another given curve.

If then  $V = f(x, y, p, q, r, \&c.)$  where  $p = \frac{dy}{dx}$ ;  $q = \frac{d^2y}{dx^2}$ ;  $r = \frac{d^3y}{dx^3}$ ; &c.;  $U = \alpha + \int dx V$ ;  $\alpha$  being an arbitrary constant; and  $U_I - U_{II}$  be the value of  $U$  between the limits  $x_I, y_I$ ;  $x_{II}, y_{II}$ : our object is to find the relation between  $x$  and  $y$ , which renders  $U_I - U_{II}$  a maximum or minimum; either when  $x_I, y_I, x_{II}, y_{II}$  are invariable, or when they are variable and connected by given equations  $y_I = \phi(x_I)$ ,  $y_{II} = \psi(x_{II})$ .

Now it is to be observed, that if we can determine the form of the function  $F(x)$  in the equation  $y = F(x)$ , which renders  $U_I - U_{II}$  a maximum or minimum, when the limits are invariable; we can also find it by the same process, when the limits are not assigned, but when equations only are given connecting them: for we may suppose that the symbols  $x_I, y_I, x_{II}, y_{II}$  which represented the limits which are supposed invariable in that process, now represent those values of the variables in the equations  $y_I = \phi(x_I)$ ,  $y_{II} = \psi(x_{II})$  which must be taken as the limits of the integral in order that  $U_I - U_{II}$  shall be a maximum or minimum. The only difference in the results we shall obtain in the two cases will be this: when the limits are assigned, we can substitute their given values for the symbols  $x_I, y_I, x_{II}, y_{II}$ , and thus find determined values for one or more of the constants in the equation  $y = F(x)$ ; whereas when we have not the values of the limits assigned, those constants which may be expressed in terms of these symbols will continue arbitrary, unless we have some method of determining the values of the limits. We will then for the present consider the limits of the integral not to change.

We supposed  $U = \alpha + \int dx V$  when the function of  $x$  represented by  $y$  is involved in  $V$ : if  $y = F(x)$  then  $U_I - U_{II}$  is the maximum or minimum value of  $\int dx V$  when taken between the limits  $x_I, y_I$  and  $x_{II}, y_{II}$ . Let  $u$  be any function of  $x$  which vanishes when  $x_I$  or  $x_{II}$  is substituted in it for  $x$ ; and let  $k$  be a very small constant quantity; also let  $W$  be the value of  $\alpha + \int dx V$  when  $y + ku$  is substituted in  $V$  for  $y$ . Then  $W_I - W_{II}$  is greater or less than  $U_I - U_{II}$  according as  $U_I - U_{II}$  is a minimum or a maximum, whatever be the function  $u$ . Now,

when  $y + ku$  is substituted for  $y$  in  $p$ , it becomes  $p + k \frac{du}{dx} = p + k u'$ ;  $q$  becomes  $q + k u''$ ;  $r$  becomes  $r + k u'''$ ; &c.; and therefore since  $W$  is the value of  $U$  when  $y + ku$  is substituted

tuted for  $y$  in  $V$ ;  $p + k u'$  for  $p$ ;  $q + k u''$  for  $q$ ;  $r + k u'''$  for  $r$ ; &c.; we have

$$\begin{aligned} W &= U + k \int dx \left\{ \frac{dV}{dy} u + \frac{dV}{dp} u' + \frac{dV}{dq} u'' + \frac{dV}{dr} u''' + \&c. \right\} \\ &+ \frac{k^2}{2} \int dx \left\{ \frac{d^2 V}{dy^2} u^2 + 2 \frac{d^2 V}{dy dp} u u' + \frac{d^2 V}{dp^2} u'^2 + 2 \frac{d^2 V}{dy dq} u u'' \right. \\ &+ \left. 2 \frac{d^2 V}{dp dq} u' u'' + \frac{d^2 V}{dq^2} u''^2 + \&c. \right\} + \&c. \\ &= U + k \int dx \left\{ Nu + Pu' + Qu'' + Ru''' + \&c. \right\} + \\ &\frac{k^2}{2} \int dx \left\{ Au^2 + 2Bu u' + Cu'^2 + 2Du u'' + 2Eu' u'' + \right. \\ &\left. Fu''^2 + \&c. \right\} + \&c. \text{ for the sake of conciseness.} \end{aligned}$$

Now  $\int dx Pu' = Pu - \int dx u P'$ : and therefore, since  $u$  vanishes at the limits,  $\int dx Pu'$  is the same as  $-\int dx u P'$ , when both are taken between the limits. Hence if we denote

the  $\int dx Pu'$  when taken between the limits by  $\int_{x''}^{x'} dx Pu'$ ,

we have  $\int_{x''}^{x'} dx Pu' = - \int_{x''}^{x'} dx u P'$ . Similarly we shall find

$\int_{x''}^{x'} dx u'' Q = \int_{x''}^{x'} dx x u Q''$ : and so on\*. Hence if we substitute the preceding results, we find

$$\begin{aligned} W_I - W_{II} &= U_I - U_{II} + k \int_{x''}^{x'} dx \{ N - P' + Q'' - R''' + \&c. \} \\ &+ \&c. \text{ the terms following those which are written down} \\ &\text{being multiplied by } k^2, k^3, \&c. \text{ Now since the term multi-} \\ &\text{plied by } k \text{ may be altered from positive to negative by only} \end{aligned}$$

\* It may be necessary to explain the notation made use of in the following investigations. The expressions  $\int dx V$  and  $\int dx \{ N - P' + Q'' - \&c. \}$  denote the functions whose differential coefficients taken with respect to  $x$  are  $V$  and  $N - P' + Q'' - \&c.$  respectively:  $\int dx$  is considered in these expressions as a mere symbol. For the sake of conciseness it is desirable to have an expression for such a function as  $\int dx V$  when a certain value, as  $x_i$  or  $x_{ii}$ , is assigned to the variable  $x$  contained in it: the expressions

$\int_{x''}^{x_i} dx V$  and  $\int_{x''}^{x_{ii}} dx V$  are used for this purpose. An expression equivalent to  $\int_{x_i}^{x_{ii}} dx V - \int_{x''}^{x_{ii}} dx V$  very often occurs, and it is therefore convenient to express it by one term, as  $\int_{x''}^{x_i} dx V$ . In the use made of these expressions

it will be seen that  $\int_{x''}^{x_i} dx$  and  $\int_{x_{ii}}^{x_i} dx$  are considered as mere symbols.

changing the sign of  $k$ , and, by properly assuming  $k$ , may be made greater than the sum of all the terms which follow it, if  $U_I - U_{II}$  be either a maximum or a minimum, it is necessary that  $N - P' + Q'' - R''' + \&c. = 0$ . This equation when integrated will give us the relation between  $x$  and  $y$ , or the equation  $y = F(x)$  which renders  $\int_{x''}^x dx V$  a maximum or a minimum.

The order of the differential equation we have just found will in general be  $2n$ , if the order of the highest differential coefficient in  $V$  be  $n$ . We may find an equation one order lower in this way :

$$\frac{d(V)}{dx} = V' = \frac{dV}{dx} + \frac{dV}{dy} \frac{dy}{dx} + \frac{dV}{dp} \frac{dp}{dx} + \frac{dV}{dq} \frac{dq}{dx} + \frac{dV}{dr} \frac{dr}{dx} + \&c.;$$

$$\text{or, } V' = M + Np + Pq + Qr + Rs + \&c.$$

but  $0 = -p \{N - P' + Q'' - R''' + \&c.\}$  by the equation last found.

$$\therefore V' = M + Pq + P'p + Qr - pQ'' + Rs + R'''p + \&c.$$

$$\text{Now, } Pq + pP' = (Pp)'; \quad Qr - pQ'' = Qr + Q'q - Q'q - Q''p = (Qq - pQ')'; \quad Rs + R'''p = Rs + rR' - rR' - qR'' + qR'' + pR''' = (Rr - qR' + pR'')';$$

$\&c. = \&c.$ : substituting these results we find

$$V' = M + (Pp)' + (Qq - pQ')' + (Rr - qR' + pR'')' + \&c., \text{ and}$$

$$\therefore V = \beta + \int dx M + Pp + Qq - pQ' + Rr - qR' + pR'' + \&c. \quad (1).$$

This equation contains one arbitrary constant  $\beta$ : and if the order of the highest differential coefficient in  $V$  be  $n$ , there will in general be  $2n$  arbitrary constants in the primitive equation between  $x$  and  $y$ . These are to be determined by means of the limits and other data which, according to the case, must be granted for that purpose. But if the limits are not given, the constants which are expressed in terms of the symbols representing them, must remain arbitrary, as was observed above. Our object then, in order to complete the solution in this case, must be to determine the actual values of the limits which these symbols represent.

Since  $y_I = \phi(x_I)$  and  $y_{II} = \psi(x_{II})$  we may consider  $U_I - U_{II}$  as a function of  $x_I$  and  $x_{II}$ : now if we substitute  $x_I + \delta x_I$  for  $x_I$  and  $x_{II} + \delta x_{II}$  for  $x_{II}$  in  $U_I - U_{II}$ , the result, which we will denote by  $\omega_I - \omega_{II}$ , is  $U_I - U_{II} + \mu \delta x_I + \pi \delta x_{II} + \&c.$ ,  $\mu$  and  $\pi$  representing the coefficients of  $\delta x_I$  and  $\delta x_{II}$  in the expanded expression, and since  $U_I - U_{II}$  is always greater or always less

less than  $\omega_i - \omega_{ii}$  we must have  $\mu \delta x_i + \pi \delta x_{ii} = 0$  by the common theory.

This equation  $\mu \delta x_i + \pi \delta x_{ii} = 0$  resolves itself into two others, since  $\delta x_i$  and  $\delta x_{ii}$  are independent of one another, in which  $x_i$  and  $x_{ii}$  are the only unknown quantities: they therefore may be determined by means of them.

Now  $\omega_i - \omega_{ii}$  being the value of  $U_i - U_{ii}$  when  $x_i + \delta x_i$  is substituted for  $x_i$  and  $x_{ii} + \delta x_{ii}$  for  $x_{ii}$  in it; and since  $x_i$  and  $x_{ii}$  enter into  $U_i$  and  $U_{ii}$  respectively, partly in consequence of their being involved in  $U$ , and partly in consequence of the change of  $x$  into  $x_i$  and  $x_{ii}$  respectively wherever  $x$  occurs in  $U$ ; it is evident that  $\omega_i$  may be obtained by substituting  $x_i + \delta x_i$  for  $x_i$  and  $x_{ii} + \delta x_{ii}$  for  $x_{ii}$  wherever they occur in  $U$ , and then substituting  $x + \delta x$  for  $x$ , and afterwards changing  $x$  and  $\delta x$  into  $x_i$  and  $\delta x_i$  in the result. In like manner we may find  $\omega_{ii}$  or the value of  $U_{ii}$  when  $x_i + \delta x_i$  and  $x_{ii} + \delta x_{ii}$  are substituted for  $x_i$  and  $x_{ii}$  in it, by first substituting  $x_i + \delta x_i$  and  $x_{ii} + \delta x_{ii}$  for  $x_i$  and  $x_{ii}$  wherever they occur in  $U$ , and then substituting  $x + \delta x$  for  $x$ , and afterwards changing  $x$  and  $\delta x$  into  $x_{ii}$  and  $\delta x_{ii}$ . All these substitutions are to be made not only where  $x_i$  and  $x_{ii}$  occur alone, but likewise where they occur involved in the functions  $y_i$  and  $y_{ii}$ . In like manner since the operation of substituting  $x + \delta x$  for  $x$  in  $U$  is equivalent to the substitution of  $x_i + \delta x_i$  for  $x_i$ , when  $\omega_i$  is obtained; and equivalent to the substitution of  $x_{ii} + \delta x_{ii}$  for  $x_{ii}$ , when  $\omega_{ii}$  is sought; it is evident that in one case  $y$  must be considered as the same function of  $x$  that  $y_i$  is of  $x_i$ ; and in the other  $y$  must be considered the same function of  $x$  that  $y_{ii}$  is of  $x_{ii}$ .

Since we only want those terms which are multiplied by the first powers of  $\delta x_i$  and  $\delta x_{ii}$ , we will carry the operations in what follows only to that extent. We will also use  $\delta y_i$  to denote the term multiplied by  $\delta x_i$  in the new value of  $y_i$  obtained by substituting  $x_i + \delta x_i$  for  $x_i$  in it. In like manner  $\delta y_{ii}$  denotes the term multiplied by  $\delta x_{ii}$  in the new value of  $y_{ii}$  when  $x_{ii} + \delta x_{ii}$  is substituted for  $x_{ii}$  in  $y_{ii}$ ; and  $\delta y$  denotes the corresponding term when  $x + \delta x$  is substituted, for  $x$  in  $y$  considered as a function of  $x$ :  $\delta p$  the term in  $p$ : and so on.

Now, if  $U$  contain  $x_i, y_i, x_{ii}, y_{ii}$ , and we substitute  $x_i + \delta x_i$  for  $x_i$  and  $x_{ii} + \delta x_{ii}$  for  $x_{ii}$ , the new value of  $U$  thus obtained is

$$U + \frac{dU}{dx_i} \delta x_i + \frac{dU}{dy_i} \delta y_i + \frac{dU}{dx_{ii}} \delta x_{ii} + \frac{dU}{dy_{ii}} \delta y_{ii} + \&c.$$

but  $U = \alpha + \int dx V$ , and  $\alpha$  may be considered either as a function of  $x_i, y_i$  or of  $x_{ii}, y_{ii}$ : we will suppose  $\alpha$  to be a function

function of  $x, y, \rho$ , and  $V$  to contain  $x, y, x_{II}, y_{II}$ , and our expression becomes

$$U + \frac{d\alpha}{dx} \delta x_I + \frac{d\alpha}{dy} \delta y_I + \delta x_I \int dx \frac{dV}{dx} + \delta y_I \int dx \frac{dV}{dy} + \delta x_{II} \int dx \frac{dV}{dx_{II}} + \delta y_{II} \int dx \frac{dV}{dy_{II}} + \&c. \quad (2)$$

Again, in order to obtain  $\omega$ , we must, as was remarked, substitute  $x + \delta x$  for  $x$  in this expression (2), and then change  $x$  and  $\delta x$  into  $x_I$  and  $\delta x_I$ : as we are only in want of the terms which are multiplied by the simple power of  $\delta x$  and  $\delta x_{II}$ , it will be sufficient to make this substitution in  $U$ . Now we found by (1) above

$$V = \beta + \int dx M + Pp + Qq - pQ' + \&c.$$

$$\begin{aligned} \text{and, } \therefore U &= \alpha + \int dx V = \alpha + \int dx \{ \beta + \int dx M \} + \\ &\quad \int dx p (P - Q') + \int dx q Q + \&c. \\ &= \alpha + \int dx \{ \beta + \int dx M \} + y (P - Q') - \\ &\quad \int dx y (P - Q')' + pQ - \int dx p Q' + \&c. \end{aligned}$$

Let  $\delta U$  denote that part of the new value of  $U$ , when  $x + \delta x$  is substituted for  $x$  in this expression, which is multiplied by the simple power of  $\delta x$ : then

$$\begin{aligned} \delta U &= \delta x \{ \beta + \int dx M \} + \delta y (P - Q') + y (P - Q')' \delta x - \\ &\quad \delta x \cdot y (P - Q')' + \delta p \cdot Q + p Q' \cdot \delta x - \delta x \cdot p Q' + \&c. \\ &= \delta x \{ \beta + \int dx M \} + \delta y (P - Q') + \delta p \cdot Q + \&c. \end{aligned}$$

$$\text{but, } V \delta x = \delta x \{ \beta + \int dx M \} + p \delta x (P - Q') + q \delta x \cdot Q + \&c. \text{ by (1)}$$

$$\therefore \delta U = V \delta x + (\delta y - p \delta x) (P - Q' + \&c.) + (\delta p - q \delta x) (Q - \&c.) + \&c.$$

If then we substitute in (2) and then change  $\delta x$  and  $\delta y$  into  $\delta x_I$  and  $\delta y_I$ , we find for the value of  $\omega$ ,

$$\begin{aligned} &\frac{d\alpha}{dx_I} \delta x_I + \frac{d\alpha}{dy_I} \delta y_I + \delta x_I \int dx \frac{dV}{dx} + \delta y_I \int dx \frac{dV}{dy} + \\ &\delta x_{II} \int dx \frac{dV}{dx_{II}} + \delta y_{II} \int dx \frac{dV}{dy_{II}} + \&c. + U_I + V_I \delta x_I + \\ &(\delta y_I - p_I \delta x_I) \{ P_I - Q_I' + \&c. \} + (\delta p_I - q_I \delta x_I) \{ Q_I - \&c. \} + \&c. \end{aligned}$$

In this expression  $\int dx \frac{dV}{dx_{II}}$  denotes the value of  $\int dx \frac{dV}{dx_{II}}$  when  $x_I$  is substituted for  $x$  in it. In like manner we shall find

$$\omega_{II} =$$

$\omega_{,,} = \frac{d\alpha}{dx_i} \delta x_i + \frac{d\alpha}{dy_i} \delta y_i + \delta x_i \int_{x_{,,}}^{x'} \frac{dV}{dx_i} + \delta y_i \int_{y_{,,}}^{y'} \frac{dV}{dy_i} +$   
 $\delta x_{,,} \int_{x_{,,}}^{x'} \frac{dV}{dx_{,,}} + \delta y_{,,} \int_{y_{,,}}^{y'} \frac{dV}{dy_{,,}} + \&c. + U_{,,} + V_{,,} \delta x_{,,} +$   
 $(\delta y_{,,} - p_{,,} \delta x_{,,}) \{P_{,,} - Q'_{,,} + \&c.\} + \{\delta p_{,,} - q_{,,} \delta x_{,,}\} \{Q_{,,} - \&c.\} + \&c.$   
 and therefore subtracting this result from the former, we find

$$\omega - \omega_{,,} = \delta x_i \int_{x_{,,}}^{x'} \frac{dV}{dx_i} + \delta y_i \int_{y_{,,}}^{y'} \frac{dV}{dy_i} + \delta x_{,,} \int_{x_{,,}}^{x'} \frac{dV}{dx_{,,}} + \delta y_{,,} \int_{y_{,,}}^{y'} \frac{dV}{dy_{,,}} + \&c. + V \delta x_i - V_{,,} \delta x_{,,} + (\delta y_i - p_i \delta x_i) (P_i - Q'_i + \&c.) - (\delta y_{,,} - p_{,,} \delta x_{,,}) (P_{,,} - Q'_{,,} + \&c.) + U_i - U_{,,} + (\delta p_i - q_i \delta x_i) (Q_i - \&c.) - (\delta p_{,,} - q_{,,} \delta x_{,,}) (Q_{,,} - \&c.) + \&c.$$

since  $\int_{x_{,,}}^{x'} \frac{dV}{dx_i} - \int_{x_{,,}}^{x'} \frac{dV}{dx_{,,}} = \int_{x_{,,}}^{x'} \frac{dV}{dx_{,,}}; \&c. = \&c.$

It appears, then, that the equation which we represented by  $\mu \delta x_i + \pi \delta x_{,,} = 0$  is

$$\delta x_i \int_{x_{,,}}^{x'} \frac{dV}{dx_i} + \delta y_i \int_{y_{,,}}^{y'} \frac{dV}{dy_i} + \delta x_{,,} \int_{x_{,,}}^{x'} \frac{dV}{dx_{,,}} + \delta y_{,,} \int_{y_{,,}}^{y'} \frac{dV}{dy_{,,}} + \&c. + V \delta x_i - V_{,,} \delta x_{,,} + (\delta y_i - p_i \delta x_i) (P_i - Q'_i + \&c.) - (\delta y_{,,} - p_{,,} \delta x_{,,}) (P_{,,} - Q'_{,,} + \&c.) + (\delta p_i - q_i \delta x_i) (Q_i - \&c.) - (\delta p_{,,} - q_{,,} \delta x_{,,}) (Q_{,,} - \&c.) + \&c. = 0 \quad (3)$$

If  $V$  contains none of the quantities  $x_i, y_i, x_{,,}, y_{,,}$ , then since  $\frac{dV}{dx_i} = 0, \frac{dV}{dy_i} = 0, \frac{dV}{dx_{,,}} = 0$  and  $\frac{dV}{dy_{,,}} = 0$ , the equation for determining the limits is

$$V_i \delta x_i - V_{,,} \delta x_{,,} + (\delta y_i - p_i \delta x_i) (P_i - Q'_i + \&c.) - (\delta y_{,,} - p_{,,} \delta x_{,,}) (P_{,,} - Q'_{,,} + \&c.) + (\delta p_i - q_i \delta x_i) (Q_i - \&c.) - (\delta p_{,,} - q_{,,} \delta x_{,,}) (Q_{,,} - \&c.) + \&c. = 0 \quad (4)$$

By means of this equation, then, we can determine the limits when they are not involved in  $V$ ; but when they are involved in  $V$ , we must use equation (3); and therefore in either case we can determine the arbitrary constants as was required.

We will now resume the consideration of the value of  $W_i - W_{,,}$  which in consequence of the preceding results is reduced to

$$U_i - U_{,,} + \frac{k^3}{2} \int_{x_{,,}}^{x'} \{A u^2 + 2 B u u' + C u'^2 + 2 D u u' + 2 E u' u'' + F u'^2 + \&c.\} + \&c.$$

Now,



Now,  $\int dx \, 2 B u u' = \int dx \, B \frac{d u^2}{dx} = B u^2 - \int dx \, u^2 B$ , and therefore  $\int_{x_{ii}}^{x_i} dx \, 2 B u u' = - \int_{x_{ii}}^{x_i} dx \, u^2 B'$ .

Similarly  $\int_{x_{ii}}^{x_i} dx \, 2 E u' u'' = - \int_{x_{ii}}^{x_i} dx \, E u'^2$ .

Again,  $\int dx \, 2 D u u'' = D \frac{d u^2}{dx} - \int dx \, 2 D u'^2 - D' u^2 + \int dx \, u^2 D''$ , and therefore  $\int_{x_{ii}}^{x_i} dx \, 2 D u u'' = \int_{x_{ii}}^{x_i} dx \, u^2 D'' - \int_{x_{ii}}^{x_i} dx \, 2 D u'^2$ ; and so on. Hence

$$W_i - W_{ii} = U_i - U_{ii} + \frac{k^2}{2} \int_{x_{ii}}^{x_i} dx \, \{ (A - B' + D') u^2 + (C - 2 D - E) u'^2 + F u''^2 + \&c. \} + \&c.$$

Now, in order that  $U_i - U_{ii}$  may be a maximum or a minimum, the term multiplied by  $k^2$  in the above expression must be either positive or negative whatever be the function  $u$ ; and  $U_i - U_{ii}$  will be a maximum or a minimum according as this term is negative or positive. It is not easy to find generally the relation of the coefficients of  $u^2$ ,  $u'^2$ ,  $u''^2$ , &c. to one another by which these conditions are fulfilled. In order to determine it in particular cases we may remark, that if the values of  $\Phi'(x)$  be found corresponding to the series of values  $x_{ii}$ ,  $x_i + h$ ,  $x_{ii} + 2h$ , &c.  $x_{ii} + \overline{n-1} \cdot h$ ;  $h$  being  $= \frac{x_i - x_{ii}}{n}$  and  $n$  a large number; then  $\Phi(x_i) - \Phi(x_{ii})$  is positive or negative, according as the sum of the positive values of  $\Phi'(x)$  found in the manner just mentioned, is greater or less than the sum of the negative values. If all these values of  $\Phi'(x)$  are positive, or all negative, then  $\Phi(x_i) - \Phi(x_{ii})$  is in the one case positive, and in the other negative, as is evident. If then we deduce the values of  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ , &c. from  $V$ , and the expression  $(A - B' + D') u^2 + (C - 2 D - E) u'^2 + F u''^2 + \&c.$ , which we will call  $\Phi'(x)$ , be reduced as much as possible by means of the equation  $N - P' + Q'' - R''' + \&c. = 0$ , we shall in many cases see whether it be possible so to assume the function  $u$  that  $\frac{k^2}{2} \int_{x_{ii}}^{x_i} dx \, \Phi'(x)$  shall be either positive or negative: if this be possible, then the equation  $N - P' + Q'' - R''' + \&c. = 0$  does not make  $\int_{x_{ii}}^{x_i} dx \, V$  either a maximum or a minimum:

num: but if it be not possible, it does. Thus if  $V =$

$$\frac{\sqrt{1+p^2}}{\sqrt{2g} \sqrt{y-y_0}}; \text{ we find } \sqrt{y-y_0} \cdot \sqrt{1+p^2} = c, \text{ and by}$$

means of this equation we can show that  $\Phi'(x)$  is always positive whatever be the function  $u$ . Thus we can in this case show analytically that we have obtained that relation between

$x$  and  $y$  which renders  $\int_{x_0}^x dx V$  a minimum; and in a simi-

lar manner we can solve several other problems. The only case in which it is obvious that the general formula for  $\Phi'(x)$  cannot change its sign, is when the coefficients of  $u^2$   $u'^2$   $u''^2$  &c., are all of them either positive or negative for all values of  $x$  between the limits of the integral.

Lincoln's Inn, July 10th, 1830.

XLIX. *Remarks on a Passage in Dr. Thomson's "Outline of the Sciences of Heat and Electricity," London, 1830. By the Rev. B. POWELL, M.A. F.R.S., Savilian Professor of Geometry, Oxford.\**

I N looking into the volume lately published by Dr. Thomson on Heat &c., my attention was immediately drawn to the chapter on radiant heat, as being a subject on which I have been particularly engaged. And I cannot but feel indebted to the distinguished author for the notice he has been pleased to take of my researches on this subject: though at the same time I trust he will allow me to make a few remarks on the mode in which the mention of them is introduced. The passage referred to is as follows:

"The conclusions from the observations of De la Roche have been called in question by Mr. Powell. He admits that when a hot body becomes luminous it gives out *heat* capable of passing directly through transparent screens. But this new heat acts more on a smooth black surface than on an absorptive white one. From this he concludes, that it is different from common radiant heat. We have no evidence that it is the same as light. It is great from red-hot metals, though the light be feeble. It exists in the solar rays, and is what produces the photometrical effect in Leslie's Photometer. But this ingenious explanation of Mr. Powell has, I think, been obviated by a very happy and instructive experiment of Mr. Ritchie," &c. &c. p. 156. Then follows a description of Mr. Ritchie's experiments in detail.

Now to a reader not previously acquainted with the sub-

\* Communicated by the Author.