SURFACES GENERATED BY THE MOTION OF AN INVARIABLE CURVE WHOSE POINTS DESCRIBE STRAIGHT LINES.

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1. A surface generated by the motion of a straight line which always meets a fixed axis and is perpendicular to it is called a right conoid. DARBOUX [1]) has shown that a family of equal circular cylinders with the axis of a conoid for an element meet the conoid in congruent curves and that these curves move into one another as the cylinders roll inside their enveloping cylinder in such a may that each point of the curve describes a generator of the conoid. This paper considers the problem of finding all ruled surfaces generated by the motion of an invariable curve whose points describe the rulings in the motion. It is shown that the right conoids and cylinders are the only surfaces possessing this property. Evidently this is a particular solution of the general problem of finding the surfaces generated by the motion of two invariable curves.

2. *General equations of a surface.* — In a recent paper [2]) we have shown that if p, q, r, t, ρ and τ are functions satisfying the conditions

$$(1)\qquad \left\{\begin{aligned} \frac{\partial q}{\partial u}-\frac{\partial p}{\partial v}-\frac{p r}{\rho}&=0,\\ \frac{\partial A_2}{\partial u}-\frac{\partial}{\partial v}\left(\frac{p}{\rho}\right)+\frac{p}{\tau}A_3&=0,\\ \frac{\partial L_3}{\partial u}+\frac{\partial}{\partial v}\left(\frac{p}{\tau}\right)+\frac{p}{\rho}A_3&=0,\end{aligned}\right.$$

[1]) G. DARBOUX, *Leçons sur la théorie générale des surfaces*, 1ère Partie, 2ème édition (Paris, Gauthier-Villars, 1914), p. 96. The proof is as follows: Let C be one of the cylinders and S its section with the one conoid. As C rolls on its envelope each point P on C describes the right line through P to the axis of the conoid. Hence all the points of S move into points of the conoid and thus in each position of S it is the section by the rolling cylinder.

[2]) L. PF. EISENHART, *One-Parameter Families of Curves* [American Journal of Mathematics, Vol. XXXVII (1915), pp. 179-191].

where

$$(2)\qquad \begin{cases} A_2 = \dfrac{1}{p}\dfrac{\partial r}{\partial u} + \dfrac{q}{\rho} + \dfrac{t}{\tau}, \\ A_3 = \dfrac{1}{p}\dfrac{\partial t}{\partial u} - \dfrac{r}{\tau}, \\ L_3 = \dfrac{\rho}{p}\dfrac{\partial A_3}{\partial u} - \dfrac{\rho}{\tau} A_2, \end{cases}$$

then the system of equations

$$(3)\qquad \begin{cases} \dfrac{\partial \alpha}{\partial u} = \dfrac{p}{\rho} l, & \dfrac{\partial \alpha}{\partial v} = A_2 l + A_3 \lambda, \\ \dfrac{\partial l}{\partial u} = -\dfrac{p}{\rho}\alpha - \dfrac{p}{\tau}\lambda, & \dfrac{\partial l}{\partial v} = -A_2 \alpha + L_3 \lambda, \\ \dfrac{\partial \lambda}{\partial u} = \dfrac{p}{\tau} l, & \dfrac{\partial \lambda}{\partial v} = -A_3 \alpha - L_3 \lambda, \end{cases}$$

is completely integrable, and the equations

$$(4)\qquad \frac{\partial x}{\partial u} = p\alpha, \qquad \frac{\partial x}{\partial v} = q\alpha + rl + t\lambda,$$

are consistent and determine a function x by quadratures.

Equations (3) admit also two sets of solutions β, m, μ; and γ, n, ν such that the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$$

is orthogonal and positive. If y and z denote the corresponding functions given by equations analogous to (4) with α, l, λ replaced by the above functions respectively, the locus of the point $P(x, y, z)$ is a surface for which α, β, γ; l, m, n; λ, μ, ν are, respectively, the direction-cosines of the tangent, principal normal and binormal at P of the curve $v =$ const. through P, and ρ and τ are the radii of first and second curvature of this curve at P.

Conversely it can be shown that the equations of any surface can be given this form when the curves $v =$ const. are neither straight lines nor minimal curves.

3. *Curves* $v =$ const. *congruent.* — When the equations of a surface are given the form of the preceding section, the curves $u =$ const. may be chosen to satisfy a suitable condition. In the case now under discussion we take the congruent curves for the curves $v =$ const. and for the curves $u =$ const. the straight lines described by the points of these curves in the motion which brings them into coincidence. Hence we may take for the parameter u the arc of a curve $v =$ const. measured from one of the generators of the surface. Accordingly the functions ρ and τ involve u alone, so that we put

$$(5)\qquad p = 1, \qquad \frac{1}{\rho} = U_1, \qquad \frac{1}{\tau} = U_2.$$

Under these conditions equations (1), (2) and (3) reduce to

$$(6)\qquad \begin{cases} \dfrac{\partial q}{\partial u} - r U_1 = 0, \\ \dfrac{\partial A_2}{\partial u} + A_3 U_2 = 0, \\ \dfrac{\partial L_3}{\partial u} + A_3 U_1 = 0, \end{cases}$$

where

$$(7)\qquad \begin{cases} A_2 = \dfrac{\partial r}{\partial u} + q U_1 + t U_2, \\ A_3 = \dfrac{\partial t}{\partial u} - r U_2, \\ L_3 = \dfrac{1}{U_1}\dfrac{\partial A_3}{\partial u} - A_2 \dfrac{U_2}{U_1}, \end{cases}$$

and

$$(8)\qquad \begin{cases} \dfrac{\partial \alpha}{\partial u} = l U_1, & \dfrac{\partial \alpha}{\partial v} = A_2 l + A_3 \lambda, \\ \dfrac{\partial l}{\partial u} = -(\alpha U_1 + \lambda U_2), & \dfrac{\partial l}{\partial v} = -A_2 \alpha + L_3 \lambda, \\ \dfrac{\partial \lambda}{\partial u} = l U_2, & \dfrac{\partial \lambda}{\partial v} = -A_3 \alpha - L_3 l. \end{cases}$$

If we put

$$(9)\qquad Q = \frac{q}{\sqrt{q^2 + r^2 + t^2}}, \qquad R = \frac{r}{\sqrt{q^2 + r^2 + t^2}}, \qquad T = \frac{t}{\sqrt{q^2 + r^2 + t^2}},$$

it follows from (4) that the direction-cosines α_1, β_1, γ_1 of the tangent to the curves $u =$ const. are of the form

$$(10)\qquad \alpha_1 = Q\alpha + Rl + T\lambda.$$

Since by hypothesis the curves $u =$ const. are straight lines, the functions α_1, β_1, γ_1 are independent of v. Equating to zero the derivatives of these functions with respect to v, the resulting equations are reducible by means of (8) to

$$(11)\qquad \begin{cases} \dfrac{\partial Q}{\partial v} - (RA_2 + TA_3) = 0, \\ \dfrac{\partial R}{\partial v} + QA_2 - TL_3 = 0, \\ \dfrac{\partial T}{\partial v} + QA_3 + RL_3 = 0. \end{cases}$$

In consequence of (6) and (7) the first of these equations maybe given the form

$$(12)\qquad \frac{\partial Q}{\partial v} = \frac{\partial}{\partial u}\sqrt{q^2 + r^2 + t^2}.$$

Making use of this result and equations (6) and (7), we find

$$(13)\qquad \begin{cases} \dfrac{\partial Q}{\partial u} = R U_1 - Q P(R A_2 + T A_3), \\ \dfrac{\partial R}{\partial u} = A_2 P - Q U_1 - T U_2 - R P(R A_2 + T A_3), \\ \dfrac{\partial T}{\partial u} = A_3 P + R U_2 - T P(R A_2 + T A_3), \end{cases}$$

where

$$(14)\qquad P = 1/\sqrt{q^2 + r^2 + t^2}.$$

The conditions of integrability of (11) and (13) are reducible to

$$(15)\qquad \begin{cases} Q R L + Q T M + (R^2 + T^2)(A_2^2 + A_3^2) = 0, \\ (Q^2 + T^2) L - R T M + R Q(A_2^2 + A_3^2) = 0, \\ R T L - (Q^2 + R^2) M - Q T(A_2^2 + A_3^2) = 0, \end{cases}$$

where we have put

$$(16)\qquad \begin{cases} L = A_2 \dfrac{\partial \log P}{\partial v} + \dfrac{\partial A_2}{\partial v} - L_3 A_3, \\ M = A_3 \dfrac{\partial \log P}{\partial v} + \dfrac{\partial A_3}{\partial v} + L_3 A_2. \end{cases}$$

4. *Special cases.* — If $Q = 0$, it follows from (6) that $R = 0$, and consequently that $\alpha_1 = \lambda$. Hence from (8) and (7) we have

$$A_3 = L_3 = A_2 = U_2 = 0.$$

Consequently λ, μ, ν are constant, and the surface is a cylinder, the generating curve being a right section.

We consider now all cases where $A_2 = 0$. From (6) it follows that either $A_3 = 0$ or $U_2 = 0$. If $A_3 = 0$, it follows from (7) that $L_3 = 0$. Hence α, β, γ; l, m, n; λ, μ, ν are functions of u alone. Therefore the curves $v =$ cost. are parallel and the surface is a cylinder.

Conversely, if the surface is generated by the translation of a curve, it is necessary and sufficient, as seen from

$$(17)\qquad \frac{\partial \alpha_1}{\partial u} = - P(R A_2 + T A_3)\alpha_1 + P(l A_2 + \lambda A_3),$$

that

$$Q(R A_2 + T A_3) = 0, \quad A_2 - R(R A_2 + T A_3) = 0, \quad A_3 - T(R A_2 + T A_3) = 0,$$

which necessitates $A_2 = A_3 = 0$.

When $A_2 = U_2 = 0$, neither T nor R can be zero. For in the first case the curves $v =$ const. would be plane asymptotic curves which is impossible; and in the second case we should have from (7) that $U_1 = 0$, that is the curves $v =$ const. would be straight lines. However, it is well known that as a generator of a quadric describes the quadric its points do not describe the generators of the other family. Hence since $A_3 \neq 0$ by hypothesis, equations (15) may be replaced by

$$QL_3 = RA_3, \qquad \frac{\partial}{\partial v}\log PA_3 = -\frac{TA_3}{Q} = -\frac{TL_3}{R}.$$

From (7) and (12) we have

$$\frac{\partial}{\partial u}\log PA_3 = U_1\frac{L_3}{A_3} - PTA_3.$$

The condition of integrability of these equations is reducible to

$$U_1\frac{\partial}{\partial v}\left(\frac{L_3}{A_3}\right) + 2\frac{P}{Q}A_3^2 = 0.$$

But by (11)

$$\frac{\partial}{\partial v}\left(\frac{L_3}{A_3}\right) = \frac{\partial}{\partial v}\left(\frac{R}{Q}\right) = 0.$$

Hence when $A_2 = 0$, so also is $A_3 = 0$.

When $A_3 = 0$, $A_2 \neq 0$, if $T = 0$, we have from (11) $RL_3 = 0$. When $T = R = 0$, the curves $v =$ const. are straight lines, which we have seen is not a solution of our problem. When $T = L_3 = 0$, we have from (7) that $U_2 = 0$, which is impossible, as the curves $v =$ const. cannot be plane asymptotic lines. Since then $T \neq 0$, and $Q \neq 0$ by hypotesis, equations (15) are equivalent to

$$QL_3 + TA_2 = 0, \qquad \frac{\partial}{\partial v}\log PA_2 + \frac{RA_2}{Q} = 0.$$

Also we have from (6)

$$\frac{\partial}{\partial u}\log PA_2 = -PRA_2.$$

The condition of integrability of these equations is reducible to

$$2A_2PQ - U_1(Q^2 + R^2) - QTU_2 = 0.$$

This equation and the one obtained by differentiating it with respect to v are together equivalent to

$$U_1 = 2A_2PQ, \qquad U_2 = 2A_2PT.$$

It is readily found that

$$\frac{\partial}{\partial u}PA_2Q = PA_2R(U_1 - 2A_2PQ) = 0, \qquad \frac{\partial}{\partial v}PA_2Q = 0,$$

$$\frac{\partial}{\partial u}PA_2T = PA_2R(U_2 - 2A_2PT) = 0, \qquad \frac{\partial}{\partial v}PA_2T = 0.$$

Hence $U_1 =$ const., and $U_2 =$ const., and consequently the curves $v =$ const. are circular helices. We discuss this case further in § 9.

5. *The general case.* — In what follows we assume that none of the quantities A_2, A_3, Q is equal to zero. Then equations (15) are equivalent to

$$(18)\quad \begin{cases} \dfrac{\partial}{\partial v}\log PA_2 = \dfrac{1}{QA_2}[A_3 W - A_2(RA_2 + TA_3)], \\ \dfrac{\partial}{\partial v}\log PA_3 = -\dfrac{1}{QA_3}[A_2 W + A_3(RA_2 + TA_2)], \end{cases}$$

where we have put

$$(19)\quad W = L_3 Q - A_3 R + A_2 T.$$

From (6), (7), (11) and (12) we obtain

$$(20)\quad \begin{cases} \dfrac{\partial}{\partial u}\log PA_2 = -\dfrac{A_3}{A_2}U_2 - P(RA_2 + TA_3), \\ \dfrac{\partial}{\partial u}\log PA_3 = \dfrac{1}{A_3}(U_1 L_3 + U_2 A_2) - P(RA_2 + TA_3). \end{cases}$$

Expressing the condition of integrability of equations (18) and (20), we get

$$(21)\quad \begin{cases} U_1(W^2 - 2A_2 TW + A_2^2) = 2PQA_2(A_2^2 + A_3^2), \\ U_1\left[\dfrac{\partial}{\partial v}\left(\dfrac{L_3}{A_3}\right) - A_2 - A_2\left(\dfrac{L_3}{A_3}\right)^2 - \dfrac{T}{Q^2 A_3^2}(A_2^2 + A_3^2)(A_3 R + L_3 Q) + \dfrac{2TL_3}{Q}\right] \\ \qquad + (A_2^2 + A_3^2)\dfrac{2P}{Q} = 0. \end{cases}$$

Eliminating U_1 from these equations, we obtain an equation which is reducible by means of (18) to

$$(22)\quad \frac{\partial}{\partial v}\log PL_3 = \frac{A_3}{L_3 Q^2}\left[-\frac{W^2}{A_2} + \left(\frac{TA_2}{A_3} - R\right)(RA_2 + TA_3)\right].$$

Also we have from (6), (12) and (11)

$$(23)\quad \frac{\partial}{\partial u}\log PL_3 = -\frac{A_3}{L_3}U_1 - P(RA_2 + TA_3).$$

By means of the first of (21) the condition of integrability of equations (22) and (23) is reducible to

$$(24)\quad \frac{A_2^2 + A_3^2}{L_3 Q^2}W(U_2 W + 3PA_2^2) = 0.$$

6. It the first of equations (21) be differentiated with respect to v, the result is reducible to

$$U_1\{(A_2^2 - W^2)[A_3 W - A_2(RA_2 + TA_3)] + A_2^2 W[A_3 + RW - T(RA_2 + TA_3)]\}$$
$$= (A_2^2 + A_3^2)A_2 PQ[WA_3 - 2A_2(RA_2 + TA_3)],$$

with the aid of

$$\frac{\partial}{\partial u}PW = -2P^2(RA_2 + TA_3)W,$$

$$\frac{\partial}{\partial v}PW = \frac{PW}{QA_2}[R(A_2^2 + A_3^2) - L_3 QA_3].$$

Eliminating U_1 between this equation and the first of (21), we get

(25) $$W\{A_3 W^2 - 2A_2(RA_2 + TA_3)W - A_2^2[A_3 - 2T(RA_2 + TA_3)]\} = 0.$$

We consider first the case where $W \neq 0$. Now (24) reduces to

$$U_2 + \frac{3PA_2^2}{W} = 0.$$

The derivative of this equation with respect to v is reducible to

(26) $$A_3 W - A_2(RA_2 + TA_3) = 0.$$

Eliminating W between equations (25) and (26), we get

$$R^2 A_2^2 + A_3^2(Q^2 + R^2) = 0.$$

This equation is satisfied if $R = A_3 = 0$, or $A_2 = A_3 = 0$, or $A_2 = Q = R = 0$, or $Q = R = 0$. These solutions were treated in § 4, where it is shown that they lead either to cylinders or to $W = 0$. Hence the only possibility in the general case is $W = 0$, which we proceed to consider.

7. *When* $W = 0$. — If we put

$$S = P\sqrt{A_2^2 + A_3^2 + L_3^2},$$

it is readily found that, when $W = 0$,

(27) $$\frac{\partial S}{\partial u} = -SP(RA_2 + TA_3), \qquad \frac{\partial S}{\partial v} = -\frac{S}{Q}(RA_2 + TA_3).$$

We denote by a, b, c, the functions defined by

$$a = \frac{P(\alpha L_3 - lA_3 + \lambda A_2)}{S}, \quad b = \frac{P(\beta L_3 - mA_3 + \mu A_2)}{S}, \quad c = \frac{P(\gamma L_3 - nA_3 + \nu V_2)}{S}.$$

Evidently a, b, c, are the direction-cosines of a line D. By means of the preceding equations it is readily found that the derivatives of a, b, c, are equal to zero, when $W = 0$.

From (10) and (27) we have

$$a\alpha_1 + b\beta_1 + c\gamma_1 = \frac{PW}{S} = 0.$$

Hence all the generators of the ruled surface S under consideration are parallel to a plane with fixed direction. If now this plane is taken for the xy-plane, we must have

$$a = \alpha L_3 - lA_3 + \lambda A_2 = 0, \qquad b = \beta L_3 - mA_3 + \mu A_2 = 0,$$

which are equivalent to

(28) $$\frac{L_3}{\gamma} = \frac{A_3}{-n} = \frac{A_2}{\nu} = \frac{S}{P}.$$

8. *S a right conoid.* — It is our purpose to show that S is a right conoid. The necessary and sufficient that S be a right conoid with the z-axis for the axis of the conoid is that there exist a function ρ such that

$$x = \rho\alpha_1, \qquad y = \rho\beta_1. \tag{29}$$

If these equations be differentiated with respect to u, the resulting equations are reducible with the aid of (17) to

$$\begin{cases} \alpha\left(NQ - \frac{1}{\rho}\right) + l(RN + A_2P) + \lambda(TN + A_3P) = 0, \\ \beta\left(NQ - \frac{1}{\rho}\right) + m(RN + A_2P) + \mu(TN + A_3P) = 0, \end{cases} \tag{30}$$

where we have put

$$N = \frac{\partial \log \rho}{\partial u} - P(RA_2 + TA_3).$$

In consequence of (28) these equations are equivalent to

$$\frac{QN - \frac{1}{\rho}}{L_3} = \frac{RN + A_2P}{-A_3} = \frac{TN + A_3P}{A_2}. \tag{31}$$

Multiplying numerators and denominators by Q, R, T, respectively and adding, we have, since the denominator is equal to zero,

$$N - \frac{Q}{\rho} + P(RA_2 + TA_3) = \frac{\partial \log \rho}{\partial u} - \frac{Q}{\rho} = 0. \tag{32}$$

The equation formed from the last two terms of (31) is reducible to

$$\left[\frac{\partial}{\partial u}\log\rho - P(RA_2 + TA_3)\right](RA_2 + TA_3) = -P(A_2^2 + A_3^2), \tag{33}$$

which in consequence of (32) is equivalent to

$$\rho = -\frac{P(RA_2 + TA_3)}{QS^2}. \tag{34}$$

Hence equations (30) may be replaced by (32) and (34). Moreover, this value is found to satisfy (32) in consequence of the first of (21) which is now

$$U_1 = \frac{2PQ}{A_2}(A_2^2 + A_3^2). \tag{35}$$

When equations (29) are differentiated with respect to v, we get

$$\frac{\partial \rho}{\partial v} = \sqrt{q^2 + r^2 + t^2}, \tag{36}$$

which is satisfied identically by ρ as given by (34). Hence the ruled surfaces satisfying the conditions of our problem are right conoids.

9. *Geometric character of the curves* $v =$ const. — Now we shall show that the curves $v =$ const. lie on circular cylinders tangent to the axis of the conoid. Such

cylinders are defined analytically by

$$x^2 + y^2 + V_1 x + V_2 y = 0, \tag{37}$$

where V_1 and V_2 are functions of v alone to be determined.

Writing this equation in the form $\rho^2 + V_1 x + V_2 y = 0$, and differentiating with respect to u, we get by means of (4) and (32)

$$2\rho Q + V_1 \alpha + V_2 \beta = 0.$$

In consequence of (29) and (34) these equations are equivalent to

$$V_1 = \frac{1}{QS}(2 Q \beta_1 - b), \qquad V_2 = -\frac{1}{QS}(2 Q \alpha_1 - \alpha).$$

Making use of (17), we show that the right-hand members of these equations are independent of u. Furthermore,

$$V_1^2 + V_2^2 = \frac{1}{Q^2 S^2}(1 - \gamma^2) = \frac{1}{S^2} + \rho^2.$$

By means of (27), (32) and (36) it is readily shown that $\frac{1}{S^2} + \rho^2$ is a constant. Hence the cylinders defined by (37) are congruent.

In § 4 we considered the particular case when $A_3 = 0$, and found that the curves $v =$ const. are circular helices. It may be shown readily that the results of §§ 5-9 are equally true when $A_3 = 0$, the equations of the case being those of the preceding sections with $A_3 = W = 0$. In this case the ruled surface is the skew helicoid with plane director.

The foregoing results may be stated in the theorem:

Cylinders and right conoids are the only ruled surfaces which may be generated by the motion of an invariable curve such that points of the curve describe the generators in the motion; in the case of the conoids these curves are the sections of the surface by circular cylinders having the axis of the conoid for a generator.

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