## DISCUSSION

**Louis Cohen** (by letter): Aside from the interesting solution of the problem that the paper deals with relating to radio frequency alternators, the great importance of the paper consists in the general method that Professor Pupin has given us for solving differential equations having variable coefficients. I believe the method will prove of great value in the solution of many other problems in electrotechnics.

I recall that I have discussed this problem with Professor Pupin about six years ago, and he told me then that he had marked out the general solution of the problem, but reserved its publication for some future time. We ought to be grateful to Mr. Liebowitz for having put it in shape for publication and presenting it before the Institute.

As an illustration of the applicability of the method developed in the paper to the solution of other problems, it may be of interest to mark out the problem of the microphone circuit.

We have in this case an inductance, a variable resistance and a continuous e. m. f. in the circuit, and the circuit equation is,

$$L\frac{dI}{dt} + RI + rI\cos\omega t = E, \tag{1}$$

where $R+r$ is the total resistance of the circuit in stationary condition.*

As far as I know the complete solution of this problem has never been given. Following, however, the method developed by Professor Pupin, we can readily obtain the solution of the problem.

Put $$I = I_0 + I_1 + I_2 + I_3 + \ldots\ldots + I_n, \tag{2}$$

and make the substitution in equation (1), we get

$$\left.\begin{aligned} &L\frac{dI_0}{dt} + RI_0 + rI_0\cos\omega t \\ &+ L\frac{dI_1}{dt} + RI_1 + rI_1\cos\omega t \\ &+ L\frac{dI_2}{dt} + RI_2 + rI_2\cos\omega t \\ &+ \ldots\ldots\ldots\ldots \\ &+ L\frac{dI_n}{dt} + RI_n + rI_n\cos\omega t = E \end{aligned}\right\} \tag{3}$$

* (R is the constant resistance of the external circuit; the resistance of the microphone, which varies periodically under the influence of a sound of frequency $\frac{\omega}{2\pi}$, is $r\cos\omega t$. The term $I(r\cos\omega t)$ in equation (1) is therefore the drop of potential at time t across the microphone.—EDITOR.)

In accordance with the method given in the paper, we can break up equation (3) into a number of independent equations, as follows:

$$\left.\begin{array}{ll}
(a) & L\dfrac{dI_0}{dt} + RI_0 = E \\
(b) & L\dfrac{dI_1}{dt} + RI_1 + rI_0\cos\omega t = 0 \\
(c) & L\dfrac{dI_2}{dt} + RI_2 + rI_1\cos\omega t = 0 \\
(d) & L\dfrac{dI_3}{dt} + RI_3 + rI_2\cos\omega t = 0 \\
 & \cdots\cdots\cdots\cdots \\
 & \cdots\cdots\cdots\cdots
\end{array}\right\} \quad (4)$$

Disregarding the transients, we have for the solution of (4a),

$$I_0 = \frac{E}{R}. \qquad (5)$$

Substituting the value of $I_0$ from (5) into (4b), we get

$$L\frac{dI_1}{dt} + RI_1 = -\frac{Er}{R}\cos\omega t \qquad (6)$$

and

$$I_1 = -\frac{Er}{RZ_1}\cos(\omega t - \theta_1) \qquad (7)$$

$$Z = \sqrt{L^2\omega^2 + R^2}, \quad \theta_1 = \tan^{-1}\frac{L\omega}{R}.$$

Substituting the value of $I_1$ into (4c), we have

$$L\frac{dI_2}{dt} + RI_2 = \frac{Er^2}{RZ_1}\cos(\omega t - \theta_1)\cos\omega t$$
$$= \frac{Er^2}{2RZ_1}\left\{\cos(2\omega t - \theta_1) + \cos\theta_1\right\}$$

and

$$I_2 = \frac{Er^2}{2RZ_1Z_2}\cos(2\omega t - \theta_1 - \theta_2) + \frac{Er^2\cos\theta_1}{2R^2Z_1}. \qquad (8)$$

Repeating the operation, we find in a similar manner,

$$I_3 = -\frac{Er^3}{4RZ_1Z_2Z_3}\cos(3\omega t - \theta_1 - \theta_2 - \theta_3)$$
$$-\frac{Er^3}{4RZ_1^2Z_2}\cos(\omega t - 2\theta_1 - \theta_2)$$
$$-\frac{Er^3}{2R^2Z_1^2}\cos\theta_1\cos(\omega t - \theta_1) \qquad (9)$$

$$\left.\begin{aligned}
I_4 = {} & \frac{E\,r^4}{8\,R\,Z_1\,Z_2\,Z_3\,Z_4}\cos(4\,\omega\,t - \theta_1 - \theta_2 - \theta_3 - \theta_4) \\
& \qquad + \frac{E\,r^4}{8\,R\,Z_1 Z_2^2 Z_3}\cos(2\,\omega\,t - \theta_1 - 2\,\theta_2 - \theta_3) \\
& + \frac{E\,r^4}{8\,R\,Z_1^2 Z_2^2}\cos(2\,\omega\,t - 2\,\theta_1 - 2\,\theta_2) \\
& \qquad + \frac{E\,r^4}{8\,R^2\,Z_1^2\,Z_2}\cos(2\,\theta_1 + \theta_2) \\
& + \frac{E\,r^4}{4\,R^2\,Z_1^2\,Z_2}\cos(2\,\omega\,t - \theta_1 - \theta_2) + \frac{E r^4 \cos^2\theta_1}{4\,R^3\,Z_1^2}.
\end{aligned}\right\} \tag{10}$$

and similarly for the other components.

If we collect separately the terms of the same frequency, and denote the results by $\eta_o$, $\eta_1$, $\eta_2$, etc., respectively, we get

$$\begin{aligned}
\eta_o &= \frac{E}{R} + \frac{E\,r^2\cos\theta_1}{2\,R^2\,Z_1} + \frac{E\,r^4}{8\,R^2\,Z_1^2\,Z_2}\cos(2\,\theta_1 + \theta_2) \\
&\qquad + \frac{E\,r^4\cos^2\theta_1}{4\,R^3\,Z_1^2} + \cdots \\
&= \frac{E}{R}\left\{1 + \frac{r^2}{2\,R\,Z_1}\cos\theta_1 + \frac{r^4}{8\,R\,Z_1^2\,Z_2}\cos(2\,\theta_1 + \theta_2)\right. \\
&\qquad \left. + \frac{r^4\cos^2\theta_1}{4\,R^2\,Z_1^2} + \cdots\right\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
-\eta_1 &= \frac{E\,r}{R\,Z_1}\left\{\cos(\omega\,t - \theta_1) + \frac{r^2}{4\,Z_1\,Z_2}\cos(\omega\,t - 2\,\theta_1 - \theta_2)\right. \\
&\qquad \left. + \frac{r^2}{2\,R\,Z_1}\cos\theta_1\cos(\omega\,t - \theta_1) + \cdots\right\}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\eta_2 &= \frac{E\,r^2}{2\,R\,Z_1\,Z_2}\left\{\cos(2\,\omega t - \theta_1 - \theta_2) + \frac{r^2}{4\,Z_2 Z_3}\cos(2\omega t - \theta_1 - 2\theta_2 - \theta_3)\right. \\
&\qquad + \frac{r^2}{4\,Z_1\,Z_2}\cos(2\,\omega\,t - 2\,\theta_1 - 2\,\theta_2) \\
&\qquad \left. + \frac{r^2}{2\,R\,Z_1}\cos(2\,\omega\,t - \theta_1 - \theta_2) + \cdots\right\}
\end{aligned} \tag{13}$$

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The total current in the circuit is

$$I = \eta_o + \eta_1 + \eta_2 + \cdots \tag{14}$$

It is seen therefore that the current is of a complex character, having a continuous current component, and currents of frequencies $\frac{\omega}{2\pi}$, $\frac{2\,\omega}{2\pi}$, etc. It is also to be noted that the amplitudes of the different components decrease as the frequencies increase.

As a partial proof we may consider the case when there is no inductance in the circuit, $L = 0$, we have then

$$\theta_1 = \theta_2 = \theta_3 = \cdots = 0$$
$$Z_1 = Z_2 = Z_3 = \cdots = R.$$

Equations (11), (12), and (13) reduce to

$$\eta_0 = \frac{E}{R}\left\{1 + \frac{r^2}{2\,R^2} + \frac{r^4}{8\,R_4} + \frac{r^4}{4\,R^4} + \cdots\cdots\right\}$$
$$-\,\eta_1 = \frac{E\,r}{R^2}\cos\omega\,t\left\{1 + \frac{r^2}{4\,R^2} + \frac{r^2}{2\,R^2} + \cdots\cdots\right\} \qquad (15)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If we put $L=0$ in equation (1) we get

$$I = \frac{E}{R + r\cos\omega\,t} = \frac{E}{R}\left\{1 + \frac{r}{R}\cos\omega\,t\right\}^{-1}. \qquad (16)$$

Expanding the above by the binomial theorem, we have

$$I = \frac{E}{R}\left\{1 - \frac{r}{R}\cos\omega\,t + \frac{r^2}{R^2}\cos^2\omega\,t - \frac{r^3}{R^3}\cos^3\omega\,t + \cdots\cdots\right\} \quad (17)$$

$$\cos^2\omega\,t = \frac{1}{2} + \frac{1}{2}\cos 2\,\omega\,t$$
$$\cos^3\omega\,t = \frac{1}{2}\cos\omega\,t + \frac{1}{4}\cos\omega\,t + \frac{1}{4}\cos 3\,\omega\,t$$
$$\cos^4\omega\,t = \frac{1}{4} + \frac{1}{2}\cos 2\,\omega\,t + \frac{1}{8} + \frac{1}{8}\cos 4\,\omega\,t$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Making these substitutions, we get

$$\because\ I = \frac{E}{R}\left\{1 + \frac{r^2}{2\,R^2} + \frac{r^4}{4\,R^4} + \frac{r^4}{8\,R^4} + \cdots\cdots\right\}$$
$$-\,\frac{E\,r}{R^2}\cos\omega\,t\left\{1 + \frac{1}{2}\frac{r^2}{R^2} + \frac{1}{4}\frac{r^2}{R^2} + \cdots\cdots\right\} \qquad (18)$$
$$+\ \cdots\cdots\cdots\cdots\cdots\cdots$$

The results by the two methods are in exact agreement.

**Benjamin Liebowitz** (by letter): Owing to the fact that Pupin's series diverge when tuned condenser circuits of low resistance are employed, great care must be exercised in interpreting the theory as applied to the Goldschmidt alternator. The theory shows that if a current of a given frequency is large, the currents of neighboring frequencies must also be large; but it also shows that by properly controlling the impedances (detuning some of the circuits, if necessary) *the series for a given frequency can be made to diverge more rapidly than any other.* There is nothing in the theory, therefore, which says that a high efficiency is impossible. On the other hand, a high efficiency would hardly be expected, owing to the inevitable large losses in the iron, and in practice the efficiency is not more than fifty-four per cent., according to Mr. Mayer.

It has been remarked that the currents of frequency $\frac{2\omega}{2\pi}$, for example, generated in the stator by successive "reflection" from the rotor, being of opposite signs, tend to neutralize each other. It must be borne in mind, however, that any power series whose ratio is greater than unity is divergent, even if the signs alternate. Therefore, all the currents tend toward infinity in an ideal machine, in spite of the differences in sign of successive amplitudes. The series will begin to converge only when the ratios $\frac{\omega M}{Z}$ become sufficiently small, and in tuned condenser circuits this cannot happen until the currents attain sufficiently large values to produce detuning, a decrease in M, and increases in the effective resistances, by the approach of saturation.

**Lester L. Israel** (by letter): From the theory developed in this paper it appears that currents of lower frequency due to reactions of the higher harmonics become increasingly large.

Since in practice the Goldschmidt alternator is quite efficient, this can hardly be so. Perhaps the apparent discrepancy may be accounted for by the fact that these induced lower harmonics are opposed in phase, together with a limitation or modification of the series representing them arising from the high energy absorption at one of the higher harmonics.