

# The Two-Copy Purity Body and the Complementary-Block Frontier for Werner Distillability

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## Abstract

For an operator  $R$  of rank at most two on  $\mathbb{C}^d \otimes \mathbb{C}^d$ , the four subset purities  $\|R\|_{\text{HS}}^2$ ,  $\|\text{Tr}_1 R\|_{\text{HS}}^2$ ,  $\|\text{Tr}_2 R\|_{\text{HS}}^2$ ,  $|\text{Tr } R|^2$  satisfy the sharp two-copy Werner inequality proved in July 2026 and the rank bound  $|\text{Tr } R|^2 \leq 2\|R\|_{\text{HS}}^2$ . We determine every linear inequality they satisfy in unrestricted local dimension and in every even dimension. In unrestricted local dimension the closed convex hull of the normalized purity vectors is a polytope with six vertices and seven facets; two of the facets are compression inequalities that the earlier bounds do not imply. In each even local dimension  $d$  the body is that polytope cut by two Cauchy–Schwarz inequalities, with vertices depending on  $d$ ; the same cut is not the body in any odd dimension, because a vertex it predicts is not the purity vector of any rank-two operator. At  $r$  copies, the certificates built from valid two-block inequalities on complementary groupings of the sites, the family used by Bharti, Gajjala and Haug, reach exactly the interval they found: an explicit dual vector shows that no combination of dimension-free two-copy inequalities on complementary blocks does better, at any copy number. The compression inequalities come from averaging the one-copy inequality over random compressions of one site; the dual vectors are assembled from vertices of the two-copy body, so the body decides the frontier.

## 1 The question

Werner states on  $\mathbb{C}^d \otimes \mathbb{C}^d$  form a one-parameter family  $\rho_\alpha = (I + \alpha F)/(d^2 + \alpha d)$ ,  $-1 \leq \alpha \leq 1$ , where  $F$  swaps the two tensor factors. They are separable exactly when  $\alpha \geq -1/d$  [1] and distillable from a single copy exactly when  $\alpha < -1/2$  [2]. Whether the states with  $-1/2 \leq \alpha < -1/d$ ,  $d \geq 3$ , can be distilled from finitely many copies is the standing question of [3], Problem 4 of [4]. A state is  $r$ -copy distillable if and only if  $(\rho^\Gamma)^{\otimes r}$  has a negative expectation on some vector of Schmidt rank at most two [2, 3]; for Werner states, reading such a vector as a rank-two matrix  $R$  on  $(\mathbb{C}^d)^{\otimes r}$ , this becomes [5, 6]

$$\rho_\alpha \text{ is } r\text{-copy distillable} \iff \sum_{S \subseteq [r]} \alpha^{|S|} \|\text{Tr}_S R\|_{\text{HS}}^2 < 0 \text{ for some } R \text{ with } \text{rank } R \leq 2, \quad (1)$$

with  $\text{Tr}_S$  the partial trace over the sites in  $S$ . Distillability at each  $\alpha$  is therefore the sign of one linear functional of the  $2^r$  subset purities  $\|\text{Tr}_S R\|_{\text{HS}}^2$  of rank-two operators, and the first thing to know about them is which linear inequalities they satisfy.

At two copies this paper answers that question in unrestricted local dimension and in every even dimension. The sharp two-copy inequality  $\|\text{Tr}_1 R\|^2 + \|\text{Tr}_2 R\|^2 \leq 2\|R\|^2 + \frac{1}{2}|\text{Tr } R|^2$ , proved in July 2026 by four groups [7, 8, 9, 10], is one facet. The rank bound  $|\text{Tr } R|^2 \leq 2\|R\|_{\text{HS}}^2$  is another. Two compression inequalities are the new facets (Lemma 3.1): compress one site of  $R$

onto a random vector, apply the one-copy inequality to the compressed operator, which still has rank at most two, and average; a fourth-moment identity turns the average into purities of  $R$ . With the three sign constraints these seven bound a polytope whose six vertices are all realized by explicit rank-one and rank-two operators, so nothing is missing (Theorem 3.2). In even local dimension the body is this polytope cut by the two Cauchy–Schwarz bounds  $|\operatorname{Tr} R|^2 \leq d \|\operatorname{Tr}_k R\|_{\text{HS}}^2$ , with vertices that move with  $d$  (Theorem 3.3); in odd dimension one vertex of that cut is not realizable, and the body there is left open.

The two-copy body then decides a question at every copy number. Bharti, Gajjala and Haug [8] derived dimension-free  $r$ -copy undistillability intervals by combining valid two-copy inequalities applied to complementary groupings of the sites, and their combination stops strictly short of the endpoint  $\alpha = -\frac{1}{2}$  for every  $r \geq 3$ . Knowing every two-copy inequality, one can ask whether any such combination does better. None does: an explicit dual vector, each of whose complementary triples is a vertex of the two-copy body, certifies that their interval is the exact frontier of the complementary-block family (Theorem 4.2).

**Relation to the companion paper.** The companion paper [13] proves that no additive split of the endpoint form into fixed-coefficient purity inequalities has a summand other than a rescaling of the form itself, and that no sum of squares in a rank-two factorization certifies the endpoint for  $d \geq 3$ . The present paper uses none of that: every proof here is from the two-copy body and the cited two-copy inequality. Nothing in the companion paper depends on the results here.

**Contributions.** Theorem 3.2: the dimension-free two-copy purity body is the polytope with the six vertices (3) and the seven facets (4). Theorem 3.3: in every even dimension  $d$  the body is the polytope (6), and that polytope strictly contains the body in every odd dimension. Theorem 4.2: the complementary-block frontier at  $r$  copies is exactly the polynomial  $p_r$  of (9)–(10), with the dual vectors (11)–(12) as the upper bound and the certificate of [8] as the lower bound.

## 2 Definitions

Throughout,  $r \geq 1$  is the number of copies (sites),  $d \geq 2$  the local dimension,  $H = (\mathbb{C}^d)^{\otimes r}$ , and  $[r] = \{1, \dots, r\}$ . Operators are complex matrices on  $H$ ;  $\|\cdot\|_{\text{HS}}$  is the Hilbert–Schmidt norm and  $\|\cdot\|_{\text{op}}$  the operator norm. For  $S \subseteq [r]$ ,  $\operatorname{Tr}_S$  is the partial trace over the sites in  $S$ , with  $\operatorname{Tr}_\emptyset R = R$  and  $\operatorname{Tr}_{[r]} R = \operatorname{Tr} R$ .

**Definition 2.1** (Subset purities and the endpoint form). For  $R$  on  $H$  and  $S \subseteq [r]$  the subset purity is  $F_S(R) = \|\operatorname{Tr}_S R\|_{\text{HS}}^2$ , and the purity vector is  $\Phi(R) = (F_S(R))_{S \subseteq [r]} \in \mathbb{R}^{2^r}$ . For  $R \neq 0$  the normalized purity vector is  $\phi(R) = \Phi(R)/F_\emptyset(R)$ , so  $\phi(R)_\emptyset = 1$ . The endpoint functional is the linear map  $\theta_r : \mathbb{R}^{2^r} \rightarrow \mathbb{R}$ ,

$$\theta_r(x) = \sum_{S \subseteq [r]} (-1)^{|S|} 2^{r-|S|} x_S,$$

and  $\Theta_r(R) = \theta_r(\Phi(R))$ .

**Definition 2.2** (Fixed-coefficient functionals). A fixed-coefficient functional is a linear map  $q : \mathbb{R}^{2^r} \rightarrow \mathbb{R}$ ,  $q(x) = \sum_S c_S x_S$  with real constants  $c_S$ . It is *valid* (at local dimension  $d$ ) if  $q(\Phi(R)) \geq 0$  for every  $R$  on  $H$  with  $\operatorname{rank} R \leq 2$ .

Because  $\Phi$  is homogeneous of degree two in  $R$ , a functional is valid if and only if it is nonnegative on normalized purity vectors of rank-at-most-two operators, and any nonnegative constant multiple of a valid functional is valid.

**Definition 2.3** (Product operators and local ratios). For local matrices  $A_1, \dots, A_r$  on  $\mathbb{C}^d$  the product operator is  $A_1 \otimes \dots \otimes A_r$ . The local ratio of a nonzero  $A$  is  $\tau(A) = |\text{Tr } A|^2 / \|A\|_{\text{HS}}^2$ . For  $\tau \in \mathbb{R}^r$  the product purity vector is

$$p_\tau \in \mathbb{R}^{2^r}, \quad (p_\tau)_S = \prod_{i \in S} \tau_i,$$

with the empty product equal to 1.

**Lemma 2.4** (Product formula). *Let  $R = A_1 \otimes \dots \otimes A_r$  with every  $A_i \neq 0$ , and let  $\tau_i = \tau(A_i)$ . Then  $\phi(R) = p_\tau$  and*

$$\Theta_r(R) = \prod_{i=1}^r (2\|A_i\|_{\text{HS}}^2 - |\text{Tr } A_i|^2) = \prod_{i=1}^r \Theta_1(A_i), \quad \theta_r(p_\tau) = \prod_{i=1}^r (2 - \tau_i).$$

*Proof.* Tracing out the sites in  $S$  removes the factors  $A_i$  with  $i \in S$  and replaces each by the scalar  $\text{Tr } A_i$ :  $\text{Tr}_S R = (\prod_{i \in S} \text{Tr } A_i) \otimes_{i \notin S} A_i$ . The Hilbert–Schmidt norm of a tensor product is the product of the norms, so  $F_S(R) = \prod_{i \in S} |\text{Tr } A_i|^2 \prod_{i \notin S} \|A_i\|_{\text{HS}}^2$ . Dividing by  $F_\emptyset(R) = \prod_i \|A_i\|_{\text{HS}}^2$  gives  $\phi(R)_S = \prod_{i \in S} \tau_i$ . For the value of  $\theta_r$ , expand the product:

$$\prod_{i=1}^r (2 - \tau_i) = \sum_{S \subseteq [r]} \prod_{i \in S} (-\tau_i) \prod_{i \notin S} 2 = \sum_S (-1)^{|S|} 2^{r-|S|} \prod_{i \in S} \tau_i = \theta_r(p_\tau).$$

Multiplying by  $\prod_i \|A_i\|_{\text{HS}}^2$  gives the operator statement.  $\square$

The lemma is the single-operator case of the exact block factorization of Bharti, Gajjala and Haug [8, Proposition 8.2].

**Definition 2.5** (Local blocks). Fix two basis vectors  $e_0, e_1$  of  $\mathbb{C}^d$  and write  $E_{ab} = |e_a\rangle\langle e_b|$ . The three local blocks are

$$B_2 = E_{00} + E_{11} \quad (\tau = 2, \text{ rank } 2), \quad B_1 = E_{00} \quad (\tau = 1, \text{ rank } 1), \quad B_0 = E_{01} \quad (\tau = 0, \text{ rank } 1).$$

**Lemma 2.6** (Product operators with prescribed ratios). *For every  $d \geq 2$  and every  $\tau$  with entries in  $\{0, 1, 2\}$ , the product operator  $R_\tau = B_{\tau_1} \otimes \dots \otimes B_{\tau_r}$  has  $\phi(R_\tau) = p_\tau$  and  $\theta_r(p_\tau) = \prod_i (2 - \tau_i)$ . Its rank is the product of the ranks of the blocks, so it is at most two exactly when at most one  $\tau_k$  equals 2.*

*Proof.* The local ratios of  $B_2, B_1, B_0$  are  $4/2 = 2, 1/1 = 1, 0/1 = 0$ , so Lemma 2.4 gives both formulas; the rank of a tensor product is the product of the ranks.  $\square$

**Lemma 2.7** (Rank bound). *For every matrix  $R$ ,  $|\text{Tr } R| \leq \|R\|_* \leq \sqrt{\text{rank } R} \|R\|_{\text{HS}}$ , where  $\|\cdot\|_*$  is the trace norm. Hence  $|\text{Tr } R|^2 \leq 2\|R\|_{\text{HS}}^2$  when  $\text{rank } R \leq 2$ , and equality holds exactly when  $R = e^{i\varphi} s (P_u + P_v)$  for orthonormal  $u, v$ , a phase  $\varphi$  and  $s > 0$ .*

*Proof.* The trace is the sum of the diagonal entries of  $R$  in a basis of right singular vectors, each bounded by the corresponding singular value, which gives the first inequality, with equality exactly when  $R = e^{i\varphi} |R|$  for a positive semidefinite  $|R|$ . The second is Cauchy–Schwarz on the singular values, with equality exactly when the nonzero singular values are equal. Both together give the stated form.  $\square$

### 3 The two-copy purity cones

At two copies there are only three normalized coordinates to plot:

$$(a, b, c) = (x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}), \quad x_{\emptyset} = 1.$$

Let  $P_d$  be the convex hull of the normalized purity vectors of nonzero rank-at-most-two operators on  $\mathbb{C}^d \otimes \mathbb{C}^d$ , and let  $P_\infty = \overline{\text{conv}}(\bigcup_{d \geq 2} P_d)$ . Convexification preserves exactly the linear inequalities under study. The unnormalized cones are the cones over these bodies with leading coordinate  $x_\emptyset$ .

#### 3.1 Compression and the seven inequalities

The sharp two-copy theorem gives  $4 - 2a - 2b + c \geq 0$  [8, Theorem 6.1, (69)]. The rank bound gives  $c \leq 2$ . The following two inequalities complete these bounds to the full body.

**Lemma 3.1** (Compression inequalities). *For every rank-at-most-two operator on a finite-dimensional bipartite space,*

$$2F_\emptyset - F_{\{1\}} + 2F_{\{2\}} - F_{\{1,2\}} \geq 0, \quad 2F_\emptyset + 2F_{\{1\}} - F_{\{2\}} - F_{\{1,2\}} \geq 0. \quad (2)$$

*Proof.* Choose independent complex coordinates  $z_i$ , each uniformly from the multiset

$$0, 0, 0, 0, \sqrt{2}, -\sqrt{2}, i\sqrt{2}, -i\sqrt{2}.$$

Their moments satisfy

$$\mathbb{E}(\bar{z}_i z_j \bar{z}_k z_l) = \delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}.$$

Indeed, independence and phase cancellation leave only paired indices;  $\mathbb{E}|z_i|^2 = 1$  and  $\mathbb{E}|z_i|^4 = 2$  give the two terms, including the coincident-index case. Compress the second site:

$$R_z = (I \otimes \langle z |) R (I \otimes | z \rangle), \quad \text{rank } R_z \leq 2.$$

Writing  $R = \sum_{i,j} R_{ij} \otimes E_{ij}$  gives  $R_z = \sum_{i,j} \bar{z}_i z_j R_{ij}$ . The moment identity yields

$$\begin{aligned} \mathbb{E} \|R_z\|_{\text{HS}}^2 &= \sum_{i,j} \|R_{ij}\|_{\text{HS}}^2 + \left\| \sum_i R_{ii} \right\|_{\text{HS}}^2 = F_\emptyset + F_{\{2\}}, \\ \mathbb{E} |\text{Tr } R_z|^2 &= \sum_{i,j} |\text{Tr } R_{ij}|^2 + \left| \sum_i \text{Tr } R_{ii} \right|^2 = F_{\{1\}} + F_{\{1,2\}}. \end{aligned}$$

Average the one-copy inequality  $2\|R_z\|_{\text{HS}}^2 - |\text{Tr } R_z|^2 \geq 0$  to obtain the first assertion. Interchanging the sites gives the second.  $\square$

In contrast with the compression bound, the two bounds used in [8, (81), (82)] are  $a + b \leq 2 + c/2$  and  $|a - b| \leq 2 - c/2$ ; the latter is Costa Rico's rank-two difference bound [5, Theorem 2, (45)]. The point  $(1, 0, 2)$  satisfies both of those bounds and has compression slack  $-1$ .

**Theorem 3.2** (The dimension-free two-copy body). *The body  $P_\infty$  is*

$$\text{conv}\{(0, 0, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2), (2, 1, 2), (1, 2, 2)\}. \quad (3)$$

*Its seven facets are exactly*

$$\begin{aligned} a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad c \leq 2, \\ 4 - 2a - 2b + c \geq 0, \quad 2 - a + 2b - c \geq 0, \quad 2 + 2a - b - c \geq 0. \end{aligned} \quad (4)$$

*The fifth facet is the sharp two-copy inequality of [7, 8, 9, 10]; the description of  $P_\infty$  rests on it, and on nothing else outside this paper.*

*Proof.* All seven inequalities hold on actual purity vectors by nonnegativity, the rank bound, the sharp two-copy theorem, and Lemma 3.1. They therefore hold on  $P_\infty$ . For the reverse inclusion between the displayed descriptions, fix  $c = 2t$ ,  $0 \leq t \leq 1$ . The inequalities become

$$a, b \geq 0, \quad a + b \leq 2 + t, \quad a - 2b \leq 2 - 2t, \quad b - 2a \leq 2 - 2t.$$

Their section is the pentagon, with degeneracies at its endpoints,

$$(0, 0), \quad (2 - 2t, 0), \quad (2, t), \quad (t, 2), \quad (0, 2 - 2t). \quad (5)$$

To verify completeness, follow the boundary from  $b = 0$  to  $a - 2b = 2 - 2t$ , then to  $a + b = 2 + t$ , then to its two reflected edges; the successive intersections are exactly the five listed points. Each point, together with  $c = 2t$ , interpolates linearly between vertices in (3) at  $t = 0$  and  $t = 1$ . Thus every section lies in their convex hull, and all six vertices satisfy the inequalities.

Five vertices are realized by the products

$$B_0 \otimes B_0, \quad B_2 \otimes B_0, \quad B_0 \otimes B_2, \quad B_2 \otimes B_1, \quad B_1 \otimes B_2,$$

respectively, omitting  $(0, 0, 2)$  from the list. For that vertex set

$$u = d^{-1/2} \sum_{j=0}^{d-1} |jj\rangle, \quad v = d^{-1/2} \sum_{j=0}^{d-1} e^{2\pi i j/d} |jj\rangle, \quad R = \frac{P_u + P_v}{\sqrt{2}}, \quad P_w = |w\rangle\langle w|.$$

The vectors  $u, v$  are orthonormal and each has both marginals  $I_d/d$ . Consequently  $\|R\|_{\text{HS}} = 1$  and  $(a, b, c) = (2/d, 2/d, 2)$ , which tends to  $(0, 0, 2)$ . All six vertices therefore belong to  $P_\infty$ .

Name them  $O, A, B, C, D, E$  in the order of (3). The equality sets of the seven inequalities contain, respectively, the triples

$$(O, B, C), \quad (O, A, C), \quad (O, A, B), \quad (C, D, E), \quad (A, B, D), \quad (A, C, D), \quad (B, C, E).$$

Each triple is affinely independent. The body is three-dimensional, since  $O, A, B, C$  are affinely independent, so each inequality defines a facet and none can be removed.  $\square$

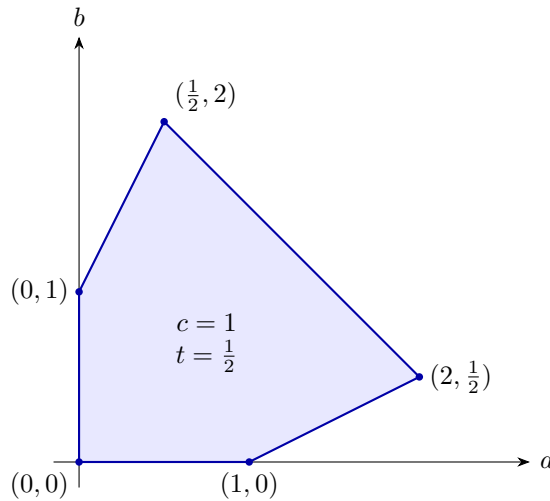


Figure 1: The exact pentagon section (5) at  $c = 2t = 1$ . Its sloping sides are the two compression inequalities and the endpoint inequality. As  $t$  runs from 0 to 1, the five vertices move linearly between the six vertices of  $P_\infty$ .

### 3.2 Even dimensions and the odd-dimensional exclusion

The limit vertex  $(0, 0, 2)$  cannot occur at finite dimension: taking the remaining trace gives  $c \leq da$  and  $c \leq db$ . These are the only extra linear constraints needed in even dimension.

**Theorem 3.3** (Finite-dimensional two-copy bodies). *For every even  $d \geq 2$ ,*

$$P_d = P_\infty \cap \{c \leq da, c \leq db\}. \quad (6)$$

*For every odd  $d \geq 3$ , the right side strictly contains  $P_d$ ; in particular  $(4/d, 2/d, 2)$  and its site swap are not in  $P_d$ .*

*Proof.* Cauchy–Schwarz gives  $|\text{Tr } X|^2 \leq d \|X\|_{\text{HS}}^2$  for a  $d \times d$  matrix. Apply it to  $\text{Tr}_1 R$  and  $\text{Tr}_2 R$  to prove the two extra inequalities. At  $c = 2t$ , put  $\ell = 2t/d$ . Intersecting the section in (5) with  $a, b \geq \ell$  gives the five vertices

$$(\ell, \ell), \quad (2 - 2t + 2\ell, \ell), \quad (2, t), \quad (t, 2), \quad (\ell, 2 - 2t + 2\ell).$$

They follow from the same successive boundary intersections as before;  $\ell \leq t$  ensures that the middle two remain feasible. Each is affine in  $t$ , so the finite-dimensional outer body has the three  $c = 0$  vertices and the five  $c = 2$  vertices

$$(2/d, 2/d, 2), \quad (4/d, 2/d, 2), \quad (2, 1, 2), \quad (1, 2, 2), \quad (2/d, 4/d, 2). \quad (7)$$

At  $d = 2$  two pairs coincide, leaving six vertices in total.

The product operators above realize the bottom vertices and  $(2, 1, 2)$ ,  $(1, 2, 2)$ ; the orthogonal maximally entangled pair realizes  $(2/d, 2/d, 2)$ . For  $d = 2m$ , write the second site as  $\mathbb{C}^m \otimes \mathbb{C}^2$  and embed  $\mathbb{C}^m$  into the first site. Let

$$\Phi_m = m^{-1/2} \sum_{j=0}^{m-1} |j\rangle_A |j\rangle_{B'}, \quad R = P_{\Phi_m} \otimes \frac{I_2}{\sqrt{2}} \quad \text{on } A \otimes (B' \otimes B'').$$

This operator has rank two and squared norm one. Its partial traces are  $(I_m/m) \otimes I_2/\sqrt{2}$  after tracing  $A$ , and  $\sqrt{2}I_m/m$  after tracing  $B$ , so

$$(a, b, c) = (1/m, 2/m, 2) = (2/d, 4/d, 2).$$

Its site swap realizes  $(4/d, 2/d, 2)$ . Hence every vertex of the outer body belongs to  $P_d$  when  $d$  is even, proving (6).

Now suppose a normalized rank-at-most-two operator realizes  $(4/d, 2/d, 2)$ . Equality  $|\text{Tr } R|^2 = 2$  in Lemma 2.7 forces

$$R = e^{i\varphi} \frac{P_u + P_v}{\sqrt{2}}, \quad u \perp v, \quad \|u\| = \|v\| = 1.$$

Indeed, its two singular values must both be  $1/\sqrt{2}$ , and equality in the trace bound makes each left singular vector a common phase times its corresponding right singular vector. Discard that phase. Write  $\rho_A^u, \rho_B^u$  for the marginals of  $P_u$ , and set  $p = \text{Tr}[(\rho_A^u)^2] = \text{Tr}[(\rho_B^u)^2]$ , with  $q$  defined from  $v$ . Then

$$\begin{aligned} a &= \frac{1}{2}(p + q + 2 \text{Tr}(\rho_B^u \rho_B^v)) \leq p + q, \\ b &= \frac{1}{2}(p + q + 2 \text{Tr}(\rho_A^u \rho_A^v)) \geq \frac{1}{2}(p + q). \end{aligned}$$

The first inequality is  $\|\rho_B^u - \rho_B^v\|_{\text{HS}}^2 \geq 0$  and the second uses positivity. Since  $a = 2b$ , both are equalities:  $\rho_B^u = \rho_B^v$  and the supports of  $\rho_A^u, \rho_A^v$  are orthogonal. Also  $b = 2/d$  implies

$$\rho_A^u + \rho_A^v = \frac{2}{d} I_d,$$

by equality in the trace inequality for this positive matrix of trace two. On their orthogonal supports, each marginal is therefore  $2/d$  times the support projection. Its trace is one, so each support has dimension  $d/2$ , impossible for odd  $d$ .

Finally, the unit-norm rank-at-most-two matrices form a compact set, and their purity image and its convex hull  $P_d$  are compact in finite dimension. The excluded point is a vertex of the outer body: it is the unique intersection of  $c = 2$ ,  $db = c$ , and  $2 - a + 2b - c = 0$ . If a convex combination of actual purity vectors equalled that vertex, every term would equal it, contradicting the argument above. Thus it is excluded from  $P_d$ , including its closure. Site interchange gives the second exclusion.  $\square$

The complete two-copy cone in odd local dimension remains open.

### 3.3 Which familiar inequalities are facets?

Set  $\vartheta = 4 - 2a - 2b + c$  and  $M = 2 - a + 2b - c$ . The identities

$$2 - a = \frac{\vartheta + M}{3}, \quad 2 - a + b - \frac{c}{2} = \frac{(2 - a) + M}{2} \quad (8)$$

show that the single-partial-trace bound and Costa Rico's difference bound are redundant in  $P_\infty$ . Their reflected versions give the other two bounds. Both displayed inequalities are tight on the edge from  $(2, 0, 0)$  to  $(2, 1, 2)$ , rather than on a facet. In contrast, the full-trace bound  $c \leq 2$  is a facet, as are the three coordinate nonnegativity inequalities and  $\vartheta \geq 0$ . For the homogeneous cone,  $x_\emptyset \geq 0$  vanishes only at the origin and is not a facet.

At finite  $d$ ,  $a = 0$  or  $b = 0$  forces  $c = 0$ , leaving an edge, so these two coordinate inequalities cease to be facets. For even  $d \geq 4$  the seven facets are

$$c \geq 0, \quad c \leq 2, \quad \vartheta \geq 0, \quad M \geq 0, \quad 2 + 2a - b - c \geq 0, \quad da - c \geq 0, \quad db - c \geq 0.$$

Their equality sets have affine dimension two: the bottom and top vertices give the first two;  $(2, 0, 0), (0, 2, 0), (2, 1, 2)$  give the third;  $(2, 0, 0), (4/d, 2/d, 2), (2, 1, 2)$  give the fourth, and reflect for the fifth;  $(0, 0, 0), (0, 2, 0), (2/d, 2/d, 2)$  give the sixth, and reflect for the seventh. At  $d = 2$  the compression inequalities become redundant, since

$$M = \frac{\vartheta + 3(2b - c)}{2}$$

and likewise after interchange. The five facets are then  $c \geq 0$ ,  $c \leq 2$ ,  $\vartheta \geq 0$ ,  $2a - c \geq 0$ , and  $2b - c \geq 0$ . The same vertex triples, with the three distinct top vertices at  $d = 2$ , verify that each is a facet.

## 4 The complementary-block frontier

The product equality family singles out the endpoint. In the interior of the one-copy interval every normalized product operator has a strictly positive margin.

**Proposition 4.1** (Product margin). *For  $-\frac{1}{2} < \alpha \leq 0$ , every nonzero product operator  $R$  of rank at most two on  $r$  sites satisfies*

$$\sum_{S \subseteq [r]} \alpha^{|S|} \phi(R)_S \geq (1 + 2\alpha)(1 + \alpha)^{r-1} > 0.$$

*Equality is attained by  $B_2 \otimes B_1^{\otimes(r-1)}$ .*

*Proof.* The ranks of the local factors multiply. Thus at most one factor has rank two; all other factors have rank one. The one-copy trace bound gives  $0 \leq \tau_i \leq 2$  at the possible rank-two site and  $0 \leq \tau_i \leq 1$  elsewhere. Expansion as in Lemma 2.4 gives  $\sum_S \alpha^{|S|} \phi(R)_S = \prod_i (1 + \alpha \tau_i)$ . Each factor is positive and decreases with  $\tau_i$ , so the minimum is the stated product, realized by the displayed blocks.  $\square$

For a nonempty proper  $J \subset [r]$ , a two-block inequality reads only  $(x_\emptyset, x_J, x_{J^c}, x_{[r]})$ . Define  $G_r$  to be the convex cone of finite sums of pullbacks of all dimension-free valid two-copy linear inequalities along these coordinate maps, where dimension-free valid means valid at every local dimension, equivalently nonnegative on  $P_\infty$ . Rectangular block dimensions give the same dimension-free inequalities, since an operator on two unequal spaces embeds into two equal spaces without changing its purities. Write

$$Q_r(a; x) = \sum_{S \subseteq [r]} (-a)^{|S|} x_S, \quad \beta_r(a) = \sup\{\beta : Q_r(a; \cdot) - \beta x_\emptyset \in G_r\}.$$

Bharti, Gajjala and Haug's polynomials [8, (85), (86)] are

$$p_{2m+1}(a) = 1 - 2 \sum_{\ell=0}^{m-1} \binom{2m+1}{2\ell+1} a^{2\ell+1}, \quad m \geq 1, \quad (9)$$

$$p_{2m}(a) = 1 + 2a^{2m} - \sum_{\substack{1 \leq j < m \\ j \text{ odd}}} \binom{2m}{j} (2a^j + a^{2m-j}) - \mathbf{1}_{m \text{ odd}} \frac{3}{2} \binom{2m}{m} a^m, \quad m \geq 1. \quad (10)$$

**Theorem 4.2** (Exact complementary-block frontier). *For  $r \geq 2$  and  $0 \leq a \leq \frac{1}{2}$ ,*

$$\beta_r(a) = p_r(a).$$

*Each  $p_r$  has a unique zero  $\gamma_r \in (0, \frac{1}{2}]$ . Hence the largest  $a$  in this interval for which  $G_r$  certifies  $Q_r(a; x) \geq 0$  is  $\gamma_r$ , or equivalently  $\alpha_r^{(2\text{block})} = -\gamma_r$ .*

*Proof.* The upper bound is a single vector in  $\mathbb{R}^{2^r}$  at which every element of  $G_r$  is nonnegative and  $Q_r(a; \cdot)$  equals  $p_r(a)$ . Define  $z_\emptyset = 1$ . If  $r$  is odd, set

$$z_{[r]} = 0, \quad z_J = \begin{cases} 2, & |J| \text{ odd}, \\ 0, & |J| \text{ even}, \end{cases} \quad \emptyset \neq J \subsetneq [r]. \quad (11)$$

Each complementary triple is  $(2, 0, 0)$  or its swap, a vertex of  $P_\infty$ . If  $r$  is even, set

$$z_{[r]} = 2, \quad z_J = \begin{cases} 0, & |J| \text{ even}, \\ 2, & |J| \text{ odd and } |J| < r/2, \\ 3/2, & |J| \text{ odd and } |J| = r/2, \\ 1, & |J| \text{ odd and } |J| > r/2, \end{cases} \quad \emptyset \neq J \subsetneq [r]. \quad (12)$$

The complementary triples are  $(0, 0, 2)$ ,  $(2, 1, 2)$ , its swap, or  $(3/2, 3/2, 2)$ , the midpoint of the last two. All lie in  $P_\infty$  by Theorem 3.2. Every element of  $G_r$  is therefore nonnegative at  $z$ . Summing the coordinates by cardinality in (11) gives (9); the odd layers below, at, and above  $r/2$  in (12) give (10). Thus  $Q_r(a; z) = p_r(a)$  and any certified  $\beta$  satisfies  $0 \leq Q_r(a; z) - \beta = p_r(a) - \beta$ . The vector  $z$  need not be a purity vector. For odd  $r$  its complementary triples  $(2, 0, 0)$  and  $(0, 2, 0)$  are realized by  $B_2 \otimes B_0$  and its swap at every  $d$ , so the same bound holds against two-copy inequalities valid at a fixed  $d$ ; for even  $r$  the triple  $(0, 0, 2)$  violates  $c \leq da$ , and the fixed- $d$  complementary-block frontier is not determined here.

For the lower bound we follow the certificate of [8, Theorem 9.1, (89)–(91)] step by step, since what has to be shown is that each of its steps is a pullback of a facet of  $P_\infty$  or a sign constraint, so that the certificate lies inside  $G_r$ ; the derivation and the root argument at the end are theirs. Use  $N = x_\emptyset$ ,  $T = x_{[r]}$ ,  $A = x_J$ ,  $B = x_{J^c}$ . The sum and difference bounds give, for  $0 \leq v \leq w$ ,

$$-wA + vB \geq -2wN + \frac{v}{2}T, \quad (13)$$

$$wA - vB \geq -2vN + \frac{v}{2}T, \quad (14)$$

$$-wA - vB \geq -2wN - \frac{v}{2}T. \quad (15)$$

For (13), write its left side as  $-(w-v)(A+B)/2 - (w+v)(A-B)/2$  and use the two bounds. For (14), use  $(w-v)A + v(A-B)$  and  $A \geq 0$ . For (15), use the negative of  $(w+v)(A+B)/2 + (w-v)(A-B)/2$ . Every slack in these derivations belongs to  $G_r$ .

If  $r = 2m + 1$ , keep the empty–full pair  $N - a^r T$  exact. Represent each other complementary pair by  $1 \leq j = |J| \leq m$ . For odd  $j$ , apply (13) with  $w = a^j$ ,  $v = a^{r-j}$ ; for even  $j$ , apply (14) with these same values. The resulting negative  $N$ -terms run through all odd sizes  $1, 3, \dots, r-2$ , giving  $p_r(a)N$ . The coefficient of  $T$  is nonnegative: singleton pairs alone give the lower bound

$$-a^r + \frac{r}{2}a^{r-1} = a^{r-1}(r/2 - a) \geq 0.$$

Discarding this multiple of  $T \geq 0$  proves  $Q_r - p_r(a)N \in G_r$ .

If  $r = 2m$ , discard the nonnegative pairs of positive even size. For odd  $j < m$ , use (15); if  $m$  is odd, the middle layer has  $\binom{2m}{m}/2$  pairs, each bounded with  $A + B \leq 2N + T/2$ . Together with  $N + a^{2m}T$  this gives the affine bound  $UN + VT$ , where

$$U = 1 - 2 \sum_{\substack{j < m \\ j \text{ odd}}} \binom{2m}{j} a^j - \mathbf{1}_{m \text{ odd}} \binom{2m}{m} a^m,$$

$$V = a^{2m} - \frac{1}{2} \sum_{\substack{j < m \\ j \text{ odd}}} \binom{2m}{j} a^{2m-j} - \mathbf{1}_{m \text{ odd}} \frac{1}{4} \binom{2m}{m} a^m.$$

Here  $V \leq 0$ . For even  $m$ , its  $j = 1$  term already gives  $V \leq a^{2m} - ma^{2m-1} \leq 0$ . For odd  $m$ , the middle term suffices, since  $a^m \leq 2^{-m} \leq \binom{2m}{m}/4$ ; the last inequality follows from  $\binom{2m}{m} = \prod_{k=1}^m (m+k)/k \geq 2^m$ . Using  $T \leq 2N$  now gives  $UN + VT \geq (U + 2V)N = p_{2m}(a)N$ . All discarded terms and the trace-bound slack belong to  $G_r$ , so  $Q_r - p_r(a)N \in G_r$  for even  $r$  as well.

Finally, as in [8, Definition 9.2],  $p_r(0) = 1$  and  $p_2(a) = (1-a)(1-2a)$ . For odd  $r \geq 3$ , every nonconstant coefficient is negative and  $p_r(1/2) \leq 1 - r < 0$ . For even  $r \geq 4$ , the singleton term gives  $p'_r(a) \leq 2ra^{r-1} - 2r < 0$  on  $(0, 1/2]$  and  $p_r(1/2) \leq 1 + 2^{1-r} - r - r2^{1-r} < 0$ . Continuity and strict decrease prove the unique-root assertion.  $\square$

This frontier covers every valid dimension-free inequality on complementary blocks. Retaining a spectator site gives a different library. For example, at  $r = 3$  the sharp two-copy inequality applied to each block  $R_{ab} = (I_{12} \otimes \langle a|)R(I_{12} \otimes |b\rangle)$ , which has rank at most two, gives after summing over  $a, b$

$$4F_\emptyset - 2F_{\{1\}} - 2F_{\{2\}} + F_{\{1,2\}} \geq 0.$$

The four sums of squared block norms are exactly the four displayed purities, since partial trace on sites 1, 2 commutes with taking the site-3 blocks. This valid inequality evaluates to  $-4$  at the vector (11) for  $r = 3$ , so it is outside  $G_3$ . The certified complementary-block interval intersects the NPT interval precisely when  $d > 1/\gamma_r$ .

## 5 Relation to earlier work

The reduction of  $r$ -copy distillability to Schmidt-rank-two witnesses is from [2, 3]; the purity-coordinate form (1) is Theorem 1 of Costa Rico [5] and, independently, is in Qi, Gupur, Wu and Guo [6]. Two-copy undistillability for  $\alpha \geq -\frac{1}{2}$  in every dimension was proved concurrently by Fu, Gao and Park [7], Song and Chen [9], Fraser, Huber, Pozsgay and Vona [10], and Bharti, Gajjala and Haug [8]; earlier partial results are the dimension-free two-copy intervals of [5, 6]. Costa Rico’s Theorem 2, (45)–(47), gives difference bounds and sum bounds, the rank-dependent sum bound there having an additional operator-class hypothesis. BGH’s Section 6 proves the sharp rank-two sum inequality, and their Section 9 combines it with the difference bound. Costa Rico and Wolf [12] prove norm inequalities between a general matrix and its partial traces under rank constraints, with the two-copy Werner problem as an application. Of the seven facets of Theorem 3.2, three are sign constraints,  $c \leq 2$  is the rank bound,  $4 - 2a - 2b + c \geq 0$  is the sharp two-copy inequality of [7, 8, 9, 10], and the two compression inequalities are proved here; the vertex list, the even-dimensional bodies and the odd-dimensional exclusion are proved here. The lower complementary-block certificate is BGH’s Theorem 9.1, and so is the argument that  $p_r$  has a unique root. The dual vectors in (11)–(12) prove its optimality over the full dimension-free complementary-block cone, including the compression inequalities. The block factorization of the endpoint form on tensor-factorized operators, of which Lemma 2.4 is the single-operator case, is [8, Proposition 8.2]. The general theory of maps positive under tensor powers, into which the all-copy endpoint embeds as tensor-stable 2-positivity of a single map [8, Theorem 3.1], is in [11].

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