On Life-Tables—*their* Construction *and* Practical Application.

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Among the matters of widely varying interest as affecting the welfare of mankind, which from time to time are brought before the notice of this Society, those connected with the subject of Vital Statistics may justly be considered as second in importance to none.

It will also, probably, be admitted without contention, that one of the most important parts of the science of Vital Statistics is that which forms the subject of this paper; for a Life-Table may be defined as a scientific instrument designed for accurately measuring the forces of life and death prevailing, whether among a whole nation or in a small town, a mere unit in the great whole, and only by the use of Life-Tables can there be made exact comparisons between nations or between the separate communities which constitute nations, as regards their vitality.

To the Fellows of this Society it is a truism almost too obvious to suggest, that for obtaining accurate results correct methods must be employed, and that for obtaining strictly comparable results similar methods should be used.

Now, the Life-Tables which have hitherto been constructed have not all been worked out by exactly the same methods, and therefore, although the discrepancies may be comparatively small, the fact remains that comparison between them is to some extent invalidated by this consideration.

The hope may be expressed that in this paper some little contribution may be made to the ultimate devising of an "ideal" method of constructing Life-Tables, so that hereafter a similar plan having been followed in each case, the results obtained may be in the fullest possible degree comparable.

But few of the Fellows of the Society immediately addressed, except those connected with the departments of public health or official statistical work, are ever likely to actually engage in the work of practically attempting the construction of a Life-Table.

There are of course distinguished Fellows of this Society who are engaged in the actuarial profession, but I should scarcely like to consider myself as venturing to address these.

Through the medium of the Royal Statistical Society I would desire to address myself chiefly to the body of Medical Officers of Health generally, believing that many of my colleagues will be ready to undertake the work of constructing local Life-Tables for their respective districts if they are aided in overcoming the initial difficulties associated with the task.

The attempt to follow such masters of the science of Vital Statistics as the late Dr. William Farr and Mr. Noel A. Humphreys, happily still surviving, who have, with such consummate ability on past occasions, presented papers to this Society on the same subject, may well give rise to hesitation and diffidence.

I should certainly never have presumed to make such an attempt had I not been honoured by a special request made by Mr. Noel A. Humphreys—such a request has acted as a stimulus to the undertaking of work additional to what had been already done, which would otherwise never have been contemplated.

Acknowledgment should also be made, at the very outset, of the great obligation which I am under to Mr. A. C. Waters, of the General Register Office, whom I must honour as my teacher in this subject, who has freely placed at my disposal his great experience in Life-Table construction, and without whose aid I should never have acquired the foundation of preliminary knowledge upon which the work to be embodied in this paper has been built up.

The plan which I propose to adopt is as follows:—

(1.) First of all to dogmatically describe the particular mode of constructing an extended Life-Table which I have arrived at, partly as the result of much laborious experimental work, and which I recommend for adoption, going very fully into details, so as if possible to anticipate and obviate for others the difficulties which I have met with, and the pitfalls into which I have stumbled. I cannot pretend to possess anything more than a very little knowledge of pure mathematics, and shall make no attempt to go more than slightly into explanation of the underlying reasons for the methods to be used. Those addressed are supposed to have merely a working knowledge of the use of seven-figure logarithms, and some patience in applying the ordinary rules of arithmetic.

(2.) Next, to describe a modification of the "short" method of constructing local Life-Tables first devised by the late Dr. William Farr, and described by him twenty-four years ago in the supplement to the Thirty-fifth Annual Report of the Registrar-General, and to demonstrate, if possible, that by the use of this modified short method a Life-Table for quinquennial age-intervals can be constructed, giving such close approximations to the results to be worked out by the previously described "extended" method, as to make it scarcely necessary for merely local Life-Tables to undertake the more elaborate and laborious method.

(3.) Afterwards to give an account of the experimental work which has led up to the special methods of calculation recommended.

(4.) And lastly, to give an outline of the practical uses of a Life-Table when constructed.

Section I.

The problem of constructing an "extended" Life-Table, that is, one for every single year of life, resolves itself chiefly into calculating the series of fractions which are tabulated in a Life-Table under the heading of the p_x column. These fractions represent the chances (or probabilities) of surviving from one year to the next. Thus, p_x stands for the chance of surviving from the exact age x to the age $x + 1$.

By far the greatest and most difficult part of the labour involved in working out a Life-Table is taken up in working out these p_x values. They are the skeleton or supporting framework on which the whole structure is built up. When once they are obtained, the after labour, although considerable, is a mere matter of easy routine calculation.

Without going at all into the mathematical theory of probability, the following considerations will be obvious:—

(1.) If we know that on 1st January, 1898, a thousand infants have been born, and that by the time new year's day of 1899 dawns, 200 of these have died, then it is clear that the chance of any individual of the original number of infants surviving to the beginning of the second year of life is 800 out of 1,000, or $\frac{800}{1,000} = 0{\cdot}8$, and so, generally, if we know the number of the living at any *exact* age x, indicated by "P_x," and the number of those dying during the following year, indicated by "d_x," then

$$p_x = \frac{P_x - d_x}{P_x} = \frac{\text{number living at end of year}}{\text{number living at beginning of year}}.$$

(2.) In actual practice, however, this simple mode of calculation (except for the first five years of life, as will afterwards be explained) cannot be adopted. The population numbers, as enumerated at each census, and the numbers returned in the death registers, do not give us the numbers of persons *at the beginning* of the several years or age-periods, but the numbers *at all ages* between certain fixed points.

Thus if, according to the census enumeration, 1,000 children are returned as living at age 4—5, this means that they are of any age between the beginning and the end of the fifth year of life. Now if twenty of these children are returned in the death register as dying at age 4—5, the problem of calculating the chance of survival becomes more complicated than the simple case just alluded to. In order to solve it we may assume two things:

(*a.*) That at the middle of the calendar year the average age of these children is $4\frac{1}{2}$ years.

(*b.*) That the number of deaths is evenly distributed during the year, half occurring in the first half of the year, and half in the second half of the year.

Therefore, on these two assumptions, which when large numbers are dealt with may be considered as approximately true for any year of life, *except the first*, the number of survivors at the beginning of the year would be 1,000 + 10, and at the end of the year 1,000 − 10, and the chance of surviving to the beginning of the sixth year of life would be expressed by the fraction $\frac{1,000 - 10}{1,000 + 10} = \frac{9\,90}{1,010} = 0{\cdot}98019$, and if we wished to calculate how many out of 100,000 at age 4 would survive to age 5, it would be done by multiplying 100,000 by 0·98019, the result being 98019.

To put the thing in a general formula, if "P_x" be the number returned at the census as living between any age x and the next age $x + 1$, they must be considered to be at the middle of the calendar year of the average age $x + \frac{1}{2}$, and if "d_x" be the number of deaths for the year at age x to $x + 1$, then the chance of surviving from age x to age $x + 1$ is expressed by the fraction $\frac{P_x - \frac{1}{2}d_x}{P^x + \frac{1}{2}d_x} = \frac{\text{number living at end of year}}{\text{number living at beginning of year}}$.

In the above example we have worked *directly* from the population and deaths. The rate of mortality per unit of the population or, as it is otherwise termed, the "central death-rate," which is expressed by the symbol "m_x" $= \frac{d_x}{P_x}$, and $\frac{1 - \frac{1}{2}m_x}{1 + \frac{1}{2}m^x}$ is only another way of expressing $\frac{P_x - \frac{1}{2}d_x}{P_x + \frac{1}{2}d_x}$.

There is no real need, therefore, as has usually been hitherto done, for working out the "m_x" values.

Data required.

Before the construction of a Life-Table can be proceeded with, the following data, or crude materials, are necessary:—

(1.) The total population numbers as enumerated at two successive censuses, say 1881 and 1891, and also the numbers classified into certain age-groups for each sex.

The age-groups referred to are:—

0— 5	15—25	45—55	75—85
5—10	25—35	55—65	85 and upwards
10—15	35—45	65—75	

In certain cases, as in those districts which contain public institutions, such as hospitals or lunatic asylums, the census numbers may need correction.

(2.) The numbers of deaths registered in the district during the ten calendar years most nearly corresponding to the census interval, as 1881-90, also arranged in similar age- and sex-groups.

The age-group 0—5, however, requires to be still further subdivided into the following groups.

0—1	under 3 months	1—2
	3 months and under 6 months	2—3
	6 „ 1 year	3—4
		4—5

It is of very special importance to secure the greatest possible accuracy in correcting the numbers of deaths, by *excluding* all such deaths as do not properly belong to the district, and also by *including* the deaths of persons properly belonging to the district which have occurred outside it, such as deaths in workhouses, &c. However, until something corresponding to an official "clearing house" is established for ensuring that all deaths are referred to their proper districts, insuperable difficulties will often be met with in securing absolute accuracy.

It must be borne in mind that an error of *one* in the death numbers will have a many times greater effect in vitiating results than an error of *ten* in the population.

(3.) It is also requisite to have returns of some of the deaths for some years preceding the decennium being dealt with, as follows:—

At age	0—1	for the years	1877-80	inclusive
„	1—2	„	'78-80	„
„	2—3	„	'79-80	
„	3—4	for the year	1880	

(4.) The numbers of male and female births in each of the years 1876-90 inclusive are also required.

To find the True Mean Total Population.

The very obvious and simple method of taking the arithmetical mean of the two census numbers is unfortunately not accurate enough, for two reasons—

(*a.*) On the assumption that population increases or decreases by a constant "rate," that is, in geometrical progression, the true mean must necessarily be *less* than the arithmetical mean.

(*b.*) The interval between two censuses is later both at its beginning and its ending by a quarter of a year than the ten calendar years most nearly corresponding.

(For a complete exposition of these points, the Supplement to the Fifty-fifth Annual Report of the Registrar-General, pp. xlii and xliii, may be consulted.)

If the population of a district have been enumerated at a certain number denoted by "P" at the census of 1881, and at an increased number denoted by "P′" at the census of 1891, then the rate of increase per unit "r" (which is assumed to be constant)

$$= \frac{P'}{P}, \text{ or } P' = rP$$

and the true mean population for the ten calendar years 1881-90

$$= \frac{rP - P}{r^{\frac{1}{40}} \times \text{hyp. log. } r}.$$

There is no need however to undertake the labour of working from this formula; for by reference to the last Decennial Report of the Registrar-General, already referred to, at pp. xliv and xlv there will be found a table called "P," which will greatly facilitate the work, and which is the result of an enormous amount of laborious calculation.

It is simply necessary to find the value of "r," and to deduce by means of this table the corresponding "factor of correction," and then to divide the arithmetical mean of the two census numbers by this factor. The result is the required true mean total population.

As it is very desirable that there shall be no error made in this most important initial calculation, it may be well to mention that there is another simple method of calculation which may be used.

If we have the population numbers for the *beginning* of 1881 and the *end* of 1890, then calling the arithmetical mean of these numbers "A," and their geometrical mean "G," a very near approximation to the true mean population for the ten calendar years 1881-90 is to be arrived at by adding together one-third of the arithmetical mean and two-thirds of the geometrical mean, or

$\frac{A + 2G}{3}$ (very nearly) = true mean population.

It will be obvious that—

log. population at beginning of 1881 = log. census number 1881 − $\frac{1}{40}$ log. r,

and

log. population at end of 1890 = log. census number 1891 − $\frac{1}{40}$ log. r.

To take an actual example—

Method I.

Population enumerated at census of 1881 = 111,343
" 1891 = 131,463

then log. r = log. 131,463 − log. 111,343 = 5·1188036 − 5·0466629 = 0·0721407.

Therefore r = 1·180703.

Common log. of r = 0·0721407, and hyp. log. r = 0·1661101

rP − P = 131,463 − 111,343 = 20,120.

Therefore—

log. 20,120 − ($\frac{1}{40}$ log. 1·180703 + log. 0·1661101) = log. true mean population.
= 4·3036280 − (0·0018035 + $\bar{1}$·2203961) = 5·0814284.

Therefore true mean population = 120,622·5.

Method II.

"r" as above found = 1·180703.

By Table "P" in Registrar-General's Decennial Supplement, the "factor of correction" corresponding to this value of r
= 1·006438 + (·006483 − ·006438) × ·703 = 1·006438 + ·000032 = 1·006470
arithmetical mean of rP and P = $\frac{1}{2}$ (111,343 + 131,463) = 121,403.
Therefore log. 121403 − log. 1·006470 = log. true mean population
= 5·0842294 − 0·0028009 = 5·0814285

Therefore true mean population = 120,622·5.

Method III.

log. 111,343 − $\frac{1}{40}$ log. 1·180703 = log. population at *beginning* of 1881;
5·0466629 − 0·0018035 = 5·0448594.

Therefore population at beginning of 1881 = 110,881·6.

Similarly—

log. 131,463 − $\frac{1}{40}$ log. 1·180703 = log. population at *end* of 1890.
5·1188036 − 0·0018035 = 5·1170001.

Therefore population at end of 1890 = 130,918·1.

"A" = $\frac{1}{2}$ (110,881·6 + 130,918·1) = 120,899·9;

log. "G" = $\frac{1}{2}$ (5·0448594 + 5·1170001) = 5·0809298.

Therefore G = 120,484·1,

and $\frac{A + 2G}{3}$ = 120,622·7.

To find the True Mean Population Numbers for the several Age and Sex-Groups.

Having obtained the true mean total population, the next step is to divide this up between the separate age- and sex-groups.

The method to be adopted may be termed the "method of "mean proportions." It is based on the assumption that in the interval between two censuses the proportion of each group to the whole has changed uniformly, that is, supposing in some particular age group the proportion is 10 per cent. at the first census and 20 per cent. at the second census, then the proportion at the *middle* of the ten years' interval would be 15 per cent. But seeing that the middle of the ten calendar years 1881-90 is only four and three-quarter years after the date of the earlier census, we should have to take $\frac{19}{20}$ of the change in five years, corresponding to $\frac{19}{40}$ of the change in ten years.

A convenient way of making the calculations is as follows:—

(*a.*) Calculate from the numbers enumerated at each of the two censuses the proportions per million in each group as existing at each census. (The sum of the numbers corresponding to all the male and female groups will of course be a million.)

(*b.*) For each age-group find the arithmetical mean of the proportionate numbers at the two censuses. This gives the mean proportion at the end of five years after the earlier census.

(*c.*) Take the difference between this mean proportion and the proportion existing at the earlier census; this gives the change of proportion in five years.

(*d.*) If the change in proportion has been an *increasing* one, $\frac{1}{20}$ of the change in five years must be *subtracted* from the mean proportion, and if the change has been a *decreasing* one, it must be *added.*

(*e.*) Having thus found the proportion *per million* for each age- and sex-group, as existing four and three-quarter years after the earlier census, a series of simple sums in the "rule of three" will give the proportions as existing in the true mean *total* number already found.

The sum of the parts should of course exactly equal the whole, if the calculations have been correctly made.

The use of logarithms renders these calculations comparatively easy.

The method may be made clearer by a numerical example:— At census 1881, out of total enumerated population of 111,343, number of males in age-group 0—5 = 7,468.

At census 1891, out of total enumerated population of 131,463, number of males in same age-group = 7,507.

(*a.*) To find the proportions per million at each census—

$$7{,}468 : 111{,}343 :: x : 1{,}000{,}000,$$
$$7{,}507 : 131{,}463 :: x' : 1{,}000{,}000.$$

$$\log. x = \log. 7{,}468 + \log. 1{,}000{,}000 - \log. 111{,}343$$
$$= 3\cdot8732043 + 6\cdot0000000 - 5\cdot0466629 = 4\cdot8265414.$$
$$\text{Therefore } x = 67{,}072\cdot01.$$
$$\log. x' = \log. 7{,}507 + \log. 1{,}000{,}000 - \log. 131{,}463$$
$$= 3\cdot8754664 + 6\cdot0000000 - 5\cdot1188036 = 4\cdot7566628.$$
$$\text{Therefore } x' = 57{,}103\cdot51.$$

(*b.*) Mean proportion per million, *i.e.*, the proportion assumed to be existing at the exact middle of the intercensal period

$$= \tfrac{1}{2}\,(57{,}103{\cdot}51 + 67{,}072{\cdot}01) = 62{,}087{\cdot}76.$$

(*c.*) Change of proportion in five years

$$= 62{,}087{\cdot}76 - 67{,}072{\cdot}01 = -\,4{,}984{\cdot}25.$$

(*d.*) Proportion existing at $4\tfrac{3}{4}$ years from census of 1881

$$= 62{,}087{\cdot}76 + \tfrac{1}{20}4{,}984{\cdot}25 = 62{,}336{\cdot}97.$$

(If the change of proportion had been an *increasing* one, the quantity $\tfrac{1}{20}4{,}984{\cdot}25$ would have had to be *subtracted.*)

(*e.*) To find the proportion of males in age-group 0—5, in the total true mean population of 120,622·5:—

$$62{,}336{\cdot}97 : 1{,}000{,}000 :: x : 120{,}622{\cdot}5.$$

$$\log. x = \log. 62{,}336{\cdot}97 + \log. 120{,}622{\cdot}5 - \log. 1{,}000{,}000$$
$$= 4{\cdot}7947457 + 5{\cdot}0814285 - 6{\cdot}0000000 = 3{\cdot}8761742.$$
$$\text{Therefore } x = 7{,}519{\cdot}2.$$

That is out of the total "years of life," or "lives at risk" for the ten years 1880-91, numbering 1,206,225, the proportion belonging to males, for the age-period 0—5, is 75,192.

The work of calculating the p_x values may now be proceeded with. A preliminary observation may be made that as the population and death numbers for ten years are being dealt with, either ten times the mean annual population numbers must be used (the "total lives at risk") with all the deaths for ten years, or one-tenth of the deaths with the mean annual population numbers. The result will, in either case, of course be the same. Perhaps the *former* course is best to adopt.

Calculation of the p_x Values for the First Five Years of Life.

Although at the census enumerations the numbers are given of those living for each of the years from 0—1 to 4—5, these numbers are found to be *altogether unreliable.* There is so much misstatement of age that more are returned as surviving at some of the later ages than at preceding ages, and more than could possibly be surviving, as shown by a direct calculation from the births and deaths for the separate years, which is absurd. These census numbers, therefore, have to be discarded—we can only use them to determine *the total mean population for the age-period* 0—5.

Having this number for males and for females, as already calculated, to work from, the numbers living at the separate ages from 0—1 to 4—5, have to be determined by processes of calculation based on the numbers of births and of deaths under 5 years of age as previously set forth in the list of data required.

The following is an explanation of the method to be employed —males and females being of course dealt with separately:—

The deaths under 1 year of age in the ten years 1881-90 must necessarily occur out of the whole number born in the nine years 1881-89, and out of *part* of those born in the year 1880, and *part* of those born in the year 1890. The deaths under 1 year of age in the ten years 1881-90 may therefore be fairly taken as occurring out of $\frac{1}{2}$ births in 1880 + all births in 1881-89 + $\frac{1}{2}$ births in 1890.

In the same way the deaths under 1 year in the ten years 1880-89 may be taken as occurring out of $\frac{1}{2}$ births in 1879 + all births in 1880-88 + $\frac{1}{2}$ births in 1889; and subtracting these deaths from the total number of births out of which they occurred, will give the number of children aged 1 year out of whom the deaths occurred in 1881-90 of children between 1 and 2 years of age.

The following is the complete scheme for the calculation:—

(*a.*) For the number *at birth* in the ten years 1881-90, take

$\frac{1}{2}$ births in 1880 + all births in 1881-89 + $\frac{1}{2}$ births in 1890.

(*b.*) For the number *at* 1 *year of age* in the ten years 1881-90, take

$\frac{1}{2}$ births in 1879 + all births in 1880-88 + $\frac{1}{2}$ births in 1889,
Less the deaths under 1 year in the ten years 1880-89.

(*c.*) For the number *at* 2 *years of age* in the ten years 1881-90, take

$\frac{1}{2}$ births in 1878 + all births in 1879-87 + $\frac{1}{2}$ births in 1888,
Less the deaths under 1 year in the ten years 1879-88,
And „ at age 1—2 „ '80-89.

(*d.*) For the number *at* 3 *years of age* in the ten years 1881-90, take

$\frac{1}{2}$ births in 1877 + all births in 1878-86 + $\frac{1}{2}$ births in 1887,
Less deaths under 1 year in 1878-87,
And „ at age 1—2 „ '79-88,
„ „ „ 2—3 „ '80-89.

(*e.*) For the number *at* 4 *years of age* in the ten years 1881-90, take

$\frac{1}{2}$ births in 1876 + all births in 1877-85 + $\frac{1}{2}$ births in 1886,
Less deaths under 1 year in 1877-86,
And deaths at age 1—2 „ '78-87,
„ „ 2—3 „ '79-88,
„ „ 3—4 „ '88-89.

We shall thus obtain a series of five numbers, *a*, *b*, *c*, *d*, *e*—calling their sum "N," then $a + b + c + d + e = N$.

It must be very carefully noted, however, that these numbers give NOT the population numbers *at all ages*, from birth to age 1, from age 1 to age 2, &c., but the numbers *actually starting* at birth, at 1 year of age, at 2 years of age, &c.

Now the total number living at the age-period 0—5, which we have already calculated from the census enumerations (which we may call "C"), represents the number living *at all ages* from birth to age 5, and corresponds to the total N *after half a year's mortality*, as well as *altered by migration.*

In order to make N and C comparable—

(1.) Either N must be *brought on* to the middle of the years by *subtracting* half the mortality for the several years.

(2.) Or C must be *carried back* half a year by restoring the numbers of those who have died in the first half of the years of life.

The latter method may be considered as best to adopt, as being the most direct, and as enabling us to calculate the p_x values by the more simple formula $\frac{P-d}{P}$.

At age 0—1 more than half the mortality occurs in the first half of the year of life. At each of the other ages it may fairly be assumed that the mortality is evenly distributed, that is, half in the first six months and half in the second six months of the years of age.

Therefore by adding to C the deaths under 6 months of age during the years 1881-90, and half the deaths during these years at ages 1—2, 2—3, 3—4, and 4—5, we shall obtain a corrected total, which may be called "T."

$$C + \text{deaths under 6 months of age in 1881-90} + \frac{\text{deaths at ages 1—2, 2—3, 3—4, 4—5, in 1881-90}}{2} = T.$$

The difference which still remains between T and N will represent chiefly the alteration due to migration. In order to eliminate this difference, the total T must be divided up in the same proportions as *a*, *b*, *c*, *d*, *e* bear to N. We shall thus finally obtain a series of five numbers, which may be called P_0, P_1, P_2, P_3, P_4.

$$\text{Thus } a : N :: P_0 : T, \text{ \&c., \&c.}$$

Having among the data the number of deaths at age 0—1 during the ten years 1881-90, "d_0," and the corresponding numbers at ages 1—2, 2—3, 3—4, and 4—5, d_1, d_2, d_3, and d_4, the p_x values may now be readily found, thus:—

$$p_0 = \frac{P_0 - d_0}{P_0}$$

$$p_1 = \frac{P_1 - d_1}{P_1} \text{ \&c., \&c.}$$

This method of calculation is of course open to the objection that the alterations due to migration may not be exactly proportionate to the numbers living at each of the first five years of life, but, at least, it gives a nearer approximation to the truth than the obviously erroneous numbers of the census enumerations. Considerations of space have prevented me from seeking to make these last described processes of calculation clearer (as I had proposed to do) by showing the working out of actual examples.

Up to this point I had already worked out a Life-Table for my own sanitary district from data which I had myself compiled. I was proposing to continue using these data for the work necessary for this paper. I found, however, that there were so many anomalies and irregularities due to the smallness of the population dealt with, that I have had to use some other figures. I have therefore taken the liberty, for which I apologise to Dr. Tatham, of using the data on which the Manchester City (males) Life-Table was founded.

Calculation of the p_x Values for the Years of Life after the First Five.

The data available from this point are the numbers of population and of deaths grouped according to age-periods.

It is obvious that a mean value of p_x for the age-period 5—10 may readily be arrived at from the population and death numbers for this age-period, thus—

$$\frac{\text{Population} - \frac{1}{2}\text{ deaths}}{\text{Population} + \frac{1}{2}\text{ deaths}} = p_{5-10},$$

that is, the mean chance of living one year in the interval from age 5 to age 10, is denoted by p_{5-10}.

Seeing that the mortality during this age-period is *decreasing*, the p_x values would be increasing year by year, if we could get at the *exact* facts, and the mean of the separate yearly values of p_x for the age-period would be *greater* than the mean value deduced from the total figures.

At some point during the age-period 10—15 the rate of mortality reaches its *lowest* point, and therefore the p_x value reaches its *highest* point.

The mean of the separate yearly values of p_x for this age-period, assuming that we could get them by an exact knowledge of the facts relating to each year, would probably not greatly differ from the mean as deduced from the total numbers for this age-period by the fraction $\frac{P - \frac{1}{2}d}{P + \frac{1}{2}d}$.

When we reach the age-period 15—25, the mortality has begun to *increase*, and the yearly values of p_x therefore to decrease, and the mean of the separate yearly values of p_x for this age-period would be *less* than the mean value of p_x calculated by the fraction $\frac{P - \frac{1}{2}d}{P + \frac{1}{2}d}$ for the whole age-period.

For the remaining age-periods, during which the mortality increases more and more, the differences of the true mean values of p_x from the means simply calculated from the total population and death numbers would become more and more marked in the direction of excess of the latter over the former.

In order therefore to obtain a series of p_x values approximating to those which we *infer* to exist, if we could get at the exact facts, an elaborate process of calculation must be adopted, known in mathematical terminology as "interpolation by the method of "finite differences."

We have to so divide up the total numbers of population and deaths for each age-period into subdivisions *belonging to each separate year*, that we shall obtain a smooth and symmetrical p_x curve, without any sudden jumps or breaks in its course.

It is by no means certain, however, that the *true* curve, if we could get it by an absolutely exact knowledge of the ages of the living and of the dying for each of the years of life, would be quite the same as that mathematically deduced, for there are certain ages or age-periods more "critical" to life than others.

However, we have the broad and general facts that the mortality decreases from birth up to some point between age 10 and age 15, and that after age 15 it increases, at first slowly, and afterwards more and more rapidly, and that the changes take place by smooth gradations from one year to another, and therefore the curve mathematically deduced does probably approach, to a sufficiently near degree, the hypothetical true curve.

It must also be clearly understood that by the process of interpolation to be presently described, *the foundation facts are not at all altered.*

The force of mortality at each age-period, equivalent to so many deaths out of so many living, is simply divided up and distributed, so as to make the series of p_x values of one age-period begin and end with appropriate relation to the preceding and following series.

The process of interpolation, as applied to any given age-period, is simply drawing a curve between two fixed points with relation to other fixed points on either side of the two being immediately dealt with.

Those who may desire to go thoroughly into the mathematical theory underlying methods of interpolation, cannot do better than consult "The Institute of Actuaries' Text Book," Part II, by Mr. George King, F.I.A., &c.

Up to this point (*i.e.*, as far as the calculation of p_4) the work of the Manchester City Life-Table was done exactly as has been here described.

In dealing with the data hereafter a different method will be adopted than that which was used in the construction of this Life-Table.

In the Manchester City Life-Table (males), the following data are given as part of the foundation numbers:—

At Age	Estimated Mean Population for the Ten Years 1881-90 (*i.e.*, Ten Times the Mean Annual Numbers).	Deaths in the Ten Years 1881-90.
4— 5	66,018*	1,129
5—10	314,343	2,396
10—15	290,034	1,075
15—25	522,994	3,227
25—35	444,524	4,901
35—45	333,934	6,528
45—55	220,426	6,865
55—65	124,294	6,762
65—75	52,964	5,437
75—85	11,842	2,158
85 and upwards	827	254

* The number given in the table on p. 17 of the "Introduction to the "Manchester City Life-Tables" is 66,582 *at age* 4. From the explanations previously given in this paper, it will be obvious that the number at age 4—5 will be found by *subtracting half the deaths* at age 4—5, and 66,582 − ½1,129 = 66,017·5.

From these columns the two others next following can be readily constructed by successive additions, beginning from below, representing (Population − ½ deaths) and (Population + ½ deaths) *at each age and upwards*.[1] These numbers must be then translated into their corresponding logarithms.

At each Age and upwards.	Population, − ½ Deaths.	Population, + ½ Deaths.	Corresponding Logs.	
			P − ½ Deaths.	P + ½ Deaths.
4	2,360,833·5	2,401,565·5	u_4 = 6·3730654	6·3804945
5	2,295,380·5	2,334,983·5	u_5 = 6·3608547	6·3682838
10	1,982,235·5	2,019,442·5	u_{10} = 6·2971553	6·3052315
15	1,692,739·0	1,728,871·0	u_{15} = 6·2285900	6·2377626
25	1,172,358·5	1,205,263·5	u_{25} = 6·0690604	6·0810820
35	730,285·0	758,289·0	u_{35} = 5·8634924	5·8798347
45	399,615·0	421,091·0	u_{45} = 5·6016418	5·6243759
55	182,621·5	197,232·5	u_{55} = 5·2615519	5·2949785
65	61,708·5	69,557·5	u_{65} = 4·7903451	4·8423439
75	11,463·0	13,875·0	u_{75} = 4·0592983	4·1422330
85	700·0	954·0	u_{85} = 2·8450980	2·9795484

The symbol u_x, it must be noted, means *at age x and upwards*.

[1] It is more convenient, as avoiding fractions of death, to take 2P − *d* and 2P + *d*.

The reasons for working with the logarithms of the numbers, and thus obtaining a *modified geometrical progression* in the interpolated quantities, instead of the *modified arithmetical progression* which would be obtained by working out the *numbers* themselves, are lucidly set forth in the "Introduction to the Manchester Life-"Tables." The chief advantage is that by the use of logarithms there is possible a *rational* continuation onwards of the series below the point at which the data terminate. These data terminate at age 85, for the recorded figures at age 95 and upwards are too unreliable to work with, and often lead to the irrational result of an *increasing* instead of a *decreasing* series of p_x values.

In working out interpolations in a series of logarithms they are dealt with as if they were numbers, and *afterwards* translated into common numbers. It is now obvious that if we can interpolate the logarithms corresponding to u_6 in both of these series, we shall, by translating them into the corresponding numbers, be easily able to calculate the value of p_5, for u_5 representing the number at age 5 and upwards, and u_6 the number at age 6 and upwards, then $u_5 - u_6$ will give the number from age 5 to age 6, and this being obtained for both the series $P - \frac{1}{2}d$ and $P + \frac{1}{2}d$, the numerator and denominator of the fraction $\frac{P - \frac{1}{2}d}{P + \frac{1}{2}d}$ are at once provided for the year 5—6.

Similarly, when u_7 is found, $u_6 - u_7$ will give the values for the year 6—7, and the means of calculating the value of p_6.

In most of the previously constructed extended Life-Tables these interpolations have only been made at certain points in the population and death numbers. Thus, u_6 having been found for population and for deaths, p_5 has been worked out; u_{16} having been found, p_{15} has been arrived at; and so having obtained a series of p_x values, p_5, p_{15}, p_{25}, &c., the p_x values for the intermediate years have been interpolated from *these*, and not *directly* from the population and deaths for each year. The chief reason for this has probably been that a smooth and symmetrical curve of p_x values has been obtainable with only three or four orders of differences; but it cannot be said that the series of p_x values thus obtained for any given age period, say p_{15} to p_{24}, represents the "force of mortality" corresponding to the total population and death numbers for the age period 15—25, in as exact a degree as if the p_x values for *each year* are worked out *directly* from interpolation in the population and death numbers. It is therefore best to adopt the latter method.

Of course the two u_x columns can be constructed by taking population and deaths *separately* instead of combining them, as in the above-given example.

More may be said on this point afterwards, but I think that the *combined* method is much to be preferred.

In order to get a good p_x curve from the population and deaths *directly*, it is necessary to take at least five orders of differences.

It would, of course, be *possible* to work out *one* scheme of ten orders of differences running through the series of eleven terms of u_x, from u_4 to u_{85} inclusive, but this would involve a very enormous amount of laborious calculation.

For five orders of differences the u_x terms have to be dealt with in series of six each.

The scheme which I have adopted, on the principle of the "survival of the fittest," is to be thus represented:—

1st Series.	2nd Series.	3rd Series.	4th Series.	5th Series.
u_4				
u_5	u_5			
u_{10}				
u_{15}	u_{15}	u_{15}		
u_{25}	u_{25}	u_{25}	u_{25}	
u_{35}	u_{35}	u_{35}	u_{35}	u_{35}
	u_{45}	u_{45}	u_{45}	u_{45}
	u_{55}	u_{55}	u_{55}	u_{55}
		u_{65}	u_{65}	u_{65}
			u_{75}	u_{75}
				u_{85}

If the series were placed *end to end*, there would be breaks or irregular transitions in the symmetry of the curve obtained. These are to be avoided.

(*a.*) By only using the *central* part of each series. In the cases of the first and fifth series this principle cannot obviously be carried out.

(*b.*) By "welding" or combining the adjoining series by a method to be afterwards described. The positions at which the welding is to be effected are indicated by $=$.

The values of u_6 to u_{14} are obtained from the 1st series, u_{16} to u_{24} by combining series 1 and 2, u_{26} to u_{34} by combining series 2 and 3, u_{36} to u_{44} by combining series 3 and 4, u_{46} to u_{54} from series 4 *alone*, u_{56} to u_{64} by combining series 4 and 5, u_{66} and onwards from series 5 alone.

The first series u_4, u_5, u_{10}, u_{15}, u_{25}, u_{35}, is the most difficult to deal with, as the intervals between the given terms are *unequal.*

The problem of interpolating a series of values with n orders of differences is to be solved—

(*a.*) By obtaining $n + 1$ consecutive terms of the series, or

(*b.*) By obtaining a line of the n differences.

The latter method requires less voluminous calculations than the former, *when once the requisite formulæ have been worked out.*

The former method will first be described for the series now being considered.

The description of the latter method, as applied to this series, will be more easily comprehended after it has been explained in relation to the more simple instance of Series 2.

We have given u_4 and u_5, and we need to work out u_6, u_7, u_8, and u_9 to have the required six consecutive terms.

To solve the problem, special formulæ have to be worked out from the general formula (or theorem) of Lagrange, by which, having given any $n+1$ terms of a series with n orders of differences, or in other words, any $n+1$ points in a curve of the nth degree, any other term in the series can be expressed in terms of the data, that is to say, any other point in the curve can be located with reference to the given fixed points.

I have worked out these formulæ as follows:—

$$u_6 = \frac{-1102u_4 + 3156{\cdot}5688u_5 + 420{\cdot}87584u_{10} - 95{\cdot}6536u_{15} + 7{\cdot}9112u_{25} - 0{\cdot}70224u_{35}}{2387}.$$

$$u_7 = \frac{-1344u_4 + 2887{\cdot}3152u_5 + 1026{\cdot}60096u_{10} - 196{\cdot}8624u_{15} + 15{\cdot}2768u_{25} - 1{\cdot}33056u_{35}}{2387}.$$

$$u_8 = \frac{-1071u_4 + 2045{\cdot}1816u_5 + 1636{\cdot}14528u_{10} - 239{\cdot}0472u_{15} + 17{\cdot}1864u_{25} - 1{\cdot}46608u_{35}}{2387}.$$

$$u_9 = \frac{-1664u_4 + 2978{\cdot}976u_5 + 6335{\cdot}1488u_{10} - 541{\cdot}632u_{15} + 35{\cdot}464u_{25} - 2{\cdot}9568u_{35}}{7161}.$$

The last of this series cannot be expressed with the denominator 2387, as 1664 is not divisible by 3. The correctness of these formulæ is to be checked by finding that the algebraical sum of all the coefficients $=1$ in each case.

The somewhat appalling series of calculations in applying these formulæ to the logarithms of u_4, u_5, &c., must be undertaken by direct multiplication and division. The work is facilitated by first setting down and carefully checking all the multiples of each of the u_x values from 2 to 9 inclusive, and also the multiples of the denominators. Then it is simply a method of addition, subtraction and division.

It is requisite to work out the results to *at least* five extra places of decimals after the seven of the logs. I have usually worked with six extra places.

Having applied the formulæ to the u_x values in the $P + \frac{1}{2}d$ column, the following series are obtained:—

	δ^1.	δ^2.	δ^3.	δ^4.	δ^5.
$\iota_4 = 6{\cdot}3804945$	−122107	−1217:930265	− 73:681560	−16:846744	+1:214391
$\iota_5 = 6{\cdot}3682838$	−123324:930265	−1291:611825	− 90:528304	−15:632353	
$\iota_6 = 6{\cdot}3559513$:069735	−124616:542090	−1382:140129	−106:160657		
$\iota_7 = 6{\cdot}3434896$:527645	−125998:682219	−1488:300786			
$\iota_8 = 6{\cdot}3308897$:845426	−127486:983005				
$\iota_9 = 6{\cdot}3181410$:862421					

The marks : are conveniently used to denote the end of seven places of decimals, and avoid the necessity of putting down a good many cyphers.

To the right hand of the series of u_x values their successive differences are placed. The operation of "differencing" consists in changing the sign of the *upper* of the two quantities differenced, and then taking their algebraical sum, that is, their *sum* if the signs are *like*, and their *difference* if the signs are *unlike*.

The first series of differences is denoted by the symbol δ^1 or δ. The same operation is then repeated on the first order of differences, and thus the second order of differences is obtained, denoted by δ^2, and so on until the last difference δ^5 is obtained.

Now it is obvious that if the calculations have been correctly performed, these differences can be carried down through the whole series of u_x values as far as u_{35}, and at each of the positions u_{10}, u_{15}, u_{25}, u_{35}, the values worked out should exactly coincide with the data worked from, at least as far as the seven decimal places of the logs., or in other words, the curve must pass exactly through the given fixed points.

In carrying the interpolating process *downwards* the differences must be successively *added;* in carrying it upwards the differences would have to be *subtracted;* that is, their signs must be changed and the sum or difference of the two quantities taken according as the signs are then like or unlike.

In the series now being dealt with the proof of correctness is very quickly reached, as the next term in the series is u_{10}.

The operation is thus proceeded with, it being more convenient to work from left to right.

$\delta^5 + \delta^4$.	$\delta^4 + \delta^3$.	$\delta^3 + \delta^2$.	$\delta^2 + \delta$.	$u_x + \delta$.
−15:632353	−106:160657	−1488:300786	−127486:983005	6·3181410:862421 = u_9
+ 1:214391	− 14:417962	− 120:578619	− 1608:879405	− 129095:862410
−14:417962	−120:578619	−1608:879405	−129095:862410	6·3052315:000011 = u_{10}
+ 1:214391	− 13:203571	− 133:782190	− 1742:661595	− 130838:524005
−13:293571	−133:782190	−1742:661595	−130838:524005	6·2921476:476006 = u_{11}
Carrying these differences downwards, the following results are successively arrived at				6·2377626:001147 = u_{15} 6·0810820:049109 = u_{25}

Usually with six extra places of decimals, the u_{10} value should only differ in the last place of decimals. If an error should be found in the third or fourth place of extra decimals at u_{10}, the work must be revised, as the cumulative effect of small errors is enormous.

Having effected the interpolation as far as u_{15} in each of the series P — ½*d* and P + ½*d*, the resulting logs. are to be translated into their corresponding numbers and differenced. Thus—

P − ½*d*.	Year.	$u_x - u_{x+1}$.		P + ½*d*.	Year	$u_x - u_{x+1}$.	
u_5 = 2,295,380·5	5 —	64,555·9		u_5 = 2,334,983·5	5 —	65,373·0	
u_6 = 2,230,824·6	6 —	63,611·6		u_6 = 2,269,610·5	6 —	64,199·0	
u_7 = 2,167,213·0	7 —	62,639·1		u_7 = 2,205,411·5	7 —	63,064·6	
u_8 = 2,104,573·9	8 —	61,657·8		u_8 = 2,142,346·9	8 —	61,974·5	
u_9 = 2,042,916·1	9 —	60,680·6	Sum = 602,641·5 = $u_5 - u_{15}$.	u_9 = 2,080,372·4	9 —	60,929·9	Sum = 606,112·5 = $u_5 - u_{15}$.
u_{10} = 1,982,235·5	10 —	59,717·5		u_{10} = 2,019,442·5	10 —	59,932·1	
u_{11} = 1,922,518·0	11 —	58,778·5		u_{11} = 1,959,510·4	11 —	58,979·1	
u_{12} = 1,863,739·5	12 —	57,867·8		u_{12} = 1,900,531·3	12 —	58,072·4	
u_{13} = 1,805,871·7	13 —	56,988·9		u_{13} = 1,842,458·9	13 —	57,207·2	
u_{14} = 1,748,882·8	14 —	56,143·8		u_{14} = 1,785,251·7	14 —	56,380·7	
u_{15} = 1,692,739·0	....	....		u_{15} = 1,673,281·2	....	....	

The data are now provided for calculating the values of p_x from p_5 to p_{14}, thus—

$$p_5 = \frac{64{,}555{\cdot}9}{65{,}373} = \begin{array}{l}\text{Logs.}\\ 4{\cdot}8099359\\ -\,4{\cdot}8153984\end{array} = \bar{1}{\cdot}9945375 = 0{\cdot}98750.$$

$$p_6 = \frac{63{,}611{\cdot}6}{64{,}199} = \begin{array}{l}4{\cdot}8035363\\ -\,4{\cdot}8075283\end{array} = \bar{1}{\cdot}9960080 = 0{\cdot}99085.$$

and so on to the end of the series.

The first series of interpolations having now been effected, the remaining four are more easily accomplished, as the terms given are *equidistant*.

The formula of Lagrange might still be employed to obtain the required consecutive number of terms, and the special formulæ would be applicable to each series.

However, there is no need to undertake the laborious work of this method, as there is a much easier and shorter way, which may be described as follows :—

Let the six terms of the series be first set down and differenced, thus—

P − ½*d*.		Δ.	Δ_2.	Δ_3.	Δ_4.	Δ_5.
u_5	= 6·3608547	−1,322,647 :	−272,649 :	−187,735 :	+ 85,293 :	−202,418 :
u_{15}	= 6·2285900	−1,595,296 :	−460,384 :	−102,442 :	−117,125 :	
u_{25}	= 6·0690604	−2,055,680 :	−562,826 :	−219,567 :		
u_{35}	= 5·8634924	−2,618,506 :	−782,393 :			
u_{45}	= 5·6016418	−3,400,899 :				
u_{55}	= 5·2615519					

The problem is to subdivide these differences for 10-yearly intervals, represented by the symbol "Δ," into smaller differences "δ," corresponding to one year, or the tenth part of the interval.

The key to the solution of this problem is the formula $\delta^n = (1 + \Delta)^n$—

The working out of the formula to five orders of differences gives the following results:—

$$\delta^5 = \cdot 00001\Delta^5.$$
$$\delta^4 = \cdot 0001\Delta^4 - 18\delta^5.$$
$$\delta^3 = \cdot 001\Delta^3 - 13\cdot 5\delta^4 - 96\cdot 75\delta^5.$$
$$\delta^2 = \cdot 01\Delta^2 - 9\delta^3 - 44\cdot 25\delta^4 - 150\delta^5.$$
$$\delta = \cdot 1\Delta - 4\cdot 5\delta^2 - 12\delta^3 - 21\delta^4 - 25\cdot 2\delta^5.$$

Now this formula must be applied to a *line of* Δ^n *values.* In this case we wish to begin at u_{15}, and it is only necessary to fill in the constant Δ^5 value in the blank space opposite u_{15}. If we wish to apply the formula to the line opposite u_{25}, the proper Δ^4 value would be found by adding together − 117,125: and − 202,418:, the sum being − 319,543: and so on with the other lines.

It is only in the fifth series that this point comes in; in all the others the interpolation has to be commenced at the second line in the series.

A very important point to regard in using these formulæ is to *take care of the signs.* A negative quantity multiplied by a negative coefficient gives a + value.

Having worked out the formula to the special case now being dealt with, we get this result:—

	δ.	δ².	δ³.	δ⁴.	δ⁵. *
$u_{15} = 6\cdot 2285900$	−142574:141989	−3230:958070	−240:359575	+24:72274	−2:02418

* These values should be more correctly marked as δu_{15}, $\delta^2 u_{15}$, &c., as showing that they all belong to the line of differences opposite u_{15}.

In order to save the possible waste of labour entailed by working through a 10-yearly series of interpolations, before the values of δ, δ², &c., are proved to be correct, it is best to first verify them by the following checking equation:—

$$u_{10}\ (i.e., u_{25}) = u_0\ (i.e., u_{15}) + 10\delta u_0 + 45\delta^2 u_0 + 120\delta^3 u_0 + 210\delta^4 u_0 + 252\delta^5 u_0.$$

In proceeding with the interpolation δ has first to be added to u_{15}, which will give u_{16}, and then $\delta + \delta^2$ added to u_{16}, &c., thus:—

				6·2285900:000,000 $=u_{15}$ − 142574:141989
			− 142574:141989 − 3230:958070	6·2143325:858001 $=u_{16}$ − 145805:100059
		− 3230:958070 − 240:359575	− 145805:100059 − 3471:317645	6·1997520:757952 $=u_{17}$ − 149276:417704
	− 240:359575 + 24:722740	− 3471:317645 − 215:636835	− 149276:417704 − 3686:954480	6·1848244:340248 $=u_{18}$ − 152963:372184
+ 24:72274 − 2:02418	− 215:636835 + 22:968560	− 3686:954480 − 192:938275	− 152963:372184 − 3879:892755	6·1695280:913064 $=u_{19}$ − 156843:264939
+ 22:69856	− 192:938275	− 3879:892755	− 156843:264939	6·1538437:703125 $=u_{20}$
and so on; the interpolation at last reaching				6·0690604:000000 $=u_{25}$

If it had been desired to work *upwards*, the process would have been commenced as follows:—

+ 24:72274 + 2:02418	− 240:359575 − 26:746920	− 3230:958070 + 267:106495	− 142574:141989 + 2963:851575	6·2285900:000000 $=u_{15}$ + 139610:290414
+ 26:74692	− 267:106495	− 2963:851575	− 139610:290414	6·2425510:290414 $=u_{14}$

The saving of labour effected by working out an interpolation by *first* obtaining the "leading differences," instead of arriving at these by means of a consecutive number of terms of the series, is so considerable, that although the work described in this paper was done by means of the formulæ already given for Series 1, for the sake of lightening the labour for others, I have, since the manuscript of the paper was written, worked out the following formulæ which have been verified as correct (in the notation u_0 means u_4, the first term of the series):—

$$\delta^5 u_0 = -\frac{20}{7161}u_4 + 0{\cdot}004u_5 + \left(\frac{124(-0{\cdot}1232u_{10} + 0{\cdot}063u_{15} - 0{\cdot}011u_{25}) + 0{\cdot}1848u_{35}}{7161}\right)$$

$$\delta^4 u_0 = +\frac{4}{231}u_4 - 0{\cdot}024u_5 + \left(\frac{2{\cdot}464u_{10} - 1{\cdot}008u_{15} + 0{\cdot}088u_{25}}{231}\right) - 5{\cdot}8\delta^5 u_0$$

$$\delta^3 u_0 = -\frac{1}{11}u_4 + 0{\cdot}12u_5 - 0{\cdot}04u_{10} + \frac{0{\cdot}12u_{15}}{11} - 3{\cdot}\delta^4 u_0 - 4{\cdot}8\delta^5 u_0$$

$$\delta^2 u_0 = +\frac{4}{11}u_4 - 0{\cdot}44u_5 + 0{\cdot}08u_{10} - \frac{0{\cdot}04u_{15}}{11} - \delta^3 u_0 + 1{\cdot}2\delta^5 u_0$$

$$\delta u_0 = +u_5 - u_4$$

In using these formulæ the labour may be diminished by the following simple device: Let u_0 (*i.e.*, u_4) be subtracted from *all* the terms of the series. Then all the terms in the formulæ containing u_4 will vanish by being reduced to zero. The *new* values must be applied then in the calculations. Thus, u_5 will be u_5-u_4, u_{10} will be $u_{10}-u_4$, &c., &c. The differences thus obtained will be *exactly the same*, as if the formulæ had been applied to the original terms. Thus:—

Original Terms.	Differences.	New Terms.	Differences.
u_4	u_5-u_4	u_4-u_4	u_5-u_4
u_5	$u_{10}-u_5$	u_5-u_4	$u_{10}-u_5$
u_{10}	....	$u_{10}-u_4$	

Having obtained the values of $\delta^5 u_0$, $\delta^4 u_0$, &c., let the original u_4 be placed as u_0. The interpolation may then be proceeded with exactly as already described for Series 2.

The values found may then be verified as correct by the following "checking equation":—

$$u_6(i.e., u_{10}) = u_0(i.e., u_4) + 6\delta u_0 + 15\delta^2 u_0 + 20\delta^3 u_0 + 15\delta^4 u_0 + 6\delta^5 u_0.$$

Process of "Welding," or combining two Series.

It remains now to explain the process by which the u_x values of two adjoining series are so blended together as to make the one series pass into the other by even gradations, instead of having the irregularity in the p_x curve, which must be obtained by simply placing different series end to end.

This process has been devised by Mr. A. C. Waters, and to him I am indebted for the knowledge of it.

The values of the blending proportions have been worked out from trigonometrical tables.

It will be noted that the sum of each of the pairs of multipliers = 1.

The process is shown below as applied to the series u_{16} to u_{24}.

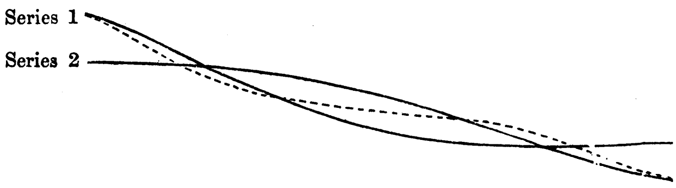
P—½d.	From Series 1.		From Series 2.		Combined Values.
u_{16} =	6·2141570 × 0·976	+	6·2143326 × 0·024	=	6·2141612
u_{17} =	6·9994423 × 0·904	+	6·1997521 × 0·096	=	6·1994720
u_{18} =	6·1844257 × 0·794	+	6·1848244 × 0·206	=	6·1845078
u_{19} =	6·1690863 × 0·654	+	6·1695281 × 0·346	=	6·1692392
u_{20} =	6·1534024 × 0·500	+	6·1538438 × 0·500	=	6·1536231
u_{21} =	6·1373525 × 0·346	+	6·1377542 × 0·654	=	6·1376152
u_{22} =	6·1209146 × 0·206	+	6·1212441 × 0·794	=	6·1211762
u_{23} =	6·1040672 × 0·096	+	6·1042997 × 0·904	=	6·1042774
u_{24} =	6·0867893 × 0·024	+	6·0869088 × 0·976	=	6·0869060

This process is facilitated by deducting the largest possible *common value* from the pairs of u_x values before multiplying, and then adding the two products to this common value. Thus—

$$u_{16} = 6·2140000 + 1570: \times 0·976 + 3326: \times 0·024 = 6·2141612.$$

The following diagram shows the curve of p_x from age 15 to age 24 as worked out (*a*) for each of the two series separately, and (*b*) from the blended values.

The dotted line shows the blended curve.



The diagram has only been roughly drawn from an original large one plotted out to scale.

For the sake of clearness the curves have been a little exaggerated.

Sufficient explanation and illustration, it may be hoped, have now been given to clearly demonstrate the whole of the processes of calculation necessary to obtain the p_x values right on to the end.

The values from age 85 onwards are to be obtained by continuing downwards the interpolations in the series ending with u_{85}. To check the accuracy of the results, the value of u_{95} can be previously obtained from the Δ^n values of the series by carrying them down one stage. The interpolation may have to be continued at least as far as u_{100}. If it is found when the l_x column comes to be worked out that it is necessary to proceed further, this can easily be done.

In working out the successive p_x values, a very important practical point is to *difference* them as you go along, and also to plot them out to scale on paper ruled into squares. If irregularities in the differences, or breaks in the symmetry of the curve are met with, *errors in calculation have been made.*

It may also be remarked that it is necessary to use *all* the seven figures to be obtained by translating the logs. into numbers, right on to the end, to get the smooth gradations and regular differences to be obtained in the p_x series by the use of the method of interpolation recommended.

Calculation of the l_x Column.

Having now obtained the logs. of the complete series of p_x values, the next stage in Life-Table construction is to proceed with the calculation of the l_x column; that is, the number of survivors at each successive year of age out of a given number supposed to set out on the journey of life at birth.

It really does not matter what number is so taken; the final results of E_x values, that is, the expectation of life at each age, might as readily be arrived at by taking one hundred as by taking a million.

It is usual, however, to divide up a hundred thousand or a million in the proportions of male and female births for the decennial period for which the Life-Table is being worked out.

The advantage of this procedure is that it is thus possible to make a combined Life-Table for *persons*, that is, males and females taken together, as regards the number of survivors at the respective ages out of the supposed original hundred thousand or million at birth.

In the example now being considered, Manchester City (males), the l_0 number is 50,764.

The point at which the new calculation is to be commenced is at age 5, at which age the number of survivors, l_5, is reduced to 34,467.

By multiplying this number by the fraction ·98750, which has been calculated as the value of p_5, the number of survivors at age 6 is obtained, and so on.

The calculations are of course made by means of logarithms. Thus:—

$l_5 = 34{,}467$			=	4·5374035
	$\times\ p_5$ =	·98750	=	$+\bar{1}$·9945375
$l_6 = 34{,}036$			=	4·5319410
	$\times\ p_6$ =	·99085	=	$+\bar{1}$·9960080
$l_7 = 33{,}725$			=	4·5279490
	$\times\ p_7$ =	·99325	=	$+\bar{1}$·9970599
$l_8 = 33{,}497$			=	4·5250089
	$\times\ p_8$ =	·99489	=	$+\bar{1}$·9977750
$l_9 = 33{,}326$			=	4·5227839
	$\times\ p_9$ =	·99591	=	$+\bar{1}$·9982193
$l_{10} = 33{,}190$			=	4·5210032

This series of calculations is proceeded with until there are no more survivors, that is, until a negative "characteristic" is obtained in the log. of l_x.

It is clear that if a million has been taken as the l_0 number, they will not all be extinct so soon as if only one-tenth part of the number has been commenced with. That is, it will require perhaps two or three more additional p_x values to get a negative characteristic in the case of a million than in the case of a hundred thousand. This mathematical necessity obviously corresponds to the actual facts, for out of an original million there is more probability of having a few survivors at extreme ages than out of a tenth part of the number.

For the sake of having a uniform standard of comparison for the number of survivors at each age, it is usual to calculate another l_x column showing the numbers of males or females surviving at each age out of a hundred thousand or million starting at birth.

This may be done—

(*a.*) By making a calculation by means of the p_x values exactly similar to that above described, but taking a hundred thousand or a million as the l_0 value.

(*b.*) Or by a series of proportions, if 34,467 survive at age 5 out of 50,764 at birth, how many would survive out of 100,000? And so on.

d_x Column.

This, although it is usually placed *before* the l_x column, must be calculated *after* it.

For the number dying in the first year of life in the Life-Table population must necessarily be the difference between those surviving at age 1 and those starting at birth, or generally, $d_x = l_x - l_{x+1}$.

P_x Column.

This column, which represents the mean number living during each year of life, is to be taken for every year of life *except the first*, as the arithmetical mean of the number beginning and the number ending the year of life.

Thus generally, $P_x = \frac{l_x + l_{x+1}}{2}$.

The same number which expresses the *mean population* will also represent the *years of life lived* during the year x to $x + 1$ by the number of survivors l_x entering upon that year.

The value of P_0, that is, the mean number living in the first year of life, may be worked out in two ways:

(*a.*) Having, in the data for the ten years for which the Life-Table is being calculated, the number of infants dying under 6 months of age out of the whole number dying under 1 year of age, it is a simple matter of proportion to find out how many of the d_0 numbers of the Life-Table have died by the middle of the year; it will be more than half, and deducting this number from l_0, will give the value of P_0.

(*b.*) Or if the numbers being dealt with are small enough to permit of a direct calculation of the *mean age at death* of those dying during the ten years under 1 year of age, then multiplying this by the number d_0 of the Life-Table, and adding the result to l_1 of the Life-Table, will give the value of P_0.

Taking the arithmetical mean of l_x and l_{x+1} as the value of P_x is exactly equivalent to allowing half a year each to those dying in the interval from age x to age $x+1$—

$$\text{Thus } P_x = l_{x+1} + \frac{d_x}{2}$$

Practically it will be found that for the first year of life 0·4 year has to be allowed instead of 0·5 as for the succeeding years.

In some Life-Tables the geometrical mean of l_x and l_{x+1} has been taken as the value of P_x.

The differences in the E_x values by this procedure are insignificant except at the later ages of life.

In the Life-Table for Haydock, published in 1898, I have given, by the help of Mr. A. C. Waters, a full discussion of this point. It would occupy too much space to repeat all this here, and it is not necessary, as the considerations presented apply more to "abbreviated" Life-Tables than to extended ones. It may suffice to simply draw attention to the fact that if the l_x values of a complete Life-Table are plotted out in a curve, this curve will be found to be *convex upwards* during the greater part of the curve, that is, from age 15 to age 70. Seeing that the problem of calculating the years of life lived by l_x from age x to age $x+1$, resolves itself, geometrically considered, into calculating the area of a figure bounded below by a horizontal line, the "abscissa," at the sides by the ordinates l_x and l_{x+1}, and at the top by the intercepted portion of the l_x curve, and that taking the arithmetical mean of l_x and l_{x+1}, is equivalent to making this intercepted portion of the l_x curve a straight line, and that taking the geometrical mean is equivalent to taking a rather deep *concave* curve, it follows that taking the arithmetical mean must give for the greater part of the Life-Table l_x curve a result rather *less* than the true one, but certainly nearer to the truth than taking the geometrical mean would give.

The Q_x Column.

When the P_x column is completed, by successive additions beginning from below, the Q_x column can be constructed.

It represents the years of life lived by l_x during the year $x+1$ and *during all the years afterwards to the end of the Life-Table.* It also represents the complete Life-Table population at each age x and upwards.

For merely local Life-Tables it is not really necessary to work out the E_x values for every separate year of life. It is sufficient to do this at quinquennial age intervals.

It is therefore a simple matter to calculate from the l_x values the values of $Q_x - Q_{x+5}$, for

$$\Sigma P_x \text{ to } P_{x+4},[2] \text{ or } Q_x - Q_{x+5} = \frac{l_x + 2(l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4}) + l_{x+5}}{2}$$

$$= \frac{l_x + l_{x+5}}{2} + l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4}.$$

Therefore the value of $Q_5 - Q_{10}$ (to take the example already given of calculating l_x values)

$$= \frac{34,467 + 33,190}{2} + 34,036 + 33,725 + 33,497 + 33,326 = 168,412\cdot5.$$

The E_x *Column.*

This column, which it is the final object of a life-table to produce, may now be readily calculated, seeing that Q_x represents the total years of life lived by l_x survivors during the year $x+1$, and during all the succeeding years to the end of the Life-Table, it is obvious that the expectation of life, or mean after-lifetime of any individual among the l_x number, is Q_x divided by l_x,

$$\text{or } E_x = \frac{Q_x}{l_x}.$$

The mode of calculation is as follows for quinquennial age periods, beginning from below and working upwards, and thus working out the Q_x and the E_x columns simultaneously.

			logs.
Q_{95}	$= 22\cdot055$	$\frac{Q_{95}}{l_{95}} =$	$\begin{array}{r}1\cdot3435071\\ -1\cdot0704136\end{array} = 0\cdot2730935 = 1\cdot88 = E_{95}.$
$+ \Sigma P_{90}$ to P_{94}	$= + 202\cdot23$		
Q_{90}	$= 224\cdot285$	$\frac{Q_{90}}{l_{90}} =$	$\begin{array}{r}2\cdot3508002\\ -1\cdot9518371\end{array} = 0\cdot3989631 = 2\cdot51 = E_{90}.$
$+ \Sigma P_{85}$ to P_{89}	$= + 1,106\cdot2$		
Q_{85}	$= 1,330\cdot5$	$\frac{Q_{85}}{l_{85}} =$	$\begin{array}{r}3\cdot1240149\\ -2\cdot6189725\end{array} = 0\cdot5050424 = 3\cdot20 = E_{85}.$
$+ \Sigma P_{80}$ to P_{84}	$= + 4,035\cdot4$		
Q_{80}	$= 5,366$	$\frac{Q_{80}}{l_{80}} =$	$\begin{array}{r}3\cdot7296507\\ -3\cdot1207205\end{array} = 0\cdot6089302 = 4\cdot06 = E_{80}.$

The above examples will show clearly how this series of calculations is to be continued to the end.

In order to avoid confusion, it should be noted that some life-tables, as for example the one for Brighton, have been worked out without bringing in the P_x column.

Instead of this a column called the Σlx_{+1} column is con-

[2] That is the sum of the values of P^x, P_{x+1}, P_{x+2}, P_{x+3}, P_{x+4}.

structed. That is, opposite the l_x number is placed the number of survivors at age $x+1$ and upwards. Then $\frac{\Sigma l_{x+1}}{l_x}$ will give what is called the "curtate expectation of life," that is, the average number of years lived by l_x, without regard to the parts of years lived by those dying in the interval from age x to age $x+1$.

The complete expectation of life will then be given by the formula $E_x = \frac{\Sigma l_{x+1}}{l_x} + 0{\cdot}5$, or, in the case of the first year of life,

$$E_0 = \frac{\Sigma l_1}{l_0} + 0{\cdot}4.$$

The method, however; previously described is to be preferred.

The work of an extended Life-Table for Manchester City (males) being now completed, the E_x values at quinquennial age intervals have been deduced as follows:—

	New Values.	Values in the Published Life-Table.	Differences of New from Published Values.
E_0	35·10	34·71	+ 0·39
E_5	46·16	45·59	+ 0·41
E_{10}	42·86	42·75	+ 0·11
E_{15}	38·62	38·78	− 0·16
E_{20}	34·63	34·62	+ 0·01
E_{25}	30·75	30·69	+ 0·06
E_{30}	27·09	27·08	+ 0·01
E_{35}	23·74	23·76	− 0·02
E_{40}	20·67	20·68	− 0·01
E_{45}	17·80	17·80	± 0·00
E_{50}	15·08	15·06	+ 0·02
E_{55}	12·53	12·49	+ 0·04
E_{60}	10·20	10·16	+ 0·04
E_{65}	8·18	8·15	+ 0·03
E_{70}	6·50	6·48	+ 0·02
E_{75}	5·15	5·11	+ 0·04
E_{80}	4·06	4·02	+ 0·04
E_{85}	3·20	3·16	+ 0·04
E_{90}	2·51	2·48	+ 0·03
E_{95}	1·88	1·86	+ 0·02

As will be made to appear afterwards, the divergences at the earlier ages depend upon different modes of interpolating the p_x values from age 5 to age 15. At the later ages they depend upon the P_x values of the published Life-Table, having been calculated from the geometric means of l_x and l_{x+1}.

Section II.

Description of a Modification of Dr. Farr's "Short" Method of constructing a Local Life-Table.

The labour involved in working out an "extended" life-table by the previously described method is, it must be admitted, very considerable, and may only too probably deter many from undertaking the task.

It is of importance, therefore, to ascertain whether there may not be a shorter way to arrive at the E_x values at quinquennial age intervals, giving for practical purposes a sufficiently near approximation to the values which *would be obtained* by the more laborious method.

The late Dr. William Farr, as has been already mentioned in the introductory remarks, devised a "short method" which, as applied by him to the data upon which his Life-Table No. 3 of England and Wales (males) was founded, gave E_x values at the earlier ages very near to the true ones, but at the later ages, and especially after age 65, the discrepancies in the direction of excess were very considerable.

Up to the point at which the p_x values for the first five years of life have been worked out, and the value of l_5 determined, the work to be done is exactly the same as for the extended Life-Table; there is no short method for the first quinquennial age-period.

If the work up to the point indicated has been completed, and it is desired to take a "royal road" beyond, and avoid what, on first view, is certainly the appalling labour of interpolation, the following is the procedure to be adopted.

From the data of population and deaths at the various age-periods, it is a very simple matter of calculation to ascertain the *mean* p_x values for each of the age-periods.

Thus for any quinquennial or decennial age-period

$$\log. \overline{P - \tfrac{1}{2}d} - \log. \overline{P + \tfrac{1}{2}d} = \log. p_{x \text{ to } x+n}$$

n being 5 or 10 as the case may be.

Still keeping to the example of Manchester City (males), the calculations are effected as follows:—

Age Periods.	$\frac{P - \frac{1}{2}d}{P + \frac{1}{2}d}$		Logs.		
5—10	$\frac{314{,}343 - 1{,}198}{314{,}343 + 1{,}198}$	$= \frac{313{,}145}{315{,}541}$	$= \begin{matrix} 5{\cdot}4957455 \\ -5{\cdot}4990558 \end{matrix}$	$=\bar{1}{\cdot}9966897$	$= 0{\cdot}99241$
10—15	$\frac{290{,}034 - 537{\cdot}5}{209{,}304 + 537{\cdot}5}$	$= \frac{289{,}496{\cdot}5}{290{,}571{\cdot}5}$	$= \begin{matrix} 5{\cdot}4616435 \\ -5{\cdot}4632531 \end{matrix}$	$=\bar{1}{\cdot}9983904$	$= 0{\cdot}99630$
15—25	$\frac{521{,}994 - 1{,}613{\cdot}5}{521{,}994 + 1{,}613{\cdot}5}$	$= \frac{520{,}380{\cdot}5}{523{,}607{\cdot}5}$	$= \begin{matrix} 5{\cdot}7163210 \\ -5{\cdot}7190058 \end{matrix}$	$=\bar{1}{\cdot}9973152$	$- 0{\cdot}99384$

We proceed thus until the value of log. p_{85-95} is obtained. As has been already explained, the data for age 95 and upwards are unreliable; whereas the value of p_{85-95} is found to be 0·72985, the value of p_{95-} from the actual data works out to 0·94286, a manifestly absurd result.

To overcome this difficulty and obtain a rational working value of p_{95-}, the following method is proposed:—

Let the logs of the four preceding p_x values be set down in a column and differenced. By carrying the differences downwards for one stage, a fifth equidistant value is to be obtained, which may be taken as the value of p_{95-} required.

		1st Differences.	2nd Differences.	3rd Difference.
Log. p_{55-65}	$= \bar{1}·9763671$	− 0·0209885	− 0·0137529	− 0·0089122
,, p_{65-75}	$= \bar{1}·9553786$	− 0·0347414	− 0·0226651	
,, p_{75-85}	$= \bar{1}·9206372$	− 0·0574065		
,, p_{85-95}	$= \bar{1}·8632307$			

Then log. p_{95-} =

$\bar{1}·8632307 - (0·0574065 + 0·0226651 + 0·0089122) = \bar{1}·774242469 = 0·59463.$

The following table shows the comparison of the mean p_x values thus deduced, with the means of the separate yearly values previously worked out by the extended method which has been described:—

Age Periods.	Mean p_x Values.		Difference of (*b*) from (*a*).
	By Short Method (*b*).	By Extended Method (*a*).	
5—10	0·99241	0·99247	− 0·00006
10—15	0·99630	0·99630	± 0·00000
15—25	0·99384	0·99380	+ 0·00004
25—35	0·98903	0·98890	+ 0·00013
35—45	0·98064	0·98037	+ 0·00027
45—55	0·96933	0·96872	+ 0·00051
55—65	0·94704	0·94514	+ 0·00190
65—75	0·90236	0·89665	+ 0·00571
75—85	0·83299	0·81742	+ 0·01557
85—95	0·72985	0·70007	+ 0·02978
95—	0·59463	0·58632	+ 0·00831

In proceeding to the next stage of the calculation, viz., the obtaining of the l_x values, it must be noted that to get l_{10}, l_5 must be multiplied by five times the value of p_{5-10}; to get l_{15}, l_{10} must be multiplied by five times the value of p_{10-15}; and to get l_{25}, l_{15} must be multiplied by ten times the value of p_{15-25}.

That is—

$$\begin{aligned} \log. l_5 &= \quad 4·5374035 \\ + \log. p_{5-10} \times 5 &= +\ \bar{1}·9834485 \\ &= \quad 4·5208520 = 33{,}178 = l_{10}, \end{aligned}$$

and so on.

We shall then obtain the series of l_x values tabulated below:—

		By Short Method (*b*).	By Extended Method (*a*).	Differences of (*b*) from (*a*).
l_{10}	=	33,178	33,190	− 12
l_{15}	=	32,569	32,580	− 11
l_{25}	=	30,617	30,615	+ 2
l_{35}	=	27,420	27,381	+ 39
l_{45}	=	22,551	22,458	+ 93
l_{55}	=	16,516	16,344	+ 172
l_{65}	=	9,585	9,296	+ 289
l_{75}	=	3,431	3,123	+ 308
l_{85}	=	552	416	+ 136
l_{95}	=	24	12	+ 12

From what has been previously said it will be easily comprehended that the next step, the calculation of the P_x values, is to be effected thus:—

$$P_{5-10} = \frac{l_5 + l_{10}}{2} \times 5$$

$$P_{10-15} = \frac{l_{10} + l_{15}}{2} \times 5$$

$$P_{15-25} = \frac{l_{15} + l_{25}}{2} \times 10$$

and so on. Having then obtained all these values, the Q_x and E_x values can be proceeded with as described in the method for the extended Life-Tables. Finally, the series of E_x values shown below are obtained:—

		By Short Method (*b*).	By Extended Method (*a*).	Differences of (*b*) from (*a*).
E_0	=	35·34	35·10	+ 0·24
E_5	=	46·52	46·16	+ 0·36
E_{10}	=	43·23	42·86	+ 0·37
E_{15}	=	38·99	38·62	+ 0·37
E_{25}	=	31·15	30·75	+ 0·40
E_{35}	=	24·20	23·74	+ 0·54
E_{45}	=	18·35	17·80	+ 0·55
E_{55}	=	13·23	12·53	+ 0·70
E_{65}	=	9·18	8·18	+ 1·00
E_{75}	=	6·68	5·15	+ 1·53
E_{85}	=	5·43	3·20	+ 2·23
E_{95}	=	5·00	1·88	+ 3·12

Now these results, while at the earlier ages giving a moderately close approximation to the true E_x values, can scarcely at the later ages be considered as altogether satisfactory.

When we come to analyse the reasons for the progressively increasing discrepancies in the direction of excess, it is found that they are of a twofold nature:

(1.) The mean p_x values of the short method, as has been previously shown, tend to become at the successive age-periods greater and greater than the corresponding mean values of the extended method. From this it follows that the l_x values also become greater and greater than those of the extended method, and the P_x values or the years of life lived by l_x in the interval between age x and age $x + 10$ *increase in greater proportion than l_x increases.*

(2.) However, even if we could get the mean p_x values to exactly correspond and thus obtain *identical* l_x values, there would still be differences due to the calculation of the years of life lived between age x and age $x + 10$ by *one* stage instead of in the ten successive stages of the extended method.

The extent of these differences can be readily measured by working out the E_x values to be derived from the l_x values of the extended method (a), by the same process as that which has been adopted for the values in the short method (b) (that is, the l_5, l_{10}, l_{15}, &c., values are taken without regard to the intermediate l_x values).

When this is done the following comparative results are obtained:—

Table showing the Differences of the P_x, Q_x, and E_x Values obtained by Calculation from the l_x Values of the Extended Life Table at Quinquennial Age Intervals for the Age Periods 5—10 *and* 10—15, *and afterwards at Decennial Age Intervals, from the Corresponding Values in the Extended Life Table.*

Age.	Differences in Years of Life or P_x Column.	Differences in Q_x Column.	Differences in E_x Column.	*
0	0	+ 2404	+ 0·05	+ 0·19
5	+ 730	+ 2404	+ 0·07	+ 0·29
10	− 48	+ 1674	+ 0·05	+ 0·32
15	− 817	+ 1722	+ 0·05	+ 0·32
25	− 1354	+ 2539	+ 0·08	+ 0·32
35	− 1133	+ 3893	+ 0·14	+ 0·40
45	− 890	+ 5026	+ 0·22	+ 0·33
55	− 571	+ 5916	+ 0·36	+ 0·34
65	+ 2668	+ 6487	+ 0·69	+ 0·31
75	+ 2950	+ 3819	+ 1·22	+ 0·31
85	+ 831	+ 869	+ 2·09	+ 0.14
95	+ 38	+ 38	+ 3·12	± 0·00

The figures in the column * are the differences of the differences in the preceding column from the total differences of the E_x values obtained by the short method, from the corresponding values of the extended Life-Table, and may therefore be considered to *measure the variation due to the differences of the p_x values.*

Simple geometrical considerations based on the shape of the life-table l_x curve will explain the varying values of P_x as given in the above analysis.

The same considerations will show that if similar calculations had been made from the l_x values of the extended Life-Table *at quinquennial age intervals throughout*, the differences would have been less marked than in the case of the ten-yearly intervals. This was demonstrated in a paper published in "Public Health" for July, 1898, by means of a table which I had worked out from the l_x values of the Life-Table for England and Wales (males). To economise space this need not be copied here.

The possibility thus suggests itself of elaborating Dr. Farr's original short method by working out mean p_x values all throughout for quinquennial age intervals.

However, although at the census enumerations the population numbers are given at quinquennial age intervals, these *numbers are too unreliable to work with*, owing to misstatements of age and the tendency to state ages in round numbers.

The first attempt at Life-Table construction which I made was for Haydock, using the census numbers at quinquennial age intervals, and the results were very rugged and uneven.

The required intermediate p_x values may be obtained by interpolations in the logs. of the u_x values of (Population $-\frac{1}{2}$ deaths) and (Population $+\frac{1}{2}$ deaths) of a much simpler and easier nature than those previously described for the extended method, as only three orders of differences need be taken.

The problem is, given u_5, u_{15}, u_{25}, u_{35}, to interpolate u_{20}. By the required special application of Lagrange's general formula—

$$u_{20} = \frac{10(u_{15} + u_{25}) - (u_5 + u_{15} + u_{25} + u_{35})}{16}$$

Similarly u_{30} can be worked out from u_{15}, &c., and so on, until u_{70} has been interpolated.

u_{80} has to be obtained from u_{65}, u_{70}, u_{75}, u_{85}. Thus—

$$u_{80} = \frac{u_{65} + u_{85}}{4} + 1\tfrac{1}{2}u_{75} - u_{70}.$$

The differences of the last four terms of the series now obtained, viz., u_{70}, u_{75}, u_{80}, u_{85} may be continued downwards, and then u_{90}, u_{95}, and u_{100} may be obtained.

I have worked out by this method also a Life-Table for Manchester City (males). To save space, only the final results are quoted below:—

Differences of E_x *Values from the Values of the Extended Life-Table.*

At Age 0.	5.	10.	15.	20.	25.	30.	35.	40.	45.
+ 0·06	+ 0·10	+ 0·10	+ 0·09	+ 0·11	+ 0·11	+ 0·13	+ 0·12	+ 0·16	+ 0·14
At Age 50.	**55.**	**60.**	**65.**	**70.**	**75.**	**80.**	**85.**	**90.**	**95.**
+ 0·15	+ 0·18	+ 0·20	+ 0·25	+ 0·32	+ 0·39	+ 0·49	+ 0·62	+ 0·77	+ 0·87

These results show a very great improvement on those previously given by the short method for decennial age intervals.

Description of the Special Modification of the Short Method Recommended.

I shall hope to show, however, that by a very simple modification of Dr. Farr's method, even still closer approximations to the E_x values of the extended method are to be obtained.

Very early in my first attempts at working out a Life-Table for Haydock, I stumbled upon the discovery that in working with mean values of p_x for ten-yearly periods, the value of P_x for the ten-yearly period is *less* if the calculation is made from l_x to l_{x+10} by *two* stages instead of *one*.

Thus assuming that we have a population of 16,000 at age 75, and that this number is reduced to one-fourth part, or 4,000, by age 85, the mean value of $p_{x \text{ to } x+10}$, or the chance of living one year in the interval from age x to age $x+10$ would be 0·87055, or $16{,}000 + (0{\cdot}87055)^{10} = 4{,}000$.

By the *one* stage calculation of Dr. Farr's method the value of

$$P_{x \text{ to } x+10} = \frac{16{,}000 + 4{,}000}{2} \times 10 = 100{,}000.$$

By calculating in *two* stages we have this result:—

$$(1.)\quad 16{,}000 \times (0{\cdot}87055)^5 = 8{,}000.$$
$$(2.)\quad 8{,}000 \times (0{\cdot}87055)^5 = 4{,}000.$$

and the years of life lived in the interval from age_{75} to age_{85}

$$= \left(\frac{16{,}000 + 8{,}000}{2} + \frac{8{,}000 + 4{,}000}{2}\right) \times 5 = 90{,}000.$$

It then occurred to me that it might be possible thus to obtain values of E_x more closely approximating to those of an extended Life-Table. On submitting the idea to the test of experiment I obtained very close approximations to the true values as far as E_{65}; after this the results were increasingly too great.

The further idea then struck me that it might be possible by further increasing the number of stages of the calculation I still might obtain better results.

Thus to take the case of a calculation in four stages:—

(1.) $16{,}000 \times (0{\cdot}87055)^{2\frac{1}{2}} = 11{,}314.$
(2.) $11{,}314 \times (0{\cdot}87055)^{2\frac{1}{2}} = 8{,}000.$
(3.) $8{,}000 \times (0{\cdot}87055)^{2\frac{1}{2}} = 5{,}656.$
(4.) $5{,}656 \times (0{\cdot}87055)^{2\frac{1}{2}} = 4{,}000.$

By this calculation the years of life lived in the interval from age 75 to age 85 would be—

$$\left(\frac{16{,}000+2(11{,}314+8{,}000+5{,}656)+4{,}000}{2}\right)\times 2\tfrac{1}{2}=87{,}425.$$

Similarly the calculation might be made in five or in ten stages, with increasing diminution in the years of life obtained.

The results may be represented more clearly in a tabular form:—

16,000 *reduced to* 4,000 *in Ten Years.*

Stages of Calculation.	Years of Life.	Stages of Calculation.	Years of Life.
1	100,000	5	87,116
2	90,000	10	86,701
4	87,425	An infinite number	86,511·1

The last value is to be obtained by the formula $\dfrac{l_x - l_{x+10}}{\text{hyp. log. } \dfrac{l_x}{l_{x+10}}}$

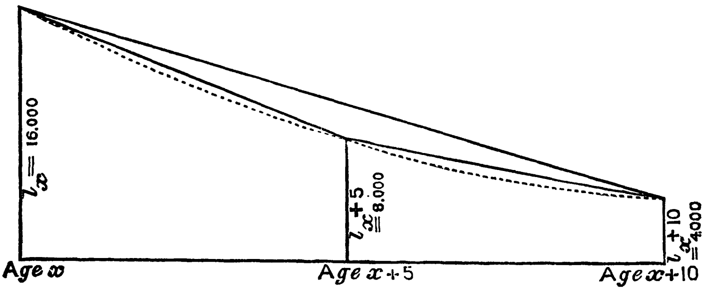
A simple geometrical construction will make this point clear.

By the hypothesis which is assumed, that the death-rate or "force of mortality" remains constant during the ten years' interval for age 75 to age 85, in calculating the years of life lived during this interval, we have to determine the area of a figure bounded below by a horizontal line, the "abscissa," upon which are erected at a distance apart corresponding to ten years two vertical lines or "ordinates" representing to scale l_x and l_{x+10}; now the curve joining the upper extremities of these ordinates would be approximately constructed by erecting successive ordinates at distances apart corresponding to one year, each one 0·87055 the length of the preceding one, and joining these extremities by a series of straight lines. There would be thus obtained an approach to the "logarithmic curve," or the "curve "of equal proportional decrements."

The *true* area of the figure would be determined by conceiving of the distances between the successive ordinates being infinitely small, and is to be obtained by the formula already given.

Taking the arithmetical mean of l_x and l_{x+10} multiplied by 10 is equivalent to making the "curve" joining the extreme ordinates a straight line.

A reference to the diagram will make it apparent that the total area of the *two* four-sided figures formed by joining the end of the middle ordinate with the ends of the extreme ordinates, is much less than the *one* figure formed by joining the ends of the extreme ordinates, and also that the greater the number of sections in which the area of the whole figure is calculated, the nearer will be the approximation to the true area which is bounded above by the logarithmic curve, which is indicated by the dotted line.



I have submitted this idea to the test of experiment to find out the most simple and most accurate application of the many possible variations in the mode of calculating the years of life for successive decennial age-periods.

As the result I propose the following scheme:—

(1.) For each of the age periods 15—25 to 65—75 inclusive, to make the calculation in two stages.

(2.) For the two age periods 75—85 and 85—95, to make the calculation in *four* stages.

(3.) From age 95 to continue the calculation to the end by *yearly* stages.

Starting again with the data (*a*) the log. of l_5, and (*b*) the logs. of the mean p_x values, as previously worked out and tabulated for Dr. Farr's original method, the following results have been obtained for Manchester City (males) by the application of the suggested scheme:—

$E_0 = 35{\cdot}10$	$E_{15} = 38{\cdot}62$	$E_{45} = 17{\cdot}80$	$E_{75} = 5{\cdot}15$
$E_5 = 46{\cdot}16$	$E_{25} = 30{\cdot}75$	$E_{55} = 12{\cdot}53$	$E_{85} = 3{\cdot}20$
$E_{10} = 42{\cdot}86$	$E_{35} = 23{\cdot}74$	$E_{65} = 8{\cdot}18$	$E_{95} = 1{\cdot}88$

From these values it is possible by a simple process of interpolation to obtain the intermediate values for the even ages 20, 30, &c.

It must be noted that the intermediate values of l_{x+5}, which have been obtained in the calculation of l_{x+10} by two stages *are* *not available for working out the* E_{x+5} *values.* If so used they give widely divergent results. They are of no further use than the one to which they have already been put.

Formulæ have to be used corresponding to those which have already been given for working out intermediate quinquennial values in the logs. of the u_x values of $(P-\frac{1}{2}d)$, and $(P+\frac{1}{2}d)$, but in this case they need only be applied to the E_x *numbers.*

$$\text{Thus } E_{30} = \frac{10\,(E_{25}+E_{35}) - (E_{15}+E_{25}+E_{35}+E_{45})}{16}.$$

This formula, with the needed obvious changes in the suffixes of E_x, is applicable as far as E_{80}.

The rule simply is, given the four equidistant terms to work with, *from ten times the sum of the two middle terms subtract the sum of all four terms and divide the remainder by* 16; *then the result is the central term required.*

I have found as a rule that it is best not to bring in E_5, so that E_{20} is to be deduced thus

$$E_{20} = \frac{E_{15}+E_{35}}{4} + 1\tfrac{1}{2}\,E_{25} - E_{30}.$$

Finally

$$E_{90} = \frac{E_{75}+E_{95}}{4} + 1\tfrac{1}{2}\,E_{85} - E_{80}.$$

These two last formulæ may be reduced to the common rule—where at the top and bottom of the series there are only three original terms to work with, together with a fourth term already interpolated, *to one-fourth of the sum of the two outside terms add one and a half times the inner term and subtract from the sum the value of the term already interpolated.*

These interpolations having been effected, the final results come out as follows :—

Differences of the E_x *Values from the Corresponding Values of the Extended Life-Table.*

At age 0.	5.	10.	15.	20.	25.	30.	35.	40.	45.
− 0·02	−0·03	−0·04	−0·04	−0·07	−0·03	+0·01	−0·01	−0·01	±0·00
At age 50.	**55.**	**60.**	**65.**	**70.**	**75.**	**80.**	**85.**	**90.**	**95.**
− 0·01	+0·02	+0·06	+0·07	+0·08	+0·05	+0·07	+0·08	+0·04	+0·04

It is thus apparent that by this extremely simple method E_x values at quinquennial age intervals are to be obtained approximating to those of the extended Life-Table with a very remarkable nearness, the differences indeed being less than the probable differences due to inaccuracies in the foundation figures from erroneous statements of the ages of the living and of the dying.

It would not be safe to generalise too much from one particular instance, but in the next section of this paper additional examples will be given which need not at this point be inserted.

Section III.

From want of space the whole of this section (including diagrams) has had to be excluded from publication in the present number of the *Journal*.

It is to appear among the "Miscellanea" of one or more succeeding numbers.

The following is a synopsis of the matters dealt with in this section:—

(1.) Reasons why the values of p_5 to p_{24} have been calculated by interpolations in the series u_4, u_5, u_{10}, u_{15}, u_{25}, u_{35}.

(*a*.) It is not a matter of indifference what series is taken, as by different series from the same data values of E_0 and E_5 are to be obtained differing from each other by as much as a whole year.

(*b*.) The "ideal" curve is considered as to its relations to the mean values of p_x to be obtained from the population and death numbers of the three age-groups 5—10, 10—15, and 15—25.

(*c*.) It is shown that the series chosen is the only one which realises this ideal.

(2.) A discussion of the reasons for making interpolations in u_x values of population and deaths *combined*, instead of dealing with them *separately*.

(3.) Experimental calculations, showing to what extent the p_x and E_x values are affected by taking different numbers of orders of differences in the interpolations.

(4.) A comparison of the results to be obtained from the same data by the "analytical" and "graphic" methods respectively, illustrated by a Life-Table worked out from the data of the Brighton Life-Table (males) by the method described in Section I.

IV.—*Practical Uses of Life-Tables.*

Having already, I fear, considerably exceeded the time and space available for this paper in describing the *construction* of Life-Tables, it will not be possible to do more than indicate in the briefest possible manner their *practical uses*, which indeed might well demand a whole paper to be at all adequately dealt with.

The use of Life-Tables in forming the basis of calculations relating to problems connected with life assurance and annuities need only be alluded to.

From the point of view of public health, the uses of a Life-Table when it has been prepared for any district may be thus summarised :—

(1.) It is possible to make exact comparisons—

(*a.*) On the one hand with the whole country, say England and Wales, and on the other, with the Selected Healthy Districts.

(*b.*) With all other districts (for the same decennial age period) for which Life-Tables have been prepared.

(*c.*) And, what is of very special importance, with succeeding Life-Tables for the *same* district, thus obtaining in the most exact way possible indications of progress or the reverse in the conditions which favour health and length of life.

The special lines along which such comparisons may be made may be thus briefly pointed out—

(*a.*) *The* p_x *Values.*

These afford the means of testing the vitality of a community for each special age or age-period, depending as they do neither upon any conditions coming before, nor upon any following after, but simply upon the force of mortality or death-rate which has prevailed at the special age-period under consideration.

Still, except for the first five years of life, such comparison is perhaps most usually and most readily made by comparing the death-rates as ordinarily worked out for the separate age-groups.

(*b.*) *The* l_x *Values.*

These depend upon *antecedent*[3] conditions, that is, upon the preceding rates of mortality. Thus high death-rates during the earlier years of life make the l_x values less at all after ages.

It must be noted too that l_x numbers can only be fairly compared between Extended Life-Tables, or between Short Life-Tables worked out by the same methods. It is fallacious to compare the l_x numbers of a Short Life-Table, especially at the later ages, with the corresponding numbers of an Extended Life-Table.

(*c.*) *The* E_x *Values.*

These are affected solely by *following*, and not by preceding conditions, depending as they do upon the death-rates for the after age periods.

For one may start the calculation of E_x values from the p_x values with any number or "radix," and of course obtain just the same results for after ages.

(*d.*) There is another, and, perhaps, the most important use

[3] The terms "antecedent" and "subsequent" are of course only used in a hypothetical sense. In a Life-Table calculated from the data for any decennial period, the actual death-rates at the different age-periods are *simultaneous*. They are assumed, however, to exist in succession as applied to the generation whose life-history is being traced.

of a life-table, which was first pointed out in the "Report on the "Health of Greater Manchester" for the years 1891-93, published by Dr. Tatham in 1894. This is associated with all the ideas connoted by the term "life-capital."

For a complete explanation and description of these ideas, reference may be made to the original report just mentioned. The following is only a brief allusion. An Extended Life-Table for males and females having been constructed for a decennial period, say 1881-90, the assumption has to be made that the age and sex distribution of its population, and the death-rates for its separate age groups remain the same during the succeeding ten years.

Having now the values expressing the expectation of life at birth for males and females, it is obvious that multiplying them by the numbers of births of males and females in any year will give the number of prospective years of life added in the year to the life capital of the community.

Since the same number which gives the estimated mean population for any year, expresses also the number of years of life lived or expended by the community in that year, the balance of loss or gain of Life-Capital can readily be struck.

The Extended Life-Table, however, gives not only the expectation of life at birth, but the mean after lifetime at each age. Hence if this can be calculated from the life-table in *groups of years*, corresponding to those usually employed in classifying deaths, the same method which has been applied to births may be applied to the estimated living population in calculating life capital.

The future lifetime of P_x persons, that is of P persons living at all ages from x to $x+1$, must be from the Life-Table equal to $Q_x - \frac{P_x}{2}$, on the assumption which is made in Life-Table construction, that the average age of the P_x persons is $x+\frac{1}{2}$ at the middle of the year. At the middle of the year they will therefore have each lived half the year on the average, and therefore $\frac{P_x}{2}$ must be deducted from the Q_x value.

It is therefore obvious that their mean expectation of life individually $= \frac{Q_x - \frac{1}{2}P_x}{P_x} = \frac{Q_x}{P_x} - \frac{1}{2}$.

Similarly the future lifetime of $P_x + P_{x+1} + \ldots . P_{x+n-1}$ persons living at all ages between x and $x+n$ is—

$$(Q_x + Q_{x+1} + \ldots . Q_{x+n-1}) - \tfrac{1}{2}(P_x + P_{x+1} + \ldots . P_{x+n-1}),$$

and their mean expectation of life—

$$= \frac{Q_x + Q_{x+1} + \ldots . Q_{x+n-1})}{P_x + P_{x+1} + \ldots . P_{x+n-1})} - \tfrac{1}{2}.$$

In actual practice of course $n = 5$ or 10.

If the mean values worked out by this method are applied to the estimated population for any year, proportionately distributed in age and sex groups, the aggregate *life-capital* of the community is arrived at.[4]

Then $\frac{\text{life-capital}}{\text{population}}$ = average life-capital or future lifetime of each individual of the population;

and since the mean population = years of life expended in the year,

$\frac{\text{population}}{\text{life-capital}} \times 100$ = proportion per cent. of life-capital expended in the year.

Finally, if a calculation be made of the number of deaths which *should have occurred* in each age group, if the mean death-rates for the ten-yearly period of the Life-Table had continued unchanged, and then a comparison be made between these numbers and the numbers of deaths which *have actually occurred* in the year being dealt with, it is a simple matter to strike the balance of gain or loss of life-capital. It is obvious also that the earlier in life the loss or gain is found to have occurred the greater will be the loss or gain of life-capital.

Such is a brief and bald statement, which can have no claim or pretention to originality, of the principal ways in which a Life-Table can be practically applied.

Before however a Life-Table can be used, it has to be *constructed*, and if this paper should make easier for others the path which I have found very difficult to find and to follow, the labour expended will have its more than sufficient reward.

[4] The Life-capital method assumes that the age-distribution of the *actual* population *within each age-group* is the same as the age-distribution of the Life-Table within the corresponding age-group. The error of this assumption is probably unimportant.

Discussion *on* Mr. Hayward's Paper.

Mr. A. H. Bailey, F.I.A., was satisfied that an immense amount of pains and trouble had been bestowed upon a subject which was not often, as on the present occasion, touched upon by a medical officer of health. He called attention to the fact as regards the construction of tables of mortality, that the materials available were not always the same. If one had to determine the mortality of a particular society, a friendly society or an assurance society, there was available the whole record of the individuals and the precise dates of birth and death. But in the investigations to which Mr. Hayward had been referring, the circumstances were not precisely the same. One had to take the numbers living and the number of deaths in particular localities, but they were not the deaths necessarily of the same people. The difficulty which arose was due to emigration and immigration. It was impossible to discuss a paper like the present at that meeting. The question of interpolation was one of considerable difficulty. They had often tried to arrive at the theoretical law of mortality. A suggestion was made, and it was borne out by some of the facts, that the rate of mortality from 15 to 55 increased in the geometrical progression of 3 per cent. per annum, and after 55 there was a sudden increase of 8 per cent. to the end of life. That was not actually borne out, but if they examined tables of mortality there seemed to be some foundation for it. If that was so it would simplify the principle of interpolation which was proposed here. There must be considerable difficulty in arriving at the rates of mortality in particular districts in a country like England, where the changes of the character which he had indicated were very considerable. In conclusion he expressed his great pleasure at listening to such an interesting paper.

Mr. N. A. Humphreys said the reader of the paper having mentioned his name in a far too flattering manner in connection with that of their eminent past President, Dr. William Farr, he wished to explain that he did not pretend to be an expert on Life-Tables, but was glad to have been able to devote a great deal of time and work during a long series of years to the promotion of improved methods for dealing with vital statistics, especially mortality statistics, and of their correct interpretation. The Life-Table method was the only one which gave approximately accurate means of comparing the mortality statistics of different communities and different countries. He had been struck with the industry and ability displayed by Mr. Hayward in the valuable experiments which he had made, and had been much interested in the satisfactory results derived from those experiments. The paper would be pre-eminently useful hereafter, and the subject appeared to him an entirely suitable one to bring before that Society. The experiments which Mr. Hayward had previously made on a very small population had led him to study the subject on a larger basis, and had resulted in an undoubtedly valuable paper; but it was as well to suggest caution against the doubtful results of the serious labour involved in calculating Life-Tables from the statistics of small populations. Referring to Mr. Bailey's point as to the disturbing influence of migration, he admitted that the Life-Table of a town was obviously affected by the fact that the population between 15 and 35 was very largely recruited by healthy immigrants from country districts, who come into towns as shop assistants, mechanics, domestic servants, &c. When these people fall ill they to a very large extent return to their country districts and often die there. The town therefore had the credit and advantage of these vigorous young immigrants during their residence therein, and the country districts were somewhat unjustly debited with their mortality. The rate of mortality between those ages in the country often indeed appeared to be in excess of that in towns, which was obviously untrue. Great caution therefore should be observed in devoting a great deal of time and labour to the construction of Life-Tables for small local populations, which were specially subject to these disturbing influences. Even the healthy district Life-Table, which was constructed by Dr. Farr, was open to some objection on the same ground. It overstated the true mortality of residents in the healthy districts, from the fact that the table included the deaths of considerable numbers of town immigrants who returned to these districts on account of health failure due to town residence. There was unfortunately no available method by which the true rate of mortality for country districts could be calculated by eliminating these cases. Whether they dealt with the recorded rates of mortality or with Life-Table calculations, the mortality in towns was inevitably understated, and in country districts it was overstated. These disturbing influences did not affect Life-Tables for the whole country, and it had been the rule in the Registrar-General's office for some time past to calculate and publish a new English Life-Table based upon the mortality statistics of each intercensal period of ten years. The materials from which local Life-Tables had to be constructed were more or less unsatisfactory and subject to the disturbing influences already alluded to. Census returns, it is moreover true, did not give entirely accurate facts as to age, especially in the first few years of life; but he had good reason to know that each succeeding census had given facts of increasing accuracy, and therefore he should not discourage the construction of Life-Tables based on such statistics, but would rather suggest that censuses should be taken at more frequent intervals. It was to be hoped that the Census Act of next session would give the necessary authority for a quinquennial, enumeration. He concluded by saying that there was one point which always puzzled him in dealing with this subject of Life-Tables, and he hoped the author would be able to throw some light upon it. There was a period of life from 10 to 25 at which the recorded death-rate for the whole of England and Wales was almost invariably below the rate of mortality shown by the Life-Table method. Was there, he asked, some flaw in the figures upon which these calculations were based, or was the method of dealing with them at fault? Before sitting down, he might again repeat that the Society was under a great obligation to Mr. Hayward for the unstinted labour which he had devoted to this important subject.

Mr. PRICE-WILLIAMS said that the paper contained a great mass of very interesting statistics, and he regretted the absence of some of our leading actuaries, who would be most capable of appreciating the value of the paper. He had been greatly impressed with the admirable way in which the hand-sized diagrams which accompanied the paper had been prepared, and how strikingly they illustrated (in a manner which no mere array of figures in tabular statements could have done) the chief points and conclusions to which the author had drawn attention. He hoped in future that in addition to the hand-sized diagrams, large diagrams to a befitting scale might be exhibited on the walls to enable members more readily to follow the description and to realise the conclusions given in the paper.

Mr. C. H. E. REA expressed his sincere appreciation of the great labour which the author had bestowed upon an important subject, and he believed that good might come from some of the suggestions put forward. With regard to the author's introductory statement, that "but few of the Fellows of the Society, except those connected with the departments of public health or official statistical work, would ever be likely to actually engage in the work of practically attempting the construction of a Life-Table," he thought that was a matter upon which they need not feel much regret. In spite of what the author had said as to his desire to address the medical rather than the actuarial profession, the paper, he thought, was more actuarial than medical, and he doubted very much whether the members of the medical profession would find it possible to extend their labours and studies so far out of their distinct province, as to follow the formulæ and methods which the author had advanced for them to carry out in forming the Life-Tables suggested; and then there was the danger that those who did venture thus far might become so fascinated with a glimpse at the higher mathematics as to neglect the requirements of those bodily and physically distressed. And, after all, local Life-Tables, formed on scanty statistics, constantly disturbed moreover by the effects of invalidings and migrations, could never be regarded in themselves as of any particular value. He ventured to think that the author was somewhat modest when he disclaimed being anything of a mathematician, for, judging from the present perusal, the paper seemed to practically treat on the subject of finite differences, with an example in point of Lagrange's formula for interpolation applied to a special case. There is also reference to the formation of the l_x and other depending columns. After pointing out a slight clerical defect in one of the mathematical expressions, he concluded by expressing an opinion that the medical profession might furnish important and valuable statistics from the various localities under special view, but he deprecated any attempt being made on their part to operate actuarially on the bare facts; it seemed to him very much like placing a stethoscope in the hands of an actuary with a full written description of the end that had to be placed to the ear. Such departures would be highly dangerous, and could only tend to interfere with the present satisfactory condition of our mortality tables and disturb that l_x column, the construction and meaning of which is ably explained by the author. In other respects the paper presented features of interest.

Mr. T. E. HAYWARD said that anyone venturing to present a paper before such a Society as that addressed, must of necessity be prepared to face the ordeal of criticism. In the main he had to be exceedingly grateful to those who had said such very kind things in regard to his paper, and he was equally grateful to those who had been good enough to point out what might be considered errors. Had time been permitted beforehand for the Fellows present to thoroughly master the paper, he had no doubt it would have been possible to bring forward still more searching criticisms. He was really quite sincere in what he said, namely, that he had not much mathematical knowledge. Literally he believed that what he had put down in the paper could be done by anyone with simply the use of logarithms and some patience in applying the ordinary rules of arithmetic. His primary object had been to show that by the use of a modified short method, results were to be obtained so very closely approximating to those that would be obtained by undertaking the labour of calculating a complete or "extended" Life-Table, as to render unnecessary for merely local Life-Tables the more laborious method. He could only show the closeness of approximation by working out an example and applying the most rigidly exact modes of calculation that could be used; and he trusted he had done this with some success. He had shown that the results by the short method could be obtained in a fraction of the time required for the extended Life-Table. If the short method which he had explained were used, the results were quickly arrived at, and they were quite near enough to those which would be obtained by the longer method, to enable them to be used with a sufficient degree of confidence. He fully recognised the difficulties with regard to the absence of data as to emigration and immigration. With regard to the comparison of the actual death-rates with Life-Table results, he could only suggest that discrepancies might have something to do with the different methods of interpolation adopted for the age-period 5 to 25. He was greatly indebted for what Mr. Price-Williams had said with regard to the diagrams; and for his own part he should like to have shown them on a larger scale. He had no wish to confound the actuarial and medical professions. What he had advanced did not apply to the medical profession generally, but merely to medical officers of health, who had to be, to some extent, statisticians. The work involved in the preparation of the paper had been considerable, but it was only subsidiary to his medical work, and he had taken it up to some extent as a recreation, and he hoped the results might not be without some practical use.

The CHAIRMAN (Major P. G. CRAIGIE) said it was his pleasing duty to move a very cordial vote of thanks to Mr. Hayward for his kindness in preparing the matter which he had laid before the Society that evening. He should not for a moment attempt to enter into any discussion upon a technical paper such as this, but he should like to say, as a very old officer of the Society, that he maintained that it was well within the scope of their functions to receive and consider papers of this nature dealing with the methods of recording and tabulating data, as well as papers of a more attractive and popular kind, representing economic deductions from published facts. The Royal Statistical Society, as a scientific body, were discharging an important duty in endeavouring to obtain and give publicity to the best possible lights on the sufficiency of the various methods of compiling Life-Tables. They had had that evening put before them certain suggestions for producing results which might prove of great value to those who were daily engaged in such work as that of the reader of the paper. He was particularly struck by what Mr. Hayward said, that a medical officer of health, who was every year becoming a factor of increasing force and influence in the social concern of life, ought to be trained to treat the data he had to handle from a broad and correctly statistical point of view. He felt sure that those present would not end their study of the paper in that room, but that they would afterwards carefully peruse it in the *Journal*, where the suggestions made by Mr. Hayward would find a wider and an attentive circle of students.