

## SOME ASPECTS OF THE DYNAMICS OF THE GYROSCOPE.

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BESIDE the algebraic simplicity due to selecting principal axes, which was probably Euler's single conscious aim, his dynamical equations for a rigid body present a notable peculiarity that he was less in a position to recognize completely. If the equations are written

$$M_1 = A \frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3, \text{ etc.}, \quad (1)$$

the angular acceleration happens to be given accurately by the relation

$$\dot{\omega} = \frac{d\omega_1}{dt} + \frac{d\omega_2}{dt} + \frac{d\omega_3}{dt}, \quad (2)^1$$

making it evident to inspection that the effects of applied force-moment are here segregated on an essentially kinematical basis. This results naturally from the trend of the original demonstration whose leading thought attaches to angular velocity and its changes;<sup>2</sup> and our compact vector methods allow us to parallel and complete Euler's analysis in a few lines. For any element located by its radius-vector from the center of mass, the relative velocity, acceleration and force are

$$\mathbf{v} = [\omega \mathbf{r}]; \dot{\mathbf{v}} = [\dot{\omega} \mathbf{r}] + [\omega \mathbf{v}]; d\mathbf{m}\dot{\mathbf{v}} = dm[\dot{\omega} \mathbf{r}] + dm[\omega \mathbf{v}]. \quad (3)$$

The corresponding total force-moment required is

$$\mathbf{M} = \int dm[\mathbf{r}\dot{\mathbf{v}}] = \int dm[\mathbf{r}[\dot{\omega} \mathbf{r}]] + \int dm[\mathbf{r}[\omega \mathbf{v}]] = \mathbf{M}' + \mathbf{M}''. \quad (4)$$

Then simple routine transformation of the integrals shows for the components of  $\mathbf{M}'$  and  $\mathbf{M}''$  respectively,

$$A \frac{d\omega_1}{dt}, \text{ etc.}; \text{ and } (C - B)\omega_2\omega_3, \text{ etc.};$$

establishing in the two types of acceleration the basis of separation in the second member of eq. (1).<sup>3</sup>

<sup>1</sup> In the absence of a standardized notation, some reliance must be placed upon the context, where there are transitions back and forth between vectorial and algebraic statements, in order to prevent confusion. Keeping this in view, the intended sense of the equations here should become clear.

<sup>2</sup> Euler, *Theorie der Bewegung* (German trans. by Wolfers, 1853), pp. 431-443.

<sup>3</sup> This interpretation of the first term ( $\mathbf{M}'$ ) is a corollary of a theorem announced by Minchin, *Nature*, Vol. 23, p. 62 (1881). The source of the second term ( $\mathbf{M}''$ ) did not escape Euler himself (loc. cit., p. 323).

But since Euler's day it has been discovered how the idea really at the root of his original plan can be extended. In fact the propositions covering the more general use of so-called "moving axes" have been standard material for at least thirty years; it is therefore somewhat remarkable that they have been so neglected in actual application (except to the special case) until their more recent vector statement supplied a renewed stimulus.<sup>1</sup> It has become a familiar truth now, however, that with the usual assumption about symmetry (*i. e.*,  $A = B$ ) in the gyroscope, we can retain Euler's scheme to the extent of continuing to use principal axes, without being limited by the condition underlying eq. (2). In other words the lines chosen to project upon may have any differential shift about the figure-axis, relative to the body itself. This plan offers an advantage in its more inclusive view, and does not deserve to be so nearly ignored as an alternative; thus the considerations that follow may be taken for a supplement to the "orthodox" presentation. At the same time, because they abandon the lines of kinematical description, and lean at some vital points rather toward directer dynamical statement, they foster a tendency to be encouraged.

We are to conceive the problem of the gyroscope in the usual way: A rotating rigid body having "universal-joint freedom" round a fixed point ( $O$ ) is under control of weight-moment. With origin at this fixed point assume the following right-handed set of principal axes (possible when  $A = B$ ): The figure-axis ( $C$ ) with a perpendicular ( $B$ ) to it in a vertical plane; and a third axis ( $A$ ) normal to that plane. The resultant force-moment ( $\mathbf{M}$ ) is then always being exerted about the instantaneous position of the line ( $A$ ), which therefore coincides also with the vector derivative of the moment of momentum ( $\mathbf{M}\dot{\mathbf{m}}$ ). This implies for components and resultants; the latter being of necessity the actual "physical values":

- (a)  $\omega_1 + \omega_2 + \omega_3 = \omega$  (angular velocity),
- (b)  $A\omega_1 + B\omega_2 + C\omega_3 = \mathbf{M}\dot{\mathbf{m}}$  (moment of momentum),
- (c)  $\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \mathbf{M} = \mathbf{M}\dot{\mathbf{m}}$  (external force-moment).

Further we adopt the standard angular coördinates:

- $\vartheta$  [measured from the downward vertical ( $Z$ ) to the figure-axis],
- $\psi$  [specifying the azimuth of the plane ( $ZOC$ )],
- $\phi$  [giving displacement relative to ( $ZOC$ ) about ( $OC$ )].

When a progressive shift of axes is supposed a difference is introduced thereby between the two time-rates

<sup>1</sup> The main proposition did not find a place in the earlier editions of Routh's Dynamics; and though Klein and Sommerfeld (*Theorie des Kreisels*, p. 113) emphasize Hayward's exhaustive treatment of the matter (1854), still their equations are exclusively of the classic type (where Cartesian) subject to the condition of eq. (2).

$$\mathbf{M}\dot{\mathbf{m}} \text{ and } \frac{d}{dt} \{A\omega_1 + B\omega_2 + C\omega_3\} \equiv \frac{d'}{dt} \mathbf{M}\mathbf{m},$$

and gives occasion for a corrective relation that may be written

$$\mathbf{M}\dot{\mathbf{m}} = \frac{d'}{dt} \mathbf{M}\mathbf{m} + [\mathbf{u}\mathbf{M}\mathbf{m}]; \quad (5)$$

where without breaking away from principal axes we can have

$$\mathbf{u} = \psi + \vartheta + k\dot{\varphi}; \quad (6)$$

and Euler's equations reappear for  $k = +1$ . For application to present conditions  $k = 0$ ; so that the difference of plan disappears when  $\dot{\varphi} = 0$ . Inserting the special value of  $(\mathbf{u})$ ; also  $A = B$ ,  $\omega_1 = \vartheta$ ; and anticipating the constant magnitude of  $(C\omega_3)$ , we find

$$\mathbf{M} = \mathbf{M}_1 = \frac{d}{dt} \{A(\vartheta + \omega_2)\} + [\vartheta + \psi, \mathbf{M}\mathbf{m}]. \quad (7)$$

Without going into details we may remark that this expression is well adapted to a perspicuous grouping of important particular cases; the ordinary weight-pendulum, the spherical pendulum, the adjustments to regular precession. It is favorable besides that  $\mathbf{M}_2$  and  $\mathbf{M}_3$  are both zero always. Thus the Cartesian expansion of eq. (7) falls at once into the form

$$-mg\bar{r} \sin \vartheta = A \frac{d^2 \vartheta}{dt^2} + \psi \sin \vartheta C\omega_3 - \psi \cos \vartheta A \dot{\psi} \sin \vartheta, \quad (8)$$

$$0 = A \frac{d\omega_2}{dt} + \psi \cos \vartheta A \dot{\vartheta} - \dot{\vartheta} C\omega_3, \quad (9)$$

$$0 = C \frac{d\omega_3}{dt} + \dot{\vartheta} A \dot{\psi} \sin \vartheta - \dot{\psi} \sin \vartheta A \dot{\vartheta}. \quad (10)$$

These only elaborate the one central fact that no vector change in moment of momentum is ever being produced, except the increment along the instantaneous position of the  $A$ -axis. The scheme of projection follows up this sole positive controlling action by means of eq. (8); from which in the equivalent form,

$$-mg\bar{r} \sin \vartheta = \frac{d}{dt} (A\dot{\vartheta}) + [\psi \mathbf{M}\mathbf{m}_{BC}], \quad (11)$$

we see at once that the external force-moment is absorbed completely into two effects: (1) Changed azimuth of moment of momentum found in the plane  $(BOC)$ ; (2) changed magnitude of the horizontal part  $(A\dot{\vartheta})$ . Any further changes therefore are in character internal readjustments; which must moreover harmonize with the absence of control expressed

by the first members of equations (9) and (10). The possibility of such automatic interchanges ceases to be puzzling when we remind ourselves that vector constancy is not in general consistent with constancy of projections upon lines subject to angular shift; indeed the vector product in expressions like eq. (5) marks and measures the need of external control when moment of momentum remains in constant relation to shifting axes. For instance, the clearest reading of eq. (9) understands the directional changes recorded in the last two terms to be made at the expense of the associated magnitude-change (or *vice versa*), simply because no external source of moment of momentum is available. The two familiar constant values characteristic of this problem are then not similar in their foundation; the moment of momentum about (*OZ*) is constant because that axis is permanently parallel to the weight, while the axis (*OC*) only happens to be unique among the shifting axes in its self-compensated process of gain and loss, owing to the condition ( $A = B$ ). The changes resulting from external cause and those entailed by internal constraint are evidently assembled in eq. (7), whose general plan in the second member is then seen to be a segregation according to the change in (1) magnitude and (2) direction, occurring in the moment of momentum. Yet plain and simple as the guiding thought here proves to be, it has been found difficult to dispel the inherited confusion in which such forms of statement remain involved, as a consequence of continuing to overlook the real intention of equations of motion; and their confessed limitations; while we shuffle the terms in them with indifference to everything but the formal mathematical validity of the results. The vital importance of clear perceptions on this issue is our justification for dwelling a little upon it.<sup>1</sup>

In their primary sense, equations of motion express a quantitative equivalence (equality) between a net total of external agency (cause) applied to a body, and the response (effect) in changes of that body's

<sup>1</sup> We can note the "physical inversion" implied in Euler's "Zentrifugalmoment" without serious abatement in our appreciation of his inventive genius and general soundness. Nor are we moved to find fault with Coriolis writing in 1835, when he formulates his particular discovery in terms of "Force centrifuge composée." But we must experience disappointment on encountering in the best modern book on the gyroscope these survivals from the early tentative period still masquerading as pseudo-causes; and imbedded incongruously enough in a treatment whose main lines follow Kelvin's lucid dynamical thought. It is psychologically interesting to observe the persistent perversion due to so slight a turn as writing eq. (1) first

$$M_1 = A \frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3,$$

and then

$$M_1 + (B - C)\omega_2\omega_3 = A \frac{d\omega_1}{dt}.$$

state of motion. Conceived in such terms, this "equality of cause and effect" introduces no metaphysical obscurity; nor is it an obstacle to flexible adaptation in the later stage of calculation; but when accurately considered, it reveals two cardinal limitations in the scope of the equations. First the equality subsists between two summations which are only equivalent in their net totals and not identical or interchangeable piecemeal; as eq. (4) or eq. (8) exemplifies. And secondly the constituent items of each summation can be varied indefinitely without disturbing those totals. The reduction to resultant force and force-moment being founded essentially on the work-equation, we recognize readily the substitution of them for the actual distribution of force as being on a basis of limited equivalence. And this is matched in the second member by a wide range in the admissible plans of describing the effects. Both sides of the equation of motion being in these respects artificialized for convenience, we must not seek to extract from either in general a "one to one correspondence" with all details of the physical conditions. The occurrence of a term in the description implies nothing conclusive about a corresponding physical action; and every suggestion becomes illusory when we obliterate the distinctions through original entry under the two rubrics of cause and effect. So the few forms found helpful toward a natural interpretation become conspicuously prominent. The terms of eq. (4) accord with one most fundamental idea: That a rigid body distributes automatically the applied external force, assigning its quota to each mass-element, calculable in terms of local acceleration. The principle illustrated in application by eq. (8) covers the direct expression of force-moment through changes in moment of momentum. But the process of superposition must not be pushed beyond the limits of its validity. This question regarding eq. (10), for example, is empty of content: Do the last two terms assert that these specific elements of force-moment are present; or is this merely a roundabout method of effectively denying the presence of any such elements? Let us accept explicitly these limitations as they may have bearing upon the matters that follow, where an attempt is made at adding dynamical significance to some results that are ordinarily left in the stage of algebraic proof.

Return to eq. (8) and recast it into the form

$$A \frac{d^2\vartheta}{dt^2} = \sin \vartheta \{ -mg\bar{r} - C\omega_s \dot{\psi} + A\dot{\psi}^2 \cos \vartheta \}. \quad (12)$$

As we may see from the discussion of eq. (11), a zero value of this first member entails complete absorption of force-moment in causing purely directional changes of moment of momentum; and then adding the condi-

tion  $\dot{\vartheta} = 0$  steadies the motion into regular precession, with the available force-moment exactly adequate to the effects then necessary.<sup>1</sup> This doubly specialized adjustment is seen from eq. (12) to be in every case possible at  $\vartheta = 0$  or  $\pi$  for any finite factors within the parenthesis, and commonsense confirms the conclusion; since, though no weight-moment is available, neither is any then necessary, because the vertical ( $OZ$ ) has become both a principal axis and the axis of rotation. But there remain the adjustments (in a fuller sense) determined by a zero value of the parenthesis itself, with their possible assignment to real and imaginary regions according to the associated parameters in  $0 = f(\dot{\psi}, \vartheta)$ . Where an imaginary region occurs it is found to separate the two types of real solution, which coalesce when it disappears; so that considerations developed previously for the conical pendulum<sup>2</sup> apply in parallel here. But the less restricted problem of the gyroscope outruns that parallelism, notably in allowing a fast or a slow precession; two adjustment-values of  $\dot{\psi}$  corresponding to elements unvaried otherwise. We shall concern ourselves next with the dynamics underlying this possibility, whose equation of condition has been presented in the two forms:<sup>3</sup>

$$\dot{\psi} = \frac{C\dot{\varphi} \pm \sqrt{(C\dot{\varphi})^2 + 4mgr\bar{r}(A - C) \cos \vartheta}}{2(A - C) \cos \vartheta}$$

and

$$\dot{\psi} = \frac{C\omega_3 \pm \sqrt{(C\omega_3)^2 + 4mgr\bar{r}A \cos \vartheta}}{2A \cos \vartheta}. \quad (13)$$

Ordinarily the comparison of the roots ( $\dot{\psi}_1, \dot{\psi}_2$ ) is best attached to the second form, on account of the constant (and therefore common) value of  $(C\omega_3)$ . The other seems to seek a mathematical purity in being formally explicit, whereas  $C\omega_3 = C(\dot{\varphi} + \dot{\psi} \cos \vartheta)$ . But this advantage is illusory in practice, if we are obliged to make allowance for variation in  $(\dot{\varphi})$  in order to collate the two pairs of roots under most natural conditions. It will fit our present purpose, however, to note the emphasis of the first form in so far as it follows the superposition

$$\omega = \dot{\psi} + \dot{\varphi} + \dot{\vartheta}, \quad (14)$$

and the building up of moment of momentum by stages that match.

<sup>1</sup> Eq. (11) follows the thought more closely, if we suppress the first right-hand term. Such "equations of adjustment" (type  $Q = R$ ) are different from "equations of equilibrium" (type  $Q + R = 0$ ), however slight the mathematical barrier between them. This is sometimes overlooked; zero = parenthesis of eq. (12) being classed erroneously as a "condition of equilibrium."

<sup>2</sup> Slate, *PHYS. REV.*, Vol. 21, p. 166.

<sup>3</sup> The first is apparently peculiar to Klein and Sommerfeld (loc. cit., p. 178) among the prominent writers on the subject.

If positions of the figure-axis on the boundary of a quadrant are excluded, the vertical is not a principal axis; and consequently at the first stage the moment of momentum comprises a horizontal part ( $H = \dot{\psi}(A - C) \sin \vartheta \cos \vartheta$ ) in the plane ( $ZOC$ ) as well as the vertical part ( $V = I_z \dot{\psi}$ ). Stopping at precession-adjustment, the scheme is completed by adding the component round the figure-axis ( $C\dot{\varphi}$ ), and the alternative roots going with a common value of  $(C\omega_3)$  must in view of eq. (11) satisfy the relation

$$\begin{aligned} -mgr \sin \vartheta_1 &= [\dot{\psi}_1, I_z \dot{\psi}_1 + \mathbf{H}_1 + C\dot{\varphi}_1] = [\dot{\psi}_2, I_z \dot{\psi}_2 + \mathbf{H}_2 + C\dot{\varphi}_2]; \\ &= [\dot{\psi}_1, \mathbf{H}_1 + C\dot{\varphi}_1] = [\dot{\psi}_2, \mathbf{H}_2 + C\dot{\varphi}_2]. \end{aligned} \quad (15)$$

On the mathematical side, therefore, this double possibility of adjustment implies merely a numerical equivalence of vector products consistent with certain variations in the factors; though our physical analysis may profitably go one step further. A single type of case will illustrate the point of view sufficiently; it is not necessary to exhaust the algebraic combinations; so let us choose the factors under the radical in eq. (13) all positive, making the roots opposite in sign. Assume  $\dot{\psi}_1 > 0$ ;  $H_1 > 0$ ; beside  $C\omega_3 > 0$ . This gives  $\dot{\psi}_2 < 0$ ;  $H_2 < 0$ ;  $|\dot{\psi}_1| > |\dot{\psi}_2|$ ;  $|H_1| > |H_2|$ ;  $\dot{\varphi}_2 > \dot{\varphi}_1$ ; and both values positive in the last inequality agrees with ratios of magnitude found in actual problems. Then the conventions adopted already, with the standard rule governing vector products, lead for the two roots to the effective signs:

$$-mgr \sin \vartheta_1 = \begin{cases} -[\dot{\psi}_1 \mathbf{H}_1] + [\dot{\psi}_1, C\dot{\varphi}_1] \\ -[\dot{\psi}_2 \mathbf{H}_2] - [\dot{\psi}_2, C\dot{\varphi}_2]. \end{cases} \quad (16)$$

In connection with the relations of absolute magnitude indicated above, eq. (16) exhibits clearly how the same total force-moment can be partitioned differently between constituents at the two adjustments; so this at once enforces the ideas immediately preceding eq. (12) and also justifies an interpretation, valuable notwithstanding some limitations there set forth, which takes its place alongside those attached to eq. (4) and eq. (7). What reappears everywhere is the subjective core of the descriptive process; reconciling simultaneous constancy and change (by superposition), and treating different sets of components, each in its turn, as equally real content of the resultant.

Again we make eq. (8) the starting-point, this time in order to observe in its first member that the total work is being done continually about the  $A$ -axis, while its second member plainly involves the exhibition of resulting kinetic energy elsewhere than in connection with the  $\vartheta$ -coordinate. The fact that work can be thus "transferred between co-

ordinates," even though they are perpendicular, is at the root of the dynamic stability which becomes so prominent in the gyroscope; and the completeness or the rapidity of such transfer enables us in a way to measure that form of stability or to set the limits of its range. The expression for the total kinetic energy is

$$E = \frac{A}{2} (\dot{\vartheta}^2 + \omega_2^2) + C \frac{\omega_3^2}{2}; \quad (17)$$

and the last term being constant, the variations or interchanges consequent upon work done are confined to the two other terms. Now referring to eq. (9) it is apparent that the initiative (so to speak) centers in the component that is in the line of the external force-moment. So long as  $\dot{\vartheta} = 0$  no change can occur in  $(\omega_2)$ ; but the vanishing of  $(\omega_2, \omega_3)$  separately or simultaneously does not prevent changes in  $(\dot{\vartheta})$ . This discriminates effectively between stable equilibrium and the "stability" here; the latter depends vitally upon the entrance of (at least) incipient displacement, and is in this respect similar to the "stoppage of motion" (necessarily incomplete) by eddy currents. This feature of gyroscopic mechanisms is always emphasized—that their efficiency is nullified by removing the essential "degree of freedom."

But the influence of the variable components  $(\dot{\vartheta}, \omega_2)$  is nevertheless mutual; for we see from eq. (12) that the first member may be made positive or negative, with weight-moment given, by due assignment of the other elements. And in the next place it is clear that the absorption of kinetic energy *away from* the  $\vartheta$ -coördinate will be more rapid; and therefore completed within smaller displacement if continued; in proportion as that first member has large absolute magnitude, while the first time-rate of  $(\dot{\vartheta})$  and the second are opposite in sign. Hence a change of sign in the second time-rate is a critical boundary between conditions favorable and those unfavorable to that process of transfer (*i. e.*, to stability); and the critical equation marking the limit of favorable inequalities is

$$0 = -mg\bar{r} \sin \vartheta - C\omega_3\dot{\psi} \sin \vartheta + A\dot{\psi}^2 \cos \vartheta \sin \vartheta. \quad (18)$$

By way of illustration, apply this line of thought to a case of central importance; the examination of stability for a gyroscope with figure-axis pointing vertically upward, and passing through that position ( $\vartheta = -\pi$ ) with angular velocity  $\dot{\vartheta} > 0$ . Then if this angular velocity is not to increase, nor even to persist, but is to be vigorously "nipped in the bud,"  $d^2\vartheta/dt^2$ , though accurately zero at  $\vartheta = -\pi$ , must become (strongly) negative at departure from the vertical, and remain negative throughout



a sufficient interval. For  $\vartheta = -\pi + \Delta\vartheta$ , this leads obviously to the inequality, if we can assume  $\cos \vartheta = -1$ ,

$$0 > +mg\bar{r} + C\omega_3\dot{\psi} + A\dot{\psi}^2. \quad (19)$$

But at coincidence of figure-axis and upward vertical there is a well-known relation, taking the form for our conventions,

$$\dot{\psi} = -\frac{C\omega_3}{2A}; \quad (20)$$

and using this we may simplify inequality (19) into either

$$(C\omega_3)^2 > 4Amg\bar{r}; \quad \text{or} \quad A\dot{\psi}^2 > mg\bar{r}. \quad (21)$$

In rough summary the lesson here amounts to this: The demands of the directional changes in moment of momentum must develop more than rapidly enough to monopolize the available force-moment as the figure-axis leaves the vertical. Of course the same method can be adapted to less special values of  $(\vartheta)$ ; and it can be shown in detail that the energy-relations conform to the requirements outlined.<sup>1</sup> The present purpose is attained in suggesting these aspects of the phenomena. But as a closing word it may be well to remark that the "transfer of work" here spoken of is far from being peculiar to the gyroscope; and it has nothing fundamentally mysterious about it, being introduced as a possibility whenever the resultant force changes direction; as the resultant force-moment does here. It is instructive to consider in parallel the trajectory (weight being regarded as a constant vector) and a planetary orbit at the perihelion. The latter shows the entire kinetic energy gained since passage through the aphelion in connection with a direction instantaneously perpendicular to the resultant force. And with any central force, too, a certain constancy of moment of momentum is dominant, as it is in the gyroscope.

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<sup>1</sup> The point of view alone is claimed as novel. For the results, cf. Klein and Sommerfeld (loc. cit., pp. 249, 321) where they are extracted from the analytic discussion of the graph of a cubic equation.