

The Dynamics of a Stellar System. Third Paper: Oblate and other Distributions. By A. S. Eddington, M.A., M.Sc., F.R.S., Plumian Professor.

1. In a former paper* I have treated of a stellar system, having spherical symmetry, in a steady state of motion, and with a distribution of velocities according to Schwarzschild's ellipsoidal law. An exact solution of this problem was obtained, so that it was possible to specify the density and law of velocities at all parts of a system of this kind. In the present paper the restriction of spherical symmetry is removed, and the investigation is extended to oblate and other systems.

In the previous work the system was supposed to move under forces arising solely from its own attraction; but we shall now find it useful to keep distinct those results which are true when the system moves under any conservative system of forces, and the additional limitations imposed when those forces are identified with its own gravitational attractions. It is possible that the latter are not the only forces concerned; for example, L. V. King† has pointed out that H. S. Jones's value of the absorption of light leads to the conclusion that the quantity of residual gas in interstellar space is such as to produce a gravitational field very much greater than that of the whole system of lucid stars. Further, by postponing the assumption that there are no other forces, we can obtain results applicable to part of the system only, *e.g.* the stars of one particular mass; in this way we can better take into account the heterogeneity of the actual stellar system. Thus, except where expressly stated, we shall not introduce the limitation that the attraction of the system is the sole force concerned; and it will be seen that results of wide generality are obtained, which may be regarded as purely kinematical consequences of the ellipsoidal law of velocities.

The objects of the investigation are twofold:—

(1) We must determine the velocity-law at all parts of the system. It is *assumed* that in any small region the frequency of a velocity (u, v, w) is proportional to $e^{-a^2u^2-b^2v^2-c^2w^2}$ conformably to the ellipsoidal law, where the exponent defines an ellipsoid known as the velocity-ellipsoid; but the velocity-ellipsoid will in general vary in size, shape, and orientation from one part of the system to another, and we wish to determine these variations. The problem is to fit together velocity-ellipsoids varying from point to point of space in such a way as to give a steady state of motion.

(2) We must also determine the density, or number of stars per unit volume, at all parts of the system.

From the various possible solutions we have then to pick out

* The Second Paper, *M.N.*, vol. lxxv. p. 366.

† *Nature*, vol. xc. p. 701.

those which conform to certain facts derived from observation. The chief facts are:—

(1) In the neighbourhood of the Sun (which may be taken to be on the galactic plane at an unknown distance from the centre) the ratios of the axes of the velocity-ellipsoid are approximately 2 : 1 : 1.

(2) The density-distribution is very oblate. At the same distance from the centre towards the galactic poles and in the galactic plane respectively, the ratio of the densities is about 1 : 4 (average value).

The mathematical discussion depends on the Equation of Continuity, which expresses that the stars at any place have travelled in from neighbouring parts in the right numbers and with the right velocities to give the density and velocity distributions which exist there. The analysis employed differs from that of the previous paper, because when there is not spherical symmetry the equations of motion cannot be integrated; in some respects the new treatment will appear simpler, because it follows more closely the usual hydrodynamical methods.

As in my former work, the stars are supposed to move undisturbed by the chance arrangement of their immediate neighbours.*

Derivation of the Fundamental Equations.

2. At any point of the system the directions of the axes of the velocity-ellipsoid determine three directions at right angles. The velocity-ellipsoids thus define three orthogonal families of curves, each curve being traced by moving step by step always in the direction of an axis of the velocity-ellipsoid at the point reached. These curves may be regarded as the intersections of a triply orthogonal family of surfaces, which we shall call the *principal velocity-surfaces*. The axes of the velocity-ellipsoid at any point are normals to the three principal velocity-surfaces through that point.

We shall define our co-ordinates with reference to these surfaces, and take parameters λ , μ , ν of the three principal velocity-surfaces through any point as the curvilinear co-ordinates of that point.

Let the elements of length in the principal directions be

$$P d\lambda, \quad Q d\mu, \quad R d\nu$$

where P , Q , R are in general functions of λ , μ , ν .

Consider the motion of a particle under a gravitational field of potential $\phi(\lambda, \mu, \nu)$. We have, for the kinetic and potential energies,

$$T = \frac{1}{2}(P^2\lambda'^2 + Q^2\mu'^2 + R^2\nu'^2)$$

$$V = -\phi$$

where λ' stands for $d\lambda/dt$.

* *Stellar Movements and the Structure of the Universe*, pp. 247–255.

Then Lagrange's equations give

$$\frac{d}{dt}(P^2\lambda') = \frac{1}{2}\left(\frac{\partial P^2}{\partial \lambda}\lambda'^2 + \frac{\partial Q^2}{\partial \lambda}\mu'^2 + \frac{\partial R^2}{\partial \lambda}\nu'^2\right) + \frac{\partial \phi}{\partial \lambda}, \text{ etc.}$$

But
$$\frac{dP^2}{dt} = \frac{\partial P^2}{\partial \lambda}\lambda' + \frac{\partial P^2}{\partial \mu}\mu' + \frac{\partial P^2}{\partial \nu}\nu'.$$

Hence

$$\left. \begin{aligned} P^2\lambda'' &= -\frac{1}{2}\frac{\partial P^2}{\partial \lambda}\lambda'^2 + \frac{1}{2}\frac{\partial Q^2}{\partial \lambda}\mu'^2 + \frac{1}{2}\frac{\partial R^2}{\partial \lambda}\nu'^2 - \frac{\partial P^2}{\partial \mu}\lambda'\mu' - \frac{\partial P^2}{\partial \nu}\lambda'\nu' + \frac{\partial \phi}{\partial \lambda} \\ Q^2\mu'' &= \frac{1}{2}\frac{\partial P^2}{\partial \mu}\lambda'^2 - \frac{1}{2}\frac{\partial Q^2}{\partial \mu}\mu'^2 + \frac{1}{2}\frac{\partial R^2}{\partial \mu}\nu'^2 - \frac{\partial Q^2}{\partial \lambda}\lambda'\mu' - \frac{\partial Q^2}{\partial \nu}\mu'\nu' + \frac{\partial \phi}{\partial \mu} \\ R^2\nu'' &= \frac{1}{2}\frac{\partial P^2}{\partial \nu}\lambda'^2 + \frac{1}{2}\frac{\partial Q^2}{\partial \nu}\mu'^2 - \frac{1}{2}\frac{\partial R^2}{\partial \nu}\nu'^2 - \frac{\partial R^2}{\partial \lambda}\lambda'\nu' - \frac{\partial R^2}{\partial \mu}\mu'\nu' + \frac{\partial \phi}{\partial \nu} \end{aligned} \right\} (1)$$

These are the equations of motion.

3. According to the ellipsoidal law, the number of stars in an element of volume $dx dy dz$ with velocities between u, v, w and $u + du, v + dv, w + dw$, is

$$N = \sigma dx dy dz \pi^{-\frac{3}{2}} abc e^{-\frac{1}{2}u^2 - \frac{1}{2}v^2 - \frac{1}{2}w^2} du dv dw$$

where σ is the density and a^{-1}, b^{-1}, c^{-1} are the semiaxes of the velocity-ellipsoid; the axes of co-ordinates are taken along the principal axes of the ellipsoid.

In curvilinear co-ordinates, since $u = P\lambda'$, etc., this becomes

$$N = \pi^{-\frac{3}{2}} abc \sigma P d\lambda Q d\mu R d\nu P d\lambda' Q d\mu' R d\nu' \exp \left\{ -\frac{1}{2}P^2\lambda'^2 - \frac{1}{2}Q^2\mu'^2 - \frac{1}{2}R^2\nu'^2 \right\} \quad (2)$$

$$\log N = \log(abc\sigma) + \log(P^2Q^2R^2 d\lambda d\mu d\nu d\lambda' d\mu' d\nu') - \frac{1}{2}P^2\lambda'^2 - \frac{1}{2}Q^2\mu'^2 - \frac{1}{2}R^2\nu'^2 - \frac{3}{2}\log \pi \quad (3)$$

Now the equation of continuity is *

$$\frac{DN}{Dt} = 0 \quad (4)$$

which merely expresses that if we follow up a particular group of stars the number of them is constant, *i.e.* no stars are created or destroyed.

First consider
$$\frac{D}{Dt}(d\lambda d\mu d\nu d\lambda' d\mu' d\nu').$$

* Stokes's operator, $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu} + \lambda'' \frac{\partial}{\partial \lambda'} + \mu'' \frac{\partial}{\partial \mu'} + \nu'' \frac{\partial}{\partial \nu'}$ represents the rate of change of a property associated with a particular group of particles; $\frac{\partial}{\partial t}$, which represents the change of a property associated with a point of space, is always zero in our work, since the motion is steady. The functions a, b, c, P, Q, R, σ do not contain λ', μ', ν' , so that for them $\frac{D}{Dt} = \lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu}$.

If in a time δt the co-ordinates of a star change from λ to λ_1 , etc.,

$$\left. \begin{aligned} \lambda_1 &= \lambda + \lambda' \delta t \\ \lambda_1 &= \lambda' + \lambda'' \delta t, \quad \text{neglecting squares of } \delta t \end{aligned} \right\} \quad (5)$$

Thus the Jacobian

$$J = \frac{\partial(\lambda_1 \mu_1 \nu_1 \lambda_1' \mu_1' \nu_1')}{\partial(\lambda \mu \nu \lambda' \mu' \nu')}$$

becomes by (5)

$$\begin{vmatrix} 1 & 0 & 0 & \delta t & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta t & 0 \\ 0 & 0 & 1 & 0 & 0 & \delta t \\ \frac{\partial \lambda''}{\partial \lambda} \delta t & \frac{\partial \lambda''}{\partial \mu} \delta t & \frac{\partial \lambda''}{\partial \nu} \delta t & 1 + \frac{\partial \lambda''}{\partial \lambda'} \delta t & \frac{\partial \lambda''}{\partial \mu'} \delta t & \frac{\partial \lambda''}{\partial \nu'} \delta t \\ \frac{\partial \mu''}{\partial \lambda} \delta t & \frac{\partial \mu''}{\partial \mu} \delta t & \frac{\partial \mu''}{\partial \nu} \delta t & \frac{\partial \mu''}{\partial \lambda'} \delta t & 1 + \frac{\partial \mu''}{\partial \mu'} \delta t & \frac{\partial \mu''}{\partial \nu'} \delta t \\ \frac{\partial \nu''}{\partial \lambda} \delta t & \frac{\partial \nu''}{\partial \mu} \delta t & \frac{\partial \nu''}{\partial \nu} \delta t & \frac{\partial \nu''}{\partial \lambda'} \delta t & \frac{\partial \nu''}{\partial \mu'} \delta t & 1 + \frac{\partial \nu''}{\partial \nu'} \delta t \end{vmatrix}$$

To the first order in δt the determinant reduces to its leading diagonal, so that the Jacobian becomes

$$J = 1 + \delta t \left(\frac{\partial \lambda''}{\partial \lambda'} + \frac{\partial \mu''}{\partial \mu'} + \frac{\partial \nu''}{\partial \nu'} \right).$$

Now $d\lambda_1 d\mu_1 d\nu_1 d\lambda_1' d\mu_1' d\nu_1' = J d\lambda d\mu d\nu d\lambda' d\mu' d\nu'$,

and by definition it $= \left(1 + \delta t \cdot \frac{D}{Dt} \right) d\lambda d\mu d\nu d\lambda' d\mu' d\nu'$.

Hence

$$\left(\frac{\partial \lambda''}{\partial \lambda'} + \frac{\partial \mu''}{\partial \mu'} + \frac{\partial \nu''}{\partial \nu'} \right) d\lambda d\mu d\nu d\lambda' d\mu' d\nu' = \frac{D}{Dt} (d\lambda d\mu d\nu d\lambda' d\mu' d\nu'),$$

and therefore

$$\frac{D}{Dt} \log (d\lambda d\mu d\nu d\lambda' d\mu' d\nu') = \frac{\partial \lambda''}{\partial \lambda'} + \frac{\partial \mu''}{\partial \mu'} + \frac{\partial \nu''}{\partial \nu'}. \quad (6)$$

From (1)

$$\begin{aligned} \frac{\partial \lambda''}{\partial \lambda'} &= \frac{1}{P^2} \left(-\frac{\partial P^2}{\partial \lambda} \lambda' - \frac{\partial P^2}{\partial \mu} \mu' - \frac{\partial P^2}{\partial \nu} \nu' \right) \\ &= -\frac{1}{P^2} \frac{DP^2}{Dt} = -\frac{D}{Dt} \log P^2. \end{aligned}$$

Similarly

$$\frac{\partial \mu''}{\partial \mu'} = -\frac{D}{Dt} \log Q^2; \quad \frac{\partial \nu''}{\partial \nu'} = -\frac{D}{Dt} \log R^2.$$

Hence (6) gives

$$\frac{D}{Dt} \log (P^2 Q^2 R^2 d\lambda d\mu d\nu d\lambda' d\mu' d\nu') = 0. \quad (7)$$

Returning to equations (3) and (4), we obtain in consequence of (7)

$$\frac{D}{Dt} \log (abc\sigma) = \frac{D}{Dt} (a^2 P^2 \lambda'^2 + b^2 Q^2 \mu'^2 + c^2 R^2 \nu'^2) \quad (8)$$

or, setting $abc\sigma = e^\kappa$ (8a)

$$\begin{aligned} \frac{\partial \kappa}{\partial \lambda} \lambda' + \frac{\partial \kappa}{\partial \mu} \mu' + \frac{\partial \kappa}{\partial \nu} \nu' &= \frac{\partial(a^2 P^2)}{\partial \lambda} \lambda'^3 + \frac{\partial(a^2 P^2)}{\partial \mu} \lambda'^2 \mu' + \frac{\partial(a^2 P^2)}{\partial \nu} \lambda'^2 \nu' \\ &\quad + \frac{\partial(b^2 Q^2)}{\partial \lambda} \lambda' \mu'^2 + \frac{\partial(b^2 Q^2)}{\partial \mu} \mu'^3 + \frac{\partial(b^2 Q^2)}{\partial \nu} \mu'^2 \nu' \\ &\quad + \frac{\partial(c^2 R^2)}{\partial \lambda} \lambda' \nu'^2 + \frac{\partial(c^2 R^2)}{\partial \mu} \mu' \nu'^2 + \frac{\partial(c^2 R^2)}{\partial \nu} \nu'^3 \\ &+ 2a^2 \left(-\frac{1}{2} \frac{\partial P^2}{\partial \lambda} \lambda'^3 + \frac{1}{2} \frac{\partial Q^2}{\partial \lambda} \lambda' \mu'^2 + \frac{1}{2} \frac{\partial R^2}{\partial \lambda} \lambda' \nu'^2 - \frac{\partial P^2}{\partial \mu} \lambda'^2 \mu' - \frac{\partial P^2}{\partial \nu} \lambda'^2 \nu' + \frac{\partial \phi}{\partial \lambda} \lambda' \right) \\ &+ 2b^2 \left(\frac{1}{2} \frac{\partial P^2}{\partial \mu} \lambda'^2 \mu' - \frac{1}{2} \frac{\partial Q^2}{\partial \mu} \mu'^3 + \frac{1}{2} \frac{\partial R^2}{\partial \mu} \mu' \nu'^2 - \frac{\partial Q^2}{\partial \lambda} \lambda' \mu'^2 - \frac{\partial Q^2}{\partial \nu} \mu'^2 \nu' + \frac{\partial \phi}{\partial \mu} \mu' \right) \\ &+ 2c^2 \left(\frac{1}{2} \frac{\partial P^2}{\partial \nu} \lambda'^2 \nu' + \frac{1}{2} \frac{\partial Q^2}{\partial \nu} \mu'^2 \nu' - \frac{1}{2} \frac{\partial R^2}{\partial \nu} \nu'^3 - \frac{\partial R^2}{\partial \lambda} \lambda' \nu'^2 - \frac{\partial R^2}{\partial \mu} \mu' \nu'^2 + \frac{\partial \phi}{\partial \nu} \nu' \right) \quad (9) \end{aligned}$$

where in the last three lines the values of $\frac{D\lambda'}{Dt}$ (or λ''), etc., have been inserted from the equations of motion (1).

Equation (9) has to be satisfied for all values of λ' , μ' , ν' ; we therefore equate coefficients on both sides.

The coefficient of λ'^3 gives

$$\frac{\partial(a^2 P^2)}{\partial \lambda} - a^2 \frac{\partial P^2}{\partial \lambda} = 0,$$

whence $\frac{\partial a^2}{\partial \lambda} = 0$ (10)

The coefficient of $\lambda' \mu'^2$ gives

$$\frac{\partial(b^2 Q^2)}{\partial \lambda} + a^2 \frac{\partial Q^2}{\partial \lambda} - 2b^2 \frac{\partial Q^2}{\partial \lambda} = 0,$$

therefore $(a^2 - b^2) \frac{\partial Q^2}{\partial \lambda} = -Q^2 \frac{\partial b^2}{\partial \lambda}$
 $= Q^2 \frac{\partial(a^2 - b^2)}{\partial \lambda}$ by (10)

Hence $\frac{\partial}{\partial \lambda} \left(\frac{a^2 - b^2}{Q^2} \right) = 0$ (11)

The coefficient of λ' gives $\frac{\partial \kappa}{\partial \lambda} = 2a^2 \frac{\partial \phi}{\partial \lambda}$ (12)

Collecting these and the symmetrical results, we obtain twelve equations, viz.

$$\left. \begin{aligned} \frac{\partial a^2}{\partial \lambda} &= 0 & \frac{\partial b^2}{\partial \mu} &= 0 & \frac{\partial c^2}{\partial \nu} &= 0 \\ \frac{\partial}{\partial \mu} \left(\frac{b^2 - c^2}{R^2} \right) &= 0 & \frac{\partial}{\partial \nu} \left(\frac{c^2 - a^2}{P^2} \right) &= 0 & \frac{\partial}{\partial \lambda} \left(\frac{a^2 - b^2}{Q^2} \right) &= 0 \\ \frac{\partial}{\partial \nu} \left(\frac{b^2 - c^2}{Q^2} \right) &= 0 & \frac{\partial}{\partial \lambda} \left(\frac{c^2 - a^2}{R^2} \right) &= 0 & \frac{\partial}{\partial \mu} \left(\frac{a^2 - b^2}{P^2} \right) &= 0 \\ \frac{\partial \phi}{\partial \lambda} &= \frac{1}{2a^2} \frac{\partial \kappa}{\partial \lambda} & \frac{\partial \phi}{\partial \mu} &= \frac{1}{2b^2} \frac{\partial \kappa}{\partial \mu} & \frac{\partial \phi}{\partial \nu} &= \frac{1}{2c^2} \frac{\partial \kappa}{\partial \nu} \end{aligned} \right\} \quad (13)$$

The first nine equations of (13) enable us to find values of a , b , c appropriate to the curvilinear co-ordinates chosen, and so to determine the velocity-law consistent with a steady state of motion. It will be noticed that they involve neither the potential of the field of force nor the density.

The last three equations determine the density (involved in κ) when the field of force is given. If the force is due solely to the attractions of the stars themselves, an additional condition, viz. Poisson's equation, must be satisfied.

The Distribution of Velocities.

4. From the first nine equations of (13) an important general theorem can be derived.

The principal velocity-surfaces must be confocal quadrics. The general proof is rather difficult, and is given in an Appendix to this paper. We shall here give a proof for the most interesting case—that of axial symmetry,—which is much simpler.

When there is axial symmetry, two of the families of surfaces will consist of surfaces of revolution, and the third family, which cuts them orthogonally, must be the planes through the axis. We may take the co-ordinate ν corresponding to these planes to be the azimuthal angle ϕ ; and all the quantities concerned must then, owing to the symmetry, be independent of ν .

Taking two of the equations of (13),

$$\frac{\partial}{\partial \lambda} \left(\frac{a^2 - b^2}{Q^2} \right) = 0 \quad \frac{\partial}{\partial \mu} \left(\frac{a^2 - b^2}{P^2} \right) = 0,$$

these give on integration

$$a^2 - b^2 = Q^2 f_1(\mu) = P^2 f_2(\lambda) \quad (14)$$

where f_1 and f_2 are unknown functions.

Now, if (ϖ, z, ϕ) are cylindrical co-ordinates, the definitions of P and Q give

$$P^2 = \left(\frac{\partial \varpi}{\partial \lambda} \right)^2 + \left(\frac{\partial z}{\partial \lambda} \right)^2$$

$$Q^2 = \left(\frac{\partial \varpi}{\partial \mu} \right)^2 + \left(\frac{\partial z}{\partial \mu} \right)^2.$$

Introduce new parameters λ_0, μ_0 defined by

$$\lambda_0 = \int \frac{d\lambda}{\sqrt{f_2(\lambda)}} \quad \mu_0 = \int \frac{d\mu}{\sqrt{f_1(\mu)}}.$$

Then
$$\frac{\partial \varpi}{\partial \lambda_0} = \sqrt{f_2(\lambda)} \frac{\partial \varpi}{\partial \lambda}, \quad \frac{\partial z}{\partial \lambda_0} = \sqrt{f_2(\lambda)} \frac{\partial z}{\partial \lambda}, \text{ etc.,}$$

so that

$$\begin{aligned} \left(\frac{\partial \varpi}{\partial \lambda_0}\right)^2 + \left(\frac{\partial z}{\partial \lambda_0}\right)^2 &= P^2 f_2(\lambda) \\ &= Q^2 f_1(\mu) \quad \text{by (14)} \\ &= \left(\frac{\partial \varpi}{\partial \mu_0}\right)^2 + \left(\frac{\partial z}{\partial \mu_0}\right)^2 \end{aligned}$$

Since λ_0 is a function of λ only, the surfaces $\lambda_0 = \text{const.}$ are identical with the surfaces $\lambda = \text{const.}$ Hence λ_0 can be used instead of λ as the parameter of the principal velocity-surfaces; so that (λ_0, μ_0, ν) could have been taken as our co-ordinates instead of (λ, μ, ν) .^{*} We shall suppose that λ_0, μ_0 have been used *ab initio*, and drop the suffixes as no longer necessary.

We have then

$$\left(\frac{\partial \varpi}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 = \left(\frac{\partial \varpi}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2.$$

This is the condition that the orthogonal transformation ϖ, z to λ, μ may be *conformal*. Hence, by the properties of conformal transformations,

$$\varpi + \iota z = f(\lambda + \iota \mu), \quad \text{where } \iota = \sqrt{-1}$$

and

$$\begin{aligned} P^2 = Q^2 &= \left(\frac{\partial \varpi}{\partial \lambda} + \iota \frac{\partial z}{\partial \lambda}\right) \left(\frac{\partial \varpi}{\partial \lambda} - \iota \frac{\partial z}{\partial \lambda}\right) \\ &= f'(\lambda + \iota \mu) f'(\lambda - \iota \mu). \end{aligned}$$

Again, by (14), since P^2 is now equal to Q^2 ,

$$\begin{aligned} a^2 - b^2 &= CP^2 \\ &= Cf'(\lambda + \iota \mu) f'(\lambda - \iota \mu), \end{aligned}$$

where C is a constant.

Operate on both sides by $\frac{\partial^2}{\partial \lambda \partial \mu}$. By (13) $\frac{\partial a^2}{\partial \lambda} = 0$, $\frac{\partial b^2}{\partial \mu} = 0$, so that the left side is annihilated. We obtain

$$\iota f'''(\lambda + \iota \mu) f'(\lambda - \iota \mu) - \iota f'''(\lambda - \iota \mu) f'(\lambda + \iota \mu) = 0.$$

Hence

$$\frac{f'''(\lambda + \iota \mu)}{f'(\lambda + \iota \mu)} = \frac{f'''(\lambda - \iota \mu)}{f'(\lambda - \iota \mu)}.$$

^{*} In defining λ, μ, ν (p. 38) we only stated that they must be parameters of the surfaces; the particular parameter to be used was left optional.

The two sides involve different variables $\lambda + \iota\mu$ and $\lambda - \iota\mu$; since these may vary independently, each side must be constant and equal to p^2 say. Thus

$$f'''(\lambda + \iota\mu) = p^2 f'(\lambda + \iota\mu).$$

This is a linear differential equation of which the complete solution is

$$\varpi + \iota z = f(\lambda + \iota\mu) = A \cosh p(\lambda + \iota\mu + \epsilon) + B. \quad (15)$$

where A, B, ϵ are constants of integration.

As is well known, the transformation (15) makes the curves $\lambda = \text{const.}$, $\mu = \text{const.}$ confocal conics; and the principal velocity-surfaces corresponding to λ and μ are formed by revolving these confocal conics about the z axis. The latter must evidently coincide with a principal axis of the conics, since otherwise a velocity-surface would intersect itself. This establishes the theorem.

In particular cases the quadrics may degenerate into spheres, cylinders, or planes.

An evident corollary is that if the system possesses a *plane of symmetry* the sections of the principal velocity-surfaces by that plane must be confocal conics. The plane of symmetry must be a principal velocity-surface, corresponding, say, to $\nu = \text{const.}$, and the argument applies unaltered so long as ν is not varied.

From these two results it appeared probable that the only possible forms of principal velocity-surfaces were confocal quadrics, and I was led to work out a general proof of this theorem, which is given in the Appendix (p. 54).

5. One of the families of principal velocity-surfaces consists of confocal ellipsoids. Take these to correspond to the λ co-ordinate. When we pass outwards along a curve orthogonal to these surfaces, $\mu = \text{const.}$, $\nu = \text{const.}$ Integrating

$$\frac{\partial \alpha^2}{\partial \lambda} = 0, \quad \frac{\partial}{\partial \lambda} \left(\frac{\alpha^2 - b^2}{Q^2} \right) = 0, \quad \text{and} \quad \frac{\partial}{\partial \lambda} \left(\frac{\alpha^2 - c^2}{R^2} \right) = 0, \quad \text{we have}$$

$$\begin{aligned} \alpha^2 &= A \\ b^2 &= A + BQ^2 \\ c^2 &= A + CR^2 \end{aligned}$$

where A, B, C are constants for the curve.

Now Q and R must increase outwards without limit, since they measure the divergence of consecutive λ curves. Evidently therefore B and C must be taken positive; otherwise b^2 and c^2 would vanish at some point, and the corresponding velocities become infinite.

It follows that b^2 and c^2 are greater than α^2 , so that the greatest axis of the velocity-ellipsoid is in the λ direction, *i.e.* more or less radial. The velocities in this—the star-streaming direction—are of the same magnitude in the outer parts of the system as near the centre, whereas the transverse velocities diminish according to the law $(A + BQ^2)^{-\frac{1}{2}}$ and $(A + CR^2)^{-\frac{1}{2}}$, which may be regarded as a

generalisation of the result $(A + Br^2)^{-\frac{1}{2}}$ found for the sphere in the previous paper.

6. Having shown that the principal velocity-surfaces are confocal quadrics, we now know the values of P, Q, R . The equations (13) are then easily solved. The general case is (f) below, but the particular cases when the quadrics degenerate into spheres, spheroids, etc., are especially useful. Only the results are given except for (d), which is worked out to show the method.

(a) *Rectangular co-ordinates* (x, y, z).

$$P = 1 \quad Q = 1 \quad R = 1.$$

We find a^2, b^2, c^2 are constants.

(b) *Cylindrical co-ordinates* (ϖ, z, ϕ).

$$P = 1 \quad Q = 1 \quad R = \varpi$$

$$a^2 = \text{const.} \quad b^2 = \text{const.} \quad c^2 = a^2(1 + \varpi^2).^*$$

(c) *Spherical co-ordinates* (r, θ, ϕ).

$$P = 1 \quad Q = r \quad R = r \sin \theta$$

$$a^2 = \text{const.} \quad b^2 = a^2(1 + r^2) \quad c^2 = a^2(1 + r^2 + Cr^2 \sin^2 \theta).$$

(d) *Prolate spheroidal co-ordinates* (ξ, η, γ).

$$x = \sinh \xi \sin \eta \cos \gamma \quad y = \sinh \xi \sin \eta \sin \gamma \quad z = \cosh \xi \cos \eta$$

$$P^2 = Q^2 = \cosh^2 \xi - \cos^2 \eta \quad R^2 = \sinh^2 \xi \sin^2 \eta.$$

From (13) we have, since $P^2 = Q^2$,

$$\frac{\partial}{\partial \xi} \left(\frac{a^2 - b^2}{P^2} \right) = 0 \quad \frac{\partial}{\partial \eta} \left(\frac{a^2 - b^2}{P^2} \right) = 0$$

$$\frac{\partial}{\partial \gamma} \left(\frac{a^2 - b^2}{P^2} \right) = - \frac{\partial}{\partial \gamma} \left(\frac{b^2 - c^2}{P^2} \right) - \frac{\partial}{\partial \gamma} \left(\frac{c^2 - a^2}{P^2} \right) = 0.$$

Hence, integrating

$$a^2 - b^2 = -BP^2 = -B(\cosh^2 \xi - \cos^2 \eta)$$

where $-B$ is a constant of integration.

Since $\partial a^2 / \partial \xi = 0 \quad \partial b^2 / \partial \eta = 0$,
this gives $a^2 = A + B \cos^2 \eta \quad b^2 = A + B \cosh^2 \xi$.

By similar use of the remaining equations it can be shown that

$$c^2 = A + B \sinh^2 \xi + B \cos^2 \eta + C \sinh^2 \xi \sin^2 \eta.$$

(e) *Oblate co-ordinates* (ξ, η, γ).

$$x = \cosh \xi \cos \eta \cos \gamma \quad y = \cosh \xi \cos \eta \sin \gamma \quad z = \sinh \xi \sin \eta$$

$$P^2 = Q^2 = \cosh^2 \xi - \cos^2 \eta \quad R^2 = \cosh^2 \xi \cos^2 \eta$$

$$a^2 = A + B \cos^2 \eta \quad b^2 = A + B \cosh^2 \xi$$

$$c^2 = A + B \cosh^2 \xi + B \cos^2 \eta + C \cosh^2 \xi \cos^2 \eta.$$

(f) *General ellipsoidal co-ordinates* (λ, μ, ν).

* We omit such constants as may be absorbed by changing the unit of measurement.

The squares of the semiaxes of the confocals through (λ, μ, ν) being

$$\begin{aligned} & \lambda^2, \lambda^2 - \beta^2, \lambda^2 - \gamma^2; \quad \mu^2, \mu^2 - \beta^2, \mu^2 - \gamma^2; \quad \nu^2, \nu^2 - \beta^2, \nu^2 - \gamma^2 \\ P^2 &= \frac{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}{(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)} \quad Q^2 = \frac{(\mu^2 - \nu^2)(\lambda^2 - \mu^2)}{(\mu^2 - \beta^2)(\gamma^2 - \mu^2)} \quad R^2 = \frac{(\lambda^2 - \nu^2)(\mu^2 - \nu^2)}{(\beta^2 - \nu^2)(\gamma^2 - \nu^2)} \\ \alpha^2 &= A + B\mu^2 + B\nu^2 + C\mu^2\nu^2 \quad b^2 = A + B\nu^2 + B\lambda^2 + C\nu^2\lambda^2 \\ c^2 &= A + B\lambda^2 + B\mu^2 + C\lambda^2\mu^2. \end{aligned}$$

The Distribution of Density.

7. When the field of force is given, the last three equations of (13) determine the density. But the force must satisfy a certain condition in order that the ellipsoidal law may be possible. We have by the last three equations of (13)

$$a^2 \frac{\partial \phi}{\partial \lambda} d\lambda + b^2 \frac{\partial \phi}{\partial \mu} d\mu + c^2 \frac{\partial \phi}{\partial \nu} d\nu = \frac{1}{2} \frac{\partial \kappa}{\partial \lambda} d\lambda + \frac{1}{2} \frac{\partial \kappa}{\partial \mu} d\mu + \frac{1}{2} \frac{\partial \kappa}{\partial \nu} d\nu = \frac{1}{2} d\kappa \quad (16)$$

so that ϕ must be of a form which makes the left side of (16) a perfect differential. This requires that

$$\frac{\partial}{\partial \mu} \left(a^2 \frac{\partial \phi}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} \left(b^2 \frac{\partial \phi}{\partial \mu} \right), \text{ etc.}$$

Hence
$$\frac{\partial a^2}{\partial \mu} \frac{\partial \phi}{\partial \lambda} - \frac{\partial b^2}{\partial \lambda} \frac{\partial \phi}{\partial \mu} + (a^2 - b^2) \frac{\partial^2 \phi}{\partial \lambda \partial \mu} = 0.$$

Or, since
$$\frac{\partial a^2}{\partial \lambda} = 0, \quad \frac{\partial b^2}{\partial \mu} = 0,$$

$$\phi \frac{\partial^2}{\partial \lambda \partial \mu} (a^2 - b^2) + \frac{\partial(a^2 - b^2)}{\partial \mu} \frac{\partial \phi}{\partial \lambda} + \frac{\partial(a^2 - b^2)}{\partial \lambda} \frac{\partial \phi}{\partial \mu} + (a^2 - b^2) \frac{\partial^2 \phi}{\partial \lambda \partial \mu} = 0.$$

Therefore
$$\frac{\partial^2}{\partial \lambda \partial \mu} \{ (a^2 - b^2) \phi \} = 0 \quad (17)$$

Consider the case of axial symmetry, so that the quantities are independent of ν . Integrating (17)

$$\phi = \frac{f(\lambda) + g(\mu)}{a^2 - b^2} \quad (18)$$

where f and g are arbitrary functions.

The value of κ is then

$$\kappa = 2 \cdot \frac{a^2 f(\lambda) + b^2 g(\mu)}{a^2 - b^2} \quad (19)$$

For, writing this in the form

$$\kappa = 2a^2 \frac{f(\lambda) + g(\mu)}{a^2 - b^2} - 2g(\mu),$$

we see that $\frac{\partial \kappa}{\partial \lambda} = 2a^2 \frac{\partial \phi}{\partial \lambda}$; and similarly $\frac{\partial \kappa}{\partial \mu} = 2b^2 \frac{\partial \phi}{\partial \mu}$ is satisfied.

σ is then given by equation (8a)

$$\kappa = \log(abc\sigma) \quad . \quad . \quad . \quad (20)$$

When the system moves under its own attraction only, Poisson's equation $\nabla^2\phi = -4\pi\sigma$ must also be satisfied. Substituting for ϕ and σ we have in this case

$$\nabla^2 \left\{ \frac{f(\lambda) + g(\mu)}{a^2 - b^2} \right\} = -\frac{4\pi}{abc} \exp. \left\{ 2 \frac{a^2 f(\lambda) + b^2 g(\mu)}{a^2 - b^2} \right\}.$$

It is easy to prove that this cannot be exactly satisfied for spheroidal co-ordinates. In fact, *the only possible solution is the case of spherical symmetry* (excluding a very artificial case of two-dimensional motion with circular symmetry). It may conveniently be added here that a superimposed rotation (considered in § 9) does not lead to any further solution.

If, on the other hand, the motion takes place under the attraction of a controlling distribution whose potential is of the form (18), there are no further conditions to be satisfied; solutions for spheroidal and ellipsoidal velocity-surfaces can be obtained. It seems best to assume that the controlling system has spherical symmetry, for otherwise there remains the awkward question how its own equilibrium is maintained; but the controlled system can take an oblate or prolate form.

8. As a first example, take the case of spherical co-ordinates, § 6 (c). Equations (18) and (19) become

$$\phi = \frac{f(r) + g(\theta)}{r^2} \quad \kappa = 2a^2 \frac{f(r) + (1+r^2)g(\theta)}{r^2}.$$

Unless $g(\theta)$ is a constant, ϕ , κ , and σ become infinite and (worse still) physically discontinuous at the centre. Thus the only practical solution is when κ is a function of r only, and therefore ϕ is a function of r only.

If the system moves under its own attraction, it follows that σ is a function of r only, and we have the case of complete spherical symmetry treated in the Second Paper (the formulæ of that paper follow immediately). But if the system moves under more general forces

$$abc\sigma = e^\kappa = \text{a function of } r \text{ only,}$$

and since

$$c^2 = 1 + r^2 + Cr^2 \sin^2 \theta, \quad \sigma \text{ will vary with } \theta \text{ unless } C \text{ is zero.}$$

Thus even when the principal velocity-surfaces are spheres we can have a prolate or oblate distribution of the stars, provided that the system is controlled by the attraction of a globular distribution of matter. Theoretically the degree of oblateness might be as great as we please, but the observational results (p. 38) require that $b^2 \simeq c^2$ in the neighbourhood of the Sun ($r = \sqrt{3}$), so that C must be taken very small, and the distribu-

tion of density will then differ only slightly from spherical symmetry.*

Before passing on to consider spheroidal velocity-surfaces, we shall examine whether it is possible to obtain greater oblateness by imposing a rotation of the whole system about the galactic axis.

9. *Effect of a Superposed Rotation.*—Assuming axial symmetry, let ν be the azimuthal angle ϕ , and let ω be the angular velocity of rotation, which will in general be a function of λ and μ . We must then replace ν' by $(\nu' - \omega)$ in Schwarzschild's ellipsoidal formula, so that (2) becomes

$$N = \pi^{-\frac{3}{2}} abc\sigma P^2 Q^2 R^2 d\lambda d\mu d\nu d\lambda' d\mu' d\nu' \exp \left\{ -a^2 P^2 \lambda'^2 - b^2 Q^2 \mu'^2 - c^2 R^2 (\nu' - \omega)^2 \right\}$$

and (8) becomes

$$\frac{D}{Dt} \log (abc\sigma) = \frac{D}{Dt} \{ a^2 P^2 \lambda'^2 + b^2 Q^2 \mu'^2 + c^2 R^2 \nu'^2 - 2c^2 R^2 \nu' \omega + c^2 R^2 \omega^2 \}.$$

Consider the additional term $D(2c^2 R^2 \nu' \omega)/Dt$. By Lagrange's equations $D(R^2 \nu')/Dt = 0$ for axial symmetry. Hence

$$\frac{D}{Dt} (2c^2 R^2 \nu' \omega) = 2R^2 \nu' \left(\lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} \right) (c^2 \omega).$$

This addition, being of the second degree in the velocities, will not combine with any of the other terms in (9) and must vanish independently of them. Thus $c^2 \omega$ must be a constant, equal to Ω , say. Thus

$$\omega = \Omega/c^2 \quad (21)$$

The other additional term $\exp(c^2 R^2 \omega^2) = \exp(R^2 \Omega^2/c^2)$ does not contain the velocities and can be incorporated in κ .

Thus, without altering any of the formulæ (13), a rotation varying as c^{-2} can be superimposed, but in place of (20) we have

$$\log (abc\sigma) = \kappa + R^2 \Omega^2/c^2.$$

10. Applying the last result to the case of spherical co-ordinates considered in § 8,

$$\log (abc\sigma) = \kappa + \frac{\Omega^2 r^2 \sin^2 \theta}{a^2 (1 + r^2 + Cr^2 \sin^2 \theta)} \quad (22)$$

and κ has been shown to be a function of r only.

We have seen that C must be small if not zero; we shall therefore take it to be zero.

Comparing the density σ_θ with the density σ_0 at the same distance along the galactic axis, equation (22) gives

$$\log \frac{\sigma_\theta}{\sigma_0} = \frac{\Omega^2}{a^2} \cdot \frac{r^2}{1 + r^2} \sin^2 \theta.$$

* A large value of C is also objectionable on theoretical grounds (see p. 51).

This increases as θ increases from 0° to 90° , so that the distribution is oblate; and the oblateness may be made as large as we please by taking Ω sufficiently great.

To obtain an estimate of the necessary amount of rotation, we use the result that the number of faint stars in the galactic plane is about four times as great as along the galactic axis. Thus for large values of r , $\sigma_{90}/\sigma_0 \simeq 4$. Hence

$$\Omega = a \sqrt{\log 4}$$

and

$$\omega = \frac{\sqrt{\log 4}}{a(1+r^2)}.$$

As in the Second Paper, the unit of r is $1/\sqrt{3}$ times the Sun's distance from the centre, and $a=1$ when the unit of velocity is about 50 km. per sec. The stars near the Sun are therefore in the mean moving round the galactic axis with a linear velocity.

$$\begin{aligned} &= \sqrt{3} \omega = \frac{\sqrt{3 \log 4}}{4} \times 50 \text{ km. per sec.} \\ &= 25 \text{ km. per sec.} \end{aligned}$$

Unless the Sun is a great distance from the centre of the system, a rotation so large as this could scarcely have escaped observation. One circumstance might, however, tend to conceal it; the linear velocity of rotation, varying with $r/(1+r^2)$, reaches a maximum of less than 30 km. per sec. at $r=1$. The law of angular velocity is, in fact, such as to give comparatively little differential motion within, say, unit distance of the Sun.

Prolate Velocity-Surfaces.

11. We shall now consider the case of prolate co-ordinates (§ 6, *d*)—the most important case, because it gives a model of the stellar system agreeing satisfactorily with observation in most respects. It may seem paradoxical that in order to obtain an *oblate* distribution of stars we should take *prolate* velocity-surfaces; but the reason is clear. The normal to a prolate spheroid is a compromise between the direction towards the centre and the direction parallel to the galactic plane (the axis of symmetry being towards the galactic poles). We thus secure that the preferential motion is more nearly parallel to the galactic plane than in the case of spherical symmetry, and this tends to keep the stars closer to the plane. With oblate velocity-surfaces, the directions of preferential motion would be diverted away from the plane, and the stars would tend to spread out at right angles to it.

We shall assume, as before, that the stars move under a field of force due to a globular controlling distribution, and further, that

Unless C is large, the system will be oblate right up to the centre. We have, of course, no observational knowledge as to whether the central parts are oblate or not.

There is one difficulty in making these formulæ fit the facts of observation which perhaps tells in favour of spherical instead of prolate velocity-surfaces. For stars near the Sun the observations give $b^2 \simeq c^2 \simeq 4a^2$. If r_0 is the distance of the Sun from the centre (the unit of distance being half the focal line of the confocals), this gives

$$A + B + Br_0^2 \simeq A + Br_0^2 + Cr_0^2 \simeq 4A.$$

It is improbable that C is other than a very small quantity; for a large value of C would mean that c^2 becomes greater than b^2 in the outer parts of the system, *i.e.* the motion would tend to take place mainly in planes through the galactic axis. I can see no way in which such an unnatural arrangement could have arisen. But if C is small it follows that r_0 must be rather large, and the confocal surfaces at and beyond the Sun are beginning to approximate to spheres. By giving a large numerical value to r_0 (diminishing the unit of distance) we are restricting the space in which the velocity surfaces are sensibly elliptic and tending to revert to the case of spherical velocity-surfaces. Perhaps, however, we are laying too much stress on the approximate equality of b^2 and c^2 ; the result comes from averaging stellar velocities distributed over a large volume of space, and may not strictly represent the velocity-ellipsoid at a *point*. Moreover, it may be significant that for type A0, for which the oblateness is especially pronounced, the equality does not hold, and both C and r_0 may be taken small.

In all my formulæ the falling off of density outwards appears, at least at first sight, to be too rapid to be reconciled with observation. By (25) the density in the galactic plane falls off at least as rapidly as $\exp(-2EA r^2)$; actually there is another factor which makes the diminution still more rapid. This can be interpreted to mean that the stars thin out in the ratio e^{-1} when we reach a distance r such that the velocity in a circular orbit of radius r in the controlling medium is equal to $\sqrt{\pi}$ times the average velocity of the stars resolved in the star-streaming direction. In the Second Paper I took this average velocity to be 28 km. per sec., which multiplied by $\sqrt{\pi}$ gives 50 km. per sec. The latter was taken as the unit of velocity, and with this unit $A = a^2 = 1$. I do not think we can take the distance at which the falling off of density e^{-1} occurs in the galactic plane to be less than 800 parsecs. These numbers give for the density of the controlling medium 110 times the Sun's mass in a sphere of five parsecs radius. This is perilously near the minimum estimate of density of the lucid stars, *viz.* 10 times the Sun's mass in the same sphere.

To meet this difficulty it may be pointed out that the heterogeneity of the stellar system affects this problem considerably. The average speed of 28 kilometres per second depends on the more luminous stars, and it is possible that these may thin out

rather rapidly. There exists also a class of intrinsically faint stars with much higher speeds, which, according to the formulæ, will be more extended in distribution. This consideration permits us to increase the calculated density of the controlling medium perhaps 4 or 5 times. Again, we have taken the controlling distribution to be of uniform density; if, however, it decreases outwards, the stellar density will not diminish so fast.

The reason why our formulæ limit the number of distant stars is that a certain proportion of them must have orbits passing near the centre, and these will acquire great velocities in their fall. Thus the number of stars allowable in the outer parts depends on the proportion of great velocities near the centre. On almost any theory relating to an approximately steady state a correspondence of this kind would occur; and it may be noted that the time required to establish a rough equilibrium relation between the inner and outer parts is probably not greater than 100 million years. If we could assert definitely that there are too many distant stars to correspond to the observed motions at the centre, it would, I think, tend to show that the system is undergoing a very rapid collapse or dispersal.

However, on the whole, it seems possible to represent the observed results with about as good a quantitative agreement as could be expected from a theory, which necessarily omits some of the complexities of the actual universe; in regard to the two points mentioned, I feel that the agreement is a little strained, and the margin of safety is smaller than is desirable.

12. *Prolate Velocity-Surfaces with Rotation.*—The two methods of obtaining an oblate distribution of density, discussed in §§ 10 and 11 respectively, can be combined, *i.e.* we can take prolate velocity-surfaces and superimpose rotation. Naturally it is easier to account for the 'large oblateness by the two causes than by either of them singly; moreover, it seems natural to expect that in an actual system both causes would act. There is also some relief to the rapid falling off of density in the galactic plane when rotation is imposed, as can be seen from the formulæ.

For prolate surfaces the angular velocity is by (21)

$$\omega = \frac{\Omega}{A + B \sinh^2 \xi + B \cos^2 \eta + C \sinh^2 \xi \sin^2 \eta},$$

which becomes for the galactic plane

$$\omega = \frac{\Omega}{A + (B + C)r^2},$$

and the rotation increases the density in the galactic plane by the factor

$$\exp \frac{r^2 \Omega^2}{A + (B + C)r^2},$$

the density towards the poles being unaltered.

The complete expressions for the density are

$$\left. \begin{aligned} \sigma_0 &= \frac{\exp\{-2E(A+B)r^2\}}{\sqrt{\{(A+B)(A+Br^2)^2\}}} \\ \sigma_{90} &= \frac{\exp\{-2EA r^2 + r^2\Omega^2/(A+Br^2+Cr^2)\}}{\sqrt{\{A(A+B+Br^2)(A+Br^2+Cr^2)\}}} \end{aligned} \right\} \quad (26)$$

It may be noticed that this last expression by a suitable choice of constants may be made to increase at first, and finally decrease as r becomes larger. We could thus have a ring formed—the Milky Way. But I doubt whether such a theory of the galaxy would work out satisfactorily in detail.

13. *Systems in an Unsteady State.*—The observed velocities are found to obey approximately the ellipsoidal law, and the present paper has dealt with systems which satisfy the law rigorously. If we attach importance to the law, it is reasonable to suppose that, however the system may change in the course of time, this approximate agreement will tend to continue; otherwise we should have to attribute the observed agreement to a coincidence, depending on the particular stage of evolution at which we happen to be living. We may therefore consider briefly a system in a changing state, but changing so as still to conform to the ellipsoidal law.

In that case σ , a , b , c , κ , ϕ will be functions of the time. Further, since the principal velocity-surfaces may be changing, there will be product terms in the exponent in equation (2) such as $-2fQR\mu'\nu'$, where f is zero now but $\frac{\partial f}{\partial t}$ is not zero. We shall also

allow in the exponent terms linear in the velocities, indicating that one part of the system is moving as a whole relatively to another part (*cf.* the superposing of a rotation, § 9); and the coefficients of these terms may also vary with the time.

This generalisation involves a great many extra terms in (9); but it will be seen on making the calculation that no addition is made to the terms of the third degree in λ' , μ' , ν' . Accordingly the first nine equations of (13) are true even for an unsteady system, and all the results relating to the velocity-distribution, §§ 4–6, remain valid.

Most of the new terms are of the second degree, and merely lead to additional conditions to be satisfied by the time-derivatives of the coefficients, etc. But there are also linear terms, so that the last three equations of (13) are modified. Accordingly the results relating to the density-distribution, §§ 7–12, will not hold in an unsteady system.

I have not followed out any further the problem of a system in a changing state; but I do not think the solution, at least for the case of axial symmetry, would be difficult. The general validity of the formulæ for the velocity-distribution makes an extraordinary simplification. The principal velocity-surfaces are confocals of revolution and can change in one way only, by varying

the distance apart of the foci. The lengths of the axes of the velocity-ellipsoids at all points are determined by the three constants A, B, C, which must now be supposed to vary with the time. Thus the most general specification of the velocity-ellipsoids is at once reduced to four parameters varying with the time.

[*Added Nov. 29.*—I have now examined this problem. The result is that no exact solution exists. We infer that, in the course of collapsing from an oblate to a spherical form, the system must necessarily deviate from the ellipsoidal law.]

APPENDIX.

General proof of the theorem that the principal velocity-surfaces must be confocal quadrics.

In the following argument the letters α, β, γ (with or without suffixes) will be used to denote arbitrary functions of λ, μ, ν respectively, and the letters a, b, c to denote functions of (μ, ν) , (ν, λ) , (λ, μ) respectively. The unsuffixed a, b, c already in use agree with this convention. It is very necessary to bear these meanings in mind in following the steps of the proof, as continual use is being made of these properties, usually without explicit mention. The letters A, B, C are used for arbitrary constants.

Integrating the middle six equations of (13), we have

$$\left. \begin{aligned} \alpha^2 - b^2 &= P^2 b_1 = Q^2 a_2 \\ b^2 - c^2 &= Q^2 c_1 = R^2 b_2 \\ c^2 - a^2 &= R^2 a_1 = P^2 c_2 \end{aligned} \right\} \quad . \quad . \quad . \quad (27)$$

$$\text{from which} \quad a_1 b_1 c_1 = a_2 b_2 c_2 \quad . \quad . \quad . \quad (28)$$

Now if a_1 contains a factor involving μ and ν inseparably (*i.e.* not of the form $\beta\gamma$), the only other term in (28) in which such a factor can occur is a_2 , so that a_2 must be of the form $a_1\beta\gamma$. We can include all possible cases by writing

$$\text{Similarly} \quad \left. \begin{aligned} a_1 &= \gamma_1 a_3 & a_2 &= \beta_2 a_3 \\ b_1 &= \alpha_1 b_3 & b_2 &= \gamma_2 b_3 \\ c_1 &= \beta_1 c_3 & c_2 &= \alpha_2 c_3 \end{aligned} \right\} \quad . \quad . \quad . \quad (29)$$

$$\text{hence by (28)} \quad \alpha_1 \beta_1 \gamma_1 = \alpha_2 \beta_2 \gamma_2$$

By a similar argument we see that a_1 can only differ from a_2 by a constant multiplier, so that

$$\alpha_1 = A\alpha_2 \quad \beta_1 = B\beta_2 \quad \gamma_1 = C\gamma_2 \quad \text{and} \quad ABC = 1.$$

$$\text{By writing} \quad \left. \begin{aligned} \gamma_1 &= \gamma_3 & \beta_1 &= B\beta_3 & a_1 &= a_3/C \\ \alpha_3 &= a_0 & b_3 &= Cb_0 & c_3 &= c_0/B \end{aligned} \right\} \quad . \quad . \quad (30)$$

the constants are absorbed and we get from (27)

$$\left. \begin{aligned} \alpha^2 - b^2 &= P^2 a_3 b_0 = Q^2 \beta_3 a_0 \\ b^2 - c^2 &= Q^2 \beta_3 c_0 = R^2 \gamma_3 b_0 \\ c^2 - a^2 &= R^2 \gamma_3 a_0 = P^2 \alpha_3 c_0 \end{aligned} \right\} \quad . \quad . \quad . \quad (31)$$

from which

$$\frac{P^2 \alpha_3}{a_0} = \frac{Q^2 \beta_3}{b_0} = \frac{R^2 \gamma_3}{c_0} = \frac{a^2 - b^2}{a_0 b_0} = \frac{b^2 - c^2}{b_0 c_0} = \frac{c^2 - a^2}{c_0 a_0} \quad (32)$$

From the last three members of (32)

$$a_0 b_0 + b_0 c_0 + c_0 a_0 = 0,$$

hence

$$a_0^{-1} + b_0^{-1} + c_0^{-1} = 0 \quad (33)$$

Differentiate with respect to λ ; we obtain

$$\frac{\partial c_0^{-1}}{\partial \lambda} = - \frac{\partial b_0^{-1}}{\partial \lambda}.$$

And since one side does not contain ν , and the other does not contain μ , each side must be a function of λ only. Call this function $-\frac{\partial \alpha}{\partial \lambda}$.

$$\frac{\partial c_0^{-1}}{\partial \lambda} = - \frac{\partial \alpha}{\partial \lambda} \quad \frac{\partial b_0^{-1}}{\partial \lambda} = \frac{\partial \alpha}{\partial \lambda},$$

whence integrating (remembering that $\partial c_0^{-1}/\partial \nu = 0$, $\partial b_0^{-1}/\partial \mu = 0$)

$$c_0^{-1} = \beta - \alpha \quad b_0^{-1} = \alpha - \gamma,$$

and from (33)

$$a_0^{-1} = \gamma - \beta.$$

Hitherto λ , μ , ν have not been exactly specified; any parameters of the principal velocity-surfaces might be used. As in § 4 we may use for our co-ordinates any functions of the original λ , μ , ν without altering the frame of reference. We shall choose α^{-1} , β^{-1} , γ^{-1} as the new λ , μ , ν . Of course in doing so we alter P, Q, R. But the new P differs only from the old P by a factor which, being a function of λ only, can be absorbed in α_3 .

$$\text{Hence} \quad a_0 = \frac{\mu \nu}{\mu - \nu} \quad b_0 = \frac{\nu \lambda}{\nu - \lambda} \quad c_0 = \frac{\lambda \mu}{\lambda - \mu} \quad (34)$$

and (32) becomes

$$\frac{P^2 \alpha_3 (\mu - \nu)}{\mu \nu} = \frac{Q^2 \beta_3 (\nu - \lambda)}{\nu \lambda} = \frac{R^2 \gamma_3 (\lambda - \mu)}{\lambda \mu} = \frac{(a^2 - b^2)(\mu - \nu)(\nu - \lambda)}{\lambda \mu \nu^2} = \dots$$

Multiplying by

$$\frac{\lambda \mu \nu}{(\lambda - \mu)(\mu - \nu)(\lambda - \nu)}$$

$$\begin{aligned} \frac{P^2 \alpha_0}{(\lambda - \mu)(\lambda - \nu)} &= \frac{Q^2 \beta_0}{(\mu - \lambda)(\mu - \nu)} = \frac{R^2 \gamma_0}{(\nu - \lambda)(\nu - \mu)} = \frac{a^2 - b^2}{\nu(\mu - \lambda)} \\ &= \frac{b^2 - c^2}{\lambda(\nu - \mu)} = \frac{c^2 - a^2}{\mu(\lambda - \nu)} = S \end{aligned} \quad (35)$$

where α_0 is written for $\alpha_3 \lambda$. S may be any function of λ , μ , ν .

We now introduce six new differential equations, which must be satisfied by P, Q, R. These are known as the Lamé relations,

and embody the geometrical conditions satisfied by a triply orthogonal system of surfaces.* The Lamé relations are

$$\frac{\partial^2 P}{\partial \mu \partial \nu} - \frac{1}{Q} \frac{\partial Q}{\partial \nu} \frac{\partial P}{\partial \mu} - \frac{1}{R} \frac{\partial R}{\partial \mu} \frac{\partial P}{\partial \nu} = 0 \quad (36)$$

$$\frac{\partial}{\partial \lambda} \left(\frac{1}{P} \frac{\partial Q}{\partial \lambda} \right) + \frac{\partial}{\partial \mu} \left(\frac{1}{Q} \frac{\partial P}{\partial \mu} \right) + \frac{1}{R^2} \frac{\partial P}{\partial \nu} \frac{\partial Q}{\partial \nu} = 0 \quad (37)$$

and the corresponding cyclic results.

We can write (36)

$$\frac{\partial^2}{\partial \mu \partial \nu} \log P = \frac{\partial \log Q}{\partial \nu} \frac{\partial \log P}{\partial \mu} + \frac{\partial \log R}{\partial \mu} \frac{\partial \log P}{\partial \nu} - \frac{\partial \log P}{\partial \nu} \frac{\partial \log P}{\partial \mu}.$$

Now from (35)

$$\left. \begin{aligned} 2 \log P &= \log (\lambda - \mu) + \log (\lambda - \nu) + \log S - \log \alpha_0 \\ 2 \log Q &= \log (\mu - \lambda) + \log (\mu - \nu) + \log S - \log \beta_0 \\ 2 \log R &= \log (\nu - \lambda) + \log (\nu - \mu) + \log S - \log \gamma_0 \end{aligned} \right\} \quad (38)$$

Thus (36) becomes

$$\begin{aligned} 2 \frac{\partial^2}{\partial \mu \partial \nu} \log S &= \left(\frac{\partial \log S}{\partial \nu} + \frac{1}{\nu - \mu} \right) \left(\frac{\partial \log S}{\partial \mu} + \frac{1}{\mu - \lambda} \right) \\ &\quad + \left(\frac{\partial \log S}{\partial \mu} + \frac{1}{\mu - \nu} \right) \left(\frac{\partial \log S}{\partial \nu} + \frac{1}{\nu - \lambda} \right) \\ &\quad - \left(\frac{\partial \log S}{\partial \nu} + \frac{1}{\nu - \lambda} \right) \left(\frac{\partial \log S}{\partial \mu} + \frac{1}{\mu - \lambda} \right) \\ &= \frac{\partial \log S}{\partial \mu} \frac{\partial \log S}{\partial \nu} + \frac{1}{\nu - \mu} \left(\frac{\partial \log S}{\partial \mu} - \frac{\partial \log S}{\partial \nu} \right) \end{aligned} \quad (39)$$

$$\text{Whence} \quad 2 \frac{\partial^2 S}{\partial \mu \partial \nu} = \frac{3}{S} \frac{\partial S}{\partial \mu} \frac{\partial S}{\partial \nu} + \frac{1}{\nu - \mu} \left(\frac{\partial S}{\partial \mu} - \frac{\partial S}{\partial \nu} \right) \quad (40)$$

Again, from (35)

$$\frac{b^2 - c^2}{\lambda} = S(\nu - \mu),$$

operate by $\frac{\partial^2}{\partial \mu \partial \nu}$, the left side vanishes, and we have

$$\frac{\partial^2 S}{\partial \mu \partial \nu} + \frac{1}{\nu - \mu} \left(\frac{\partial S}{\partial \mu} - \frac{\partial S}{\partial \nu} \right) = 0 \quad (41)$$

From (40) and (41)

$$\frac{\partial^2 S}{\partial \mu \partial \nu} = \frac{1}{S} \frac{\partial S}{\partial \mu} \frac{\partial S}{\partial \nu},$$

from which (dividing through by $\frac{\partial S}{\partial \mu}$ and integrating with respect to ν)

$$\log \frac{\partial S}{\partial \mu} = \log S + c_0,$$

hence

$$\frac{\partial S}{\partial \mu} = c_1 S.$$

* See, for example, Forsyth, *Differential Geometry*, p. 418.

Since this is of the same form as (35)* with $S=1$, we see that the two solutions will lead to the same frame of reference, the difference being that in one case certain quantities are taken as the parameters of the surfaces and in the other case their reciprocals.

We can therefore without loss of generality put $S=1$.

The second Lamé relation (37) may be written

$$-\frac{1}{P^2} \frac{\partial \log P}{\partial \lambda} \frac{\partial \log Q}{\partial \lambda} - \frac{1}{Q^2} \frac{\partial \log P}{\partial \mu} \frac{\partial \log Q}{\partial \mu} + \frac{1}{R^2} \frac{\partial \log P}{\partial \nu} \frac{\partial \log Q}{\partial \nu} + \frac{1}{P^2} \left(\frac{1}{Q} \frac{\partial^2 Q}{\partial \lambda^2} \right) + \frac{1}{Q^2} \left(\frac{1}{P} \frac{\partial^2 P}{\partial \mu^2} \right) = 0.$$

Hence by (35) or (38), with $S=1$, we have

$$-\frac{1}{P^2} \left\{ \left(\frac{1}{2(\lambda-\mu)} + \frac{1}{2(\lambda-\nu)} - \frac{1}{2\alpha_0} \frac{\partial \alpha_0}{\partial \lambda} \right) \frac{1}{2(\lambda-\mu)} + \frac{1}{4(\mu-\lambda)^2} \right\} \\ - \frac{1}{Q^2} \left\{ \left(\frac{1}{2(\mu-\lambda)} + \frac{1}{2(\mu-\nu)} - \frac{1}{2\beta_0} \frac{\partial \beta_0}{\partial \mu} \right) \frac{1}{2(\mu-\lambda)} + \frac{1}{4(\mu-\lambda)^2} \right\} \\ + \frac{1}{R^2} \frac{1}{4(\nu-\lambda)(\nu-\mu)} = 0.$$

Multiplying through by $(\lambda-\mu)^2(\mu-\nu)^2(\nu-\mu)^2$ and substituting for P^2 , Q^2 , R^2 , this becomes

$$(\lambda-\mu)^2 \gamma_0 = \frac{2}{\lambda-\mu} \{ (\nu-\mu)^2(\lambda-\nu)\alpha_0 + (\nu-\lambda)^2(\nu-\mu)\beta_0 \} \\ + (\mu-\nu)^2 \left(\alpha_0 + (\nu-\lambda) \frac{\partial \alpha_0}{\partial \lambda} \right) + (\lambda-\nu)^2 \left(\beta_0 + (\nu-\mu) \frac{\partial \beta_0}{\partial \mu} \right) \quad (45)$$

In this equation for γ_0 , the right-hand side can be arranged as a polynomial in ν of the third degree. The coefficients must, of course, reduce to constants multiplied by $(\lambda-\mu)^2$, since γ_0 cannot contain λ or μ .

This form of the coefficients requires that the divisor $(\lambda-\mu)$ of the first term should cancel with a similar factor in the numerator. Thus $((\mu-\nu)\alpha_0 + (\nu-\lambda)\beta_0)$ must be divisible by $(\lambda-\mu)$. Putting $\lambda=\mu$, we find α_0 must equal β_0 , which means that α_0 is the same function of λ that β_0 is of μ . Similarly γ_0 is the same function of ν .

We have therefore shown that α_0 , β_0 , γ_0 are polynomials of the third degree in λ , μ , ν respectively, the coefficients being the same in each case. It is not difficult to verify that this result satisfies equation (45); but it is unnecessary to do so, because, as we shall see, the resulting expressions for P , Q , R are of a familiar form which is known to satisfy the Lamé relations.

Write accordingly

$$\alpha_0 = (\lambda + \epsilon_1)(\lambda + \epsilon_2)(\lambda + \epsilon_3) \\ \beta_0 = (\mu + \epsilon_1)(\mu + \epsilon_2)(\mu + \epsilon_3) \\ \gamma_0 = (\nu + \epsilon_1)(\nu + \epsilon_2)(\nu + \epsilon_3)$$

where ϵ_1 , ϵ_2 , ϵ_3 are constants.

* The last three equations are altered, but they are not required any more.

We obtain finally from (35), since $S = 1$,

$$P^2 = \frac{(\lambda - \mu)(\lambda - \nu)}{(\lambda + \epsilon_1)(\lambda + \epsilon_2)(\lambda + \epsilon_3)} \quad Q^2 = \frac{(\mu - \lambda)(\mu - \nu)}{(\mu + \epsilon_1)(\mu + \epsilon_2)(\mu + \epsilon_3)} \\ R^2 = \frac{(\nu - \lambda)(\nu - \mu)}{(\nu + \epsilon_1)(\nu + \epsilon_2)(\nu + \epsilon_3)} \quad . \quad . \quad . \quad (46)$$

These are the well-known expressions corresponding to confocal quadrics, the quantities $\lambda + \epsilon_1$, etc., being the squares of the axes of the confocals through the point. This establishes the theorem.

It will be noticed that the result is derived from equations which are true whether the system is in a steady state or not, and whether it is moving under its own attraction or under any other field of force.

Summary and Conclusions.

This paper is an investigation of the possible forms of stellar systems in which Schwarzschild's ellipsoidal law of velocities is rigorously obeyed. In general the system is assumed to be in a steady state; but the chief results as to the distribution of velocities hold good also when the system is changing, provided it continues to satisfy the ellipsoidal law.

A possible superposed rotation of the system is considered.

Starting with the consideration of systems of all possible degrees of complexity, it is shown that the ellipsoidal law can be satisfied by only one type, viz., that in which the axes of the velocity-ellipsoids are oriented along the normals to a system of confocal quadrics. The field of discussion is thus narrowed.

The direction of preferential motion (star-streaming) is more or less radial, and the average motion in this direction does not diminish as we pass to the outer parts of the system. The cross-motions decrease as we pass outwards, so that the velocity-ellipsoids become more elongated. (For a more precise statement, see § 5.) These are features of the distribution which were noticed in discussing the particular case of spherical symmetry.

(a) *Systems moving solely under their own Attraction.*—The only exact solution is the case of globular symmetry discussed in the second paper (apart from a two-dimensional solution of no interest). No further solution is obtained by admitting a rotation of the system.

(b) *Systems moving under Forces other than their own Attraction.*—We may suppose the oblate system of the earlier-type stars to be controlled by the more globular system of late-type stars; or that the lucid stars move mainly under the gravitational field of a cosmic gas (producing absorption of light in space), their own attractions being insignificant in comparison. If, in either case, we take the controlling system to be globular, we evade the chief difficulty as to how its own equilibrium is maintained. Under a

controlling distribution, we can obtain an oblate distribution of the stars, either by taking prolate principal velocity-surfaces, or by superposing a rotation of the whole system, or by a combination of the two causes. It appears to be possible to construct a system in this way in fairly good quantitative accordance with the chief results of observation, though one or two points remain doubtful.

It is not easy to estimate how far the results would be modified by admitting small deviations from the ellipsoidal law (such as are indicated by the observations). In the case of a system moving under its own attraction, where there is no exact solution, it is possible that an approximate solution might suffice, if we relaxed this condition slightly. It is certainly not permissible to assert that the properties of these mathematical stellar systems must necessarily be found in the actual universe; but, if we do not misuse it, an ideal system can help towards a better understanding of the interrelations of the phenomena.

Erratum in the "Second Paper."

Mr. Hartley points out to me that at the beginning of § 5 (vol. lxxv. p. 370) I have integrated with respect to u from $-\infty$ to ∞ , whereas I have previously treated u as a signless quantity (introducing a factor 2 on p. 368 to allow for this). The integration should have been from 0 to ∞ , and the necessary correction can be made by writing $\frac{B}{2}$ for B in equations (14) and (15). The factor 2 is then absorbed in the choice of units along with B , and the rest of the work stands without alteration.

Note on Professor Dale's Method of Periodic Analysis.

By H. C. Plummer, M.A.

The method proposed by Professor Dale (*M.N.*, lxxiv. p. 628) for the resolution of a compound periodic function is of great interest. It may be pointed out that the main propositions on which the method rests are capable of a simpler and more direct proof.

1. In place of the operator E , which has another meaning in finite differences, we may use M to denote the mean of the preceding and following tabular entry, so that

$$Mu_n = \frac{1}{2}(u_{n-1} + u_{n+1}).$$

Since secular terms can be removed by preliminary differencing, the function may be taken to be

$$f(\theta) = \sum A \sin(p\theta + a)$$