Review
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furnished a comparatively new field in the discussion of the properties of the new transcendants he has thus defined. M. Fouët, however, fails to give any account of Painlevé's methods, and ignores the work of Picard and of Mittag Leffler dealing with the same subject (*C. R.* vols. CII., CIV., etc. *Acta Math.* vols. XVII., XVIII., etc.). In this first chapter M. Fouët gives certain theorems on the existence of integrals of partial equations, and concludes with theorems connected with Dirichlet's Problem.

The last chapter of Book I. is concerned with functions defined by functional properties, such as, for example, properties of periodicity. Two pages are devoted to the consideration of doubly periodic functions.

In the three chapters of Book II. given in this volume, analytic functions are discussed from the points of view of Cauchy, Weierstrass and Riemann.

M. Fouët's main difficulty has apparently arisen in connection with the arrangement of the work. He is perhaps justified in his attempt to give separate discussions of analytic functions from the three different standpoints, and this attempt necessitates a general account of function theoretic ideas on the lines of that given in Book I. But the effect of his arrangement is a little unfortunate in such cases as the following:—

The discussion of the function $\Gamma(z)$ is divided into two parts, one of which appears in Book I. and the other in Book II. The elementary discussion of infinite products, which is essentially connected with Weierstrass' point of view of function theory, is separated from it by such matter as a general discussion of differential equations, and an account of Dirichlet's problem. This problem itself, which is intimately connected with Riemann's, is not treated in connection with the account of that work, but in connection with differential equations; and an account of the properties of harmonic functions, which should be given in connection with the discussion of Laplace's Equation, is reserved to make an addition to the chapter on Riemann's theory of functions. If M. Fouët intends to discuss in detail such an important branch of his subject as elliptic functions, it would be an advantage to have the account of the whole subject in one part of the treatise. Instead of this he has considered functions possessing an algebraic addition theorem in Vol. I., p. 168, and in Vol. II., pp. 83, 84, while the Weierstrassian σ function is given as a product in Vol. II., p. 188, general discussion being apparently reserved for a later volume.

The work is chiefly interesting for its account of many of the outlying parts of the subject, such as, for example, that of minima surfaces (Vol. II., pp. 285 *sqq.*). It is unfortunate that, owing to the scope of the work, such accounts are necessarily brief. J. E. WRIGHT.

Leçons sur les séries à termes positifs. Par. É. BOREL. Pp. 91. 1902. (Paris, Gauthier-Villars.)

This is a reproduction of 20 lectures given by M. Borel at the *Collège de France* in the session 1900-1901; they have been collected and edited by one of M. Borel's class, M. Robert d'Adhémar. It forms a part of the series of works entitled *Nouvelles leçons sur la théorie des fonctions*, in which have appeared M. Borel's lectures on integral functions and on divergent series.

In chapter 1 we are concerned mainly with the familiar logarithmic criteria of convergence, which M. Borel attributes to Bertrand, though they appear to have been given first by de Morgan (see Chrystal's *Algebra*, vol. 2, historical note to chap. xxvi.). It is not until we reach the middle of chapter 2 that M. Borel introduces us to the idea which forms the central thread of the book; this idea is that of the *croissance* (degree of rapidity of increase) of a function always increases with the independent variable x. For the simple power x^n, the *croissance* is represented by the index n; for the exponential e^x, a symbol ω is introduced, which must [in virtue of the property $\lim_{x=\infty}(x^n/e^x)=0$] be greater than any ordinary index; the symbol ω has also been used by G. Cantor to represent the *transfinite number*, which gives a useful analogy for the exponential *croissance*. The laws of combination of the symbol ω are investigated, and the results are tabulated on p. 47; they agree only partially with the algebraic laws of ordinary numbers.*

* *E.g. Multiplication* is defined as in the following examples: ωn is the *croissance* of exp. (x^n), but $n\omega$ is that of (exp. $x)^n$; ω^2 is that of exp. (exp. x); and so on. Thus $\omega n \neq n\omega$; but $\omega^\alpha\omega^\beta=\omega^{\alpha+\beta}$.

Passing to chapter 4 we find two simple criteria for the convergence of double series (with positive terms) which may be quoted here; taking the series $\Sigma\Sigma v_{mn}$ (m, $n=1, 2, \dots \infty$) we have a *sufficient* test of convergence if, after a certain stage, we can say that $v_{mn} < (m+n)^{-(2+p)}$, where $p > 0$; or that $v_{mn} < \Sigma 1/(m^{2+p}+n^{2+p})$. But if $v_{mn} > (m+n)^{-2}$, the series diverges. Similar conditions are then obtained for double integrals, in which the area of integration extends to infinity.

In chapter 5 we take up the question of the *croissance* of a power-series $\Sigma a_n x^n$; it is shown that if $\phi(n) = a_n{}^{-1/n}$ is an increasing function, of order n^p, then the *croissance* of the function defined by the series is $\omega(1/p)$; but the inverse problem is much harder.* Another interesting result is given for the case in which $\phi(n)$ tends to the limit 1 as n tends towards ∞; taking the series Σa_n to be divergent, the behaviour of the function $\Sigma a_n x^n$ near $x=1$ is determined. For instance,

$$\lim_{x=1}[(1-x)^p(1^{p-1}x+2^{p-1}x^2+\dots+n^{p-1}x^n+\dots)]=\Gamma(p).$$

The remainder of the chapter can hardly be appreciated without a direct reference to the original; the book terminates with a short account of analogous properties of the double series $\Sigma a_{mn}x^m y^n$.

We have no hesitation in recommending those who are interested in modern analysis to add this book to their shelves; it is a worthy successor to M. Borel's earlier text-books; a higher recommendation cannot be given.

T. J. I'a BROMWICH.

Théorie élémentaire des séries. Par M. GODEFROY. Pp. 266. 1903. (Paris, Gauthier-Villars.)

This book differs entirely in its aims from M. Borel's; as appears both from the title and the method of treatment, it is mainly intended for beginners in the field of analysis. It is somewhat similar to the treatment given in Chrystal's *Algebra* and Hobson's *Trigonometry*, except that the complex variable is nowhere used;† in this respect following the example of the first edition of Stolz's *Allgemeine Arithmetik*. However, M. Godefroy assumes a knowledge of the elements of the Differential Calculus, and so is able to give some theorems which are not mentioned in the treatises named; for example (p. 75) we have the theorem that a linear differential equation is soluble by power-series which converge within the same interval as the power-series which represent the co-efficients. Some of the simpler properties of Bessel's and Legendre's functions (of the first kind) are obtained.

It is something of a novelty to find in a professedly elementary book, a very clear and concise discussion of the Weierstrassian continuous, non-differentiable, function

$$\Sigma r^n \cos(a^n \pi x),$$

where $0 < r < 1$ and a is an an odd integer such that $ar > 1 + \frac{3}{2}\pi$. According to a footnote to this article, M. Poincaré has remarked that a hundred years ago a function such as this would have been thought an outrage on common sense; but we fear that in England a much more recent date might be assigned.

M. Godefroy concludes his book with 50 pages on the gamma-function; for the most part, these articles appear to be reprints of those in his earlier work bearing that title (see *Math. Gazette*, March, 1902).

As a whole the work should be found extremely useful by teachers who have to lecture on the elementary theory of infinite series; and I have made considerable use of it in my course of lectures on the subject. T. J. I'a BROMWICH.

Theorie und Praxis der Reihen. By C. RUNGE. Pp. 266. 1904. (Goschen, Leipzig.)

In this admirable monograph Professor Runge has devoted his attention to the practical use of series at the expense of the general theory and all its details. The following is a summary of the five chapters into which the book is divided.

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* For the function $\phi(n)$ may be changed so as to increase quite irregularly without in any way affecting the *croissance* of the series.

† This will be regarded as an advantage by those teachers who think it well to begin the study of power-series before introducing the complex variable.