



149. Note on Some Inequalities Connected with the Expressions $w = \frac{\sum a}{\sum (a-b)(a-c)}$, $n = \frac{a+b+c}{a-b+c}$, Where a, b, c Are Three Positive Quantities in Order of Magnitude

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The parametric representation of the surface is given by

$$\frac{x}{uv} = \frac{2y}{u+v} = \frac{z}{1} = \frac{2t}{u-v},$$

so that the points (u, v) and (v, u) , which in the plane were identical, are separated on the surface and are reflexions of each other in $t=0$.

By what has been shown, we have proved the fundamental theorem: *the equation of any curve on the surface is $\phi(u, v)=0$, where ϕ is a general polynomial in its arguments.*

The parametric curves $u=k, v=l$, are the generators

$$\left. \begin{aligned} y+t &= kz \\ y-t &= k^{-1}x \end{aligned} \right\} \quad \text{and} \quad \left. \begin{aligned} y-t &= lz \\ y+t &= l^{-1}x \end{aligned} \right\}$$

respectively, intersecting in the point (k, l) . If $\phi(u, v)$ is of degrees m in u and n in v , it represents a curve of order $m+n$ cutting each generator $u=\text{const.}$ in n points and each generator $v=\text{const.}$ in m points.

Suppose $m=n$; a typical term in $\phi(u, v)$ is $u^r v^s$, and if $r > s$ we have

$$\begin{aligned} z^m u^r v^s &= z^{m-r} (y+t)^{r-s} x^s, \\ z^m u^r v^s &= z^{m-s} (y-t)^{s-r} x^r. \end{aligned}$$

but if $r < s$ we have

$$\text{In this way the equation} \quad z^m \phi(u, v) = 0$$

becomes that of a surface of order m cutting the quadric in the $2m$ -ic curve $\phi(u, v)=0$. Hence every equation of the same degree in u and v represents a complete intersection. From this follows Halphen's theorem* that any curve on a quadric together with a certain number of generators is a complete intersection; for if $m > n$ the function $\phi(u, v)(v-k_1) \dots (v-k_{m-n})$ is of the same degree m in u and in v , showing that the curve $\phi=0$ and any $m-n$ generators of the system $v=\text{const.}$ lie on an m -ic surface.

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MATHEMATICAL NOTES.

149. [A. 1. b.] Note on some inequalities connected with the expressions

$$w = \frac{\Sigma a \Sigma (a-b)(a-c)}{\Sigma a(a-b)(a-c)}, \quad n = \frac{a+b+c}{a-b+c},$$

where a, b, c are three positive quantities in order of magnitude.

(i) n, w both lie between 1 and ∞ .

(ii) n lies between w and 3; proved by showing that $n-w$ and $n-3$ are of opposite signs.

(iii) w lies between n and $\frac{1}{4}(n^2+3)$; proved by assuming n constant while w varies, or otherwise.

(iv) n lies between w and $(4w-3)^{\frac{1}{2}}$; proved from (iii), etc.

Hence the following series is in order of magnitude, the first term being ∞ or 1 according as the series is in descending or ascending order:

$$\infty \text{ or } 1, \quad \frac{1}{4}(n^2+3), \quad w, \quad n, \quad (4w-3)^{\frac{1}{2}}, \quad 3.$$

w and n are equal if either of them is equal to 1, 3, or ∞ .

If $b+c=a', c+a=b', a+b=c'$, then a', b', c' are sides of a triangle; and conversely, if a', b', c' are sides of a triangle in order of magnitude we can put $a'=b+c, b'=c+a, c'=a+b$. Substituting for a, b, c in terms of a', b', c' in the above result, and making some slight changes, and finally dropping the dashes, we obtain the following:

If a, b, c are sides of a triangle in order of magnitude, and w, n stand for

* Bull. de la Soc. math. de France, vol. I., p. 19.

the same expressions as above, then the following series is in order of magnitude (commencing with ∞ or 2 according as it descends or ascends)

$$\infty \text{ or } 2, n + \frac{(n-2)(n-3)}{n-1}, w, n, \frac{2(w+6)^{\frac{1}{2}}}{(w+6)^{\frac{1}{2}} - (w-2)^{\frac{1}{2}}}, 3.$$

w and n both lie between 2 and ∞ , and are equal if either of them is equal to 2, 3, or ∞ . F. S. MACAULAY.

150. [K. 20. f.] Napier's Rule of Circular Parts.

Let ABC be a spherical triangle right-angled at C . Then another triangle having the same circular parts may be found as follows.

(1) Take a triangle symmetrically equal to ABC and place the two together with their sides AC in contact. They will form an isosceles triangle whose sides are $2a, c, c$, whose angles are $2A, B, B$ and whose altitude is b .

(2) Take the polar of this triangle. Its sides are $\pi - 2A, \pi - B, \pi - B$ and its angles $\pi - 2a, \pi - c, \pi - c$, and its altitude is the supplement of the altitude of the previous triangle, i.e. $\pi - b$ (as can easily be shown).

(3) Take the colunar triangle of the last triangle. Its sides are $\pi - 2A, B, B$, and its angles $\pi - 2a, c, c$, and its altitude b .

(4) Split this triangle into two right-angled triangles. The angles of these will be $c, \frac{1}{2}\pi - a, \frac{1}{2}\pi$, and the sides opposite them $b, \frac{1}{2}\pi - A, B$.

If $A'B'C'$ denote this new triangle, we have therefore

$$a' = b, b' = \frac{1}{2}\pi - A, \frac{1}{2}\pi - A' = \frac{1}{2}\pi - c, \frac{1}{2}\pi - c' = \frac{1}{2}\pi - B \text{ and } \frac{1}{2}\pi - B' = a.$$

The triangle $A'B'C'$ has therefore the same circular parts as ABC , moved one place round.

In the penultimate step of this process we had an isosceles triangle formed of $A'B'C'$ and a symmetrically equal triangle with their sides a' in contact. By placing two such triangles with their sides b' in contact and repeating the processes, we obtain a third triangle having the same circular parts moved one place further round, and so on.

This method is not so elegant as the geometric one given in text-books on spherical trigonometry, but I would say that it is easier to remember, as I never find it possible to demonstrate the geometric method to a class without referring to the text-book. G. H. BRYAN.

151. [P. 3. b.] Note on Successive Inversion.

[The Apollonian circles of a triangle ABC are the three loci $b.PB=c.PC, c.PC=a.PA$, and $a.PA=b.PB$. They have for diameters the segments of the sides between the feet of the internal and external bisectors of opposite angles. They intersect in two real points, inverse to the circum-circle, lying on the line joining the circum-centre and symmedian point. The pedal triangles of these points are equilateral.]

Inverting the property that the medians of an equilateral triangle intersect at angles of 120° , it follows that the A -circles of a triangle are coaxal and cut each other at angles of 120° .

If any point P be taken on the sides of an equilateral triangle whose centroid is O , the concentric circle of radius OP will intersect the sides in five other points, symmetrically situated two on each side. No matter what their number or what their order, successive reflections of P with respect to the medians give these five points and no more.

Inverting; no matter what their number or what their order, successive inversions of a point P with respect to the A -circles of the triangle ABC give five new points and no more.

The formal proof of the following property presents no difficulty and need only be stated; the pedal triangle of any point is similar to the pedal