

XXV.—*On Minding's Theorem.* By Professor TAIT.

(Revised June 23, 1880.)

The following paper contains a short digest of investigations communicated to the Society on several occasions during the past, and the present, session. The work had been for some months laid aside, but my attention was recalled to it by Professor CHRYSTAL's valuable paper, in which he treats MINDING'S Theorem as an example of PLÜCKER'S methods, and also by the help of RODRIGUES' co-ordinates. I am induced to publish a few of my results in full, as I think that a comparison of the analysis employed by CHRYSTAL, with the very different analysis employed by myself, may be useful as well as interesting, especially from the point of view of the simplicity of the quaternion method. Even when the quaternion processes are written out at full length, they are in general shorter than the most condensed forms of ordinary analysis; and there can be no doubt that they are much more easily interpretable into the corresponding geometrical ideas.

A hastily-written proof of the main theorem, somewhat on the same lines as the first of those now given, was printed in the "Proceedings of the London Mathematical Society," No. 147. But the present version is much simpler; and it is requisite for the intelligibility of the rest of the paper which, I repeat, is given mainly for the sake of the quaternion processes involved.

I commence with a few preliminary transformations. This would be altogether needless if quaternion methods were at all as familiar to the majority of mathematical readers as are the more usual ones.

1. In what follows we have a good deal of use to make of certain properties of linear and vector functions, so that some of the less obvious of them are here briefly stated.

Let $a_1, a_2, \&c., \beta_1, \beta_2, \&c.$, be any two sets of vectors, and let us consider the vector

$$\kappa = \Sigma V\beta a. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we operate by $V.\sigma$, where σ is any vector whatever, we have

$$\begin{aligned} V\sigma\kappa &= V.\sigma\Sigma V\beta a \\ &= \Sigma(aS\beta\sigma - \beta Sa\sigma) \\ &= (\phi - \phi')\sigma \quad . \quad . \quad . \quad . \quad . \quad (2) \\ &= 2V\epsilon\sigma \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

if $V. \epsilon$ be the impure part of the strain

$$\phi = \Sigma a S \beta() \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence if ϕ be put (as can always be done) in the normal form

$$\zeta S i() + \eta S j() + \theta S k() ,$$

where i, j, k form a rectangular unit system ; we have

$$\kappa = \Sigma V \beta a = V(i \zeta + j \eta + k \theta) \quad . \quad . \quad . \quad (5)$$

In the particular case which we shall chiefly require, it will be found that there is a certain vector $\bar{\beta}$ such that

$$\phi \bar{\beta} = 0.$$

Hence we may write ϕ in the form

$$\gamma' S \gamma() + \delta' S \delta()$$

where γ, δ are any two unit vectors perpendicular to each other and to $\bar{\beta}$. If, now, we change

$$\gamma \text{ to } \gamma \cos \vartheta + \delta \sin \vartheta ,$$

$$\text{and} \quad \delta \text{ to } -\gamma \sin \vartheta + \delta \cos \vartheta ,$$

(which are still unit vectors, perpendicular to one another, and to $\bar{\beta}$)

$$\gamma' \text{ becomes } \gamma' \cos \vartheta - \delta' \sin \vartheta ,$$

$$\text{and} \quad \delta' \quad , \quad \gamma' \sin \vartheta + \delta' \cos \vartheta .$$

These are at right angles to one another if

$$\tan 2\vartheta = \frac{2S\gamma'\delta'}{\delta'^2 - \gamma'^2} .$$

This always gives real values of ϑ , corresponding to two definite directions at right angles to one another. Hence we may always take

$$\phi = \gamma' S \gamma() + \delta' S \delta() \quad . \quad . \quad . \quad . \quad (4')$$

where γ and δ are as before, and γ' and δ' are vectors at right angles to one another.

Another point to be borne in mind is that rotation of a rigid system may

be expressed by a special linear and vector function, χ , which possesses the following characteristic properties ;

$$S\chi^a\chi\beta = S\alpha\beta,$$

(of which a particular case is

$$T\chi^a = T\alpha,)$$

and

$$V\chi^a\chi\beta = \chi V\alpha\beta.$$

Also the conjugate of χ is its reciprocal, or

$$\chi' = \chi^{-1}.$$

These premised, we may attack the question.

2. When any number of forces act on a rigid system ; β_1 at the point α_1 , β_2 at α_2 , &c., their resultant consists of the single force

$$\tilde{\beta} = \Sigma \beta$$

acting at the origin, and the couple

$$\kappa = \Sigma V\beta\alpha. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If these can be reduced to a single force, the equation of the line in which the force acts is evidently

$$V\tilde{\beta}\rho = \Sigma V\beta\alpha. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now suppose the system of forces to turn about, preserving their magnitudes, their points of application, and their mutual inclinations, then MINDING'S Theorem, proved (in CRELLE'S "Journal," vols. xiv., xv.) by an excessively elaborate process, assigns certain fixed curves in space, each of which is intersected by the line (5) in every one of the infinite number of its positions.

3. To prove this, and to find the curves in question, we may proceed as follows :—

Operating on (5) by $V.\tilde{\beta}$, it becomes

$$\rho\tilde{\beta}^2 - \tilde{\beta}S\tilde{\beta}\rho = \phi\tilde{\beta} - \phi'\tilde{\beta}$$

Let the tensors of γ' and δ' be e_1, e_2 respectively, and let β' be a unit vector perpendicular to them, then we may write

$$bt\rho = x\beta + e_1e_2\beta' - \varpi\beta, \quad . \quad . \quad . \quad . \quad (8)$$

Operating by $(\varpi - x)^{-1}$, and noting that

$$\varpi\beta' = 0,$$

we have

$$bt(\varpi - x)^{-1}\rho = -\beta - \frac{e_1e_2}{x}\beta', \quad . \quad . \quad . \quad . \quad (8')$$

Taking the scalar of the product of (8) and (8') we have

$$b^2t^2S\rho(\varpi - x)^{-1}\rho = -\frac{1}{x}(x\beta + e_1e_2\beta')^2 - S\beta\varpi\beta.$$

But by (7') we have

$$t^2 = S\beta\varpi\beta + e_1^2 + e_2^2 - 2e_1e_2S\beta\beta' \quad . \quad . \quad . \quad (9)$$

so that, finally,

$$b^2S\rho(\varpi - x)^{-1}\rho = -1 + \frac{(e_1^2 - x)(e_2^2 - x)}{xt^2} \quad . \quad . \quad . \quad (10)$$

5. Equation (10), in which t^2 is given by (9) in terms of β , is true for every point of every single resultant. But we get an immense simplification by assuming for x either of the particular values e_1^2 or e_2^2 . For then the right hand side of (10) is reduced to negative unity, and the equation represents one or other of the focal conics of the system of confocal surfaces

$$S\rho(\varpi - h)^{-1}\rho = -\frac{1}{b^2},$$

a point of each of which must therefore lie on the line (8). This is MINDING'S Theorem.

6. A singular form, in which it can be expressed, appears at once from equation (5'). For that equation is obviously the condition that the linear and vector function

$$-b\rho S\beta() + \gamma'S\gamma() + \delta'S\delta()$$

shall denote a pure strain.

Hence the following problem:—*Given a set of rectangular unit vectors, which may take any initial position: let two of them, after a homogeneous strain,*

become given vectors at right angles to one another, find what the third must become that the strain may be pure. The locus of the extremity of the third is, for every initial position, one of the single resultants of MINDING'S system; and therefore passes through each of the fixed conics.

Thus we see another very remarkable analogy between strains and couples, which is in fact suggested at once by the general expression for the impure part of a linear and vector function.

7. The scalar t , which was introduced in equations (7'), is shown by (9) to be a function of β alone. In this connection it is interesting to study the surface of the fourth order

$$S\tau\varpi\tau - (e_1^2 + e_2^2)\tau^2 - 2e_1e_2T\tau S\beta'\tau = 1,$$

where
$$\tau = \frac{1}{t} \beta.$$

But this may be left as an exercise.

Another form of t (by 7') is $S\gamma\gamma' + S\delta\delta'$.

Meanwhile (9) shows that for any assumed value of β there are but two corresponding MINDING lines.

If, on the other hand, ρ be given there are in general four values of β . For variety we may take a different mode of attacking equations (7) and (5'), which contain the whole matter. In what follows b will be merged in ρ .

8. Operating by $V.\beta$ we transform (5') into

$$\rho + \beta S\beta\rho = -(\gamma S\gamma'\beta + \delta S\delta'\beta) \quad . \quad . \quad . \quad (5'')$$

Squaring both sides we have

$$\rho^2 + S^2\beta\rho = S\beta\varpi\beta \quad . \quad . \quad . \quad (11)$$

Since β is a unit vector, this may be taken as the equation of a cyclic cone; and every central axis through the point ρ lies upon it. For we have not yet taken account of (7), which is the condition that there shall be no couple.

To introduce (7), operate on (5'') by $S.\gamma'$ and by $S.\delta'$. We thus have, by a double employment of (7),

$$\left. \begin{aligned} S\gamma'\rho + S\gamma'\beta S\beta\rho &= S\gamma\varpi\beta \\ S\delta'\rho + S\delta'\beta S\beta\rho &= S\delta\varpi\beta \end{aligned} \right\} \quad . \quad . \quad . \quad (12)$$

Next, multiplying (11) by $S\beta\varpi\beta$, and adding to it the squares of (12), we have

$$\rho^2 S\beta\varpi\beta - 2S\beta\rho S\beta\varpi\rho - S\rho\varpi\rho = -S\beta\varpi^2\beta. \quad . \quad . \quad (13)$$

If we look on β as given, while ρ is to be found, (11) is the equation of a right cylinder, and (13) that of a central surface of the second order.

$$\beta_{\|\rho_1 - \rho\|}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (14)$$
$$-\rho^2\rho_1^2+S^2\rho\rho_1=S(\rho_1-\rho)\varpi(\rho_1-\rho) \quad . \quad . \quad . \quad (11')$$

From (13) we obtain the corresponding symmetrical result

These equations become very much simplified if we assume ρ and ρ_1 to lie respectively in any two conjugate planes; specially in the planes of the focal conics, so that $S\delta'\rho=0$, and $S\gamma'\rho_1=0$.

$$S_{\rho} \varpi_{\rho_1} = 0,$$

and if, besides, they be those of the focal conics,

$$S\rho\rho_1 = -S\beta'\rho S\beta'\rho_1,$$

and the equations are

$$-\rho^2\rho_1^2+S^2\rho\rho_1=S\rho_1\varpi\rho_1+S\rho\varpi\rho, \quad (11'')$$

$$\rho^2 S \rho_1 \varpi \rho_1 + \rho_1^2 S \rho \varpi \rho = -S \rho_1 \varpi^2 \rho_1 - S \rho \varpi^2 \rho. \quad (13'')$$
$$\rho_1 = p\delta'$$

and eliminate p between the equations. We get the focal conic in the plane of

β' , γ' . In this way we see that MINDING lines pass through each point of each of the two curves; and by a similar process that every line joining two points, one on the one curve the other on the other, is a MINDING line.

Another process is more instructive. Note that, by the equations of condition above, we have

$$S^2\rho\rho_1 = \left(\frac{S\rho_1\varpi\rho_1}{e_2^2} - \rho_1^2\right)\left(\frac{S\rho\varpi\rho}{e_1^2} - \rho^2\right).$$

Then our equations become

$$\frac{S\rho\varpi\rho S\rho_1\varpi\rho_1}{e_1^2 e_2^2} - \frac{\rho_1^2 + e_1^2}{e_1^2} S\rho\varpi\rho - \frac{\rho^2 + e_2^2}{e_2^2} S\rho_1\varpi\rho_1 = 0,$$

and

$$(\rho^2 + e_2^2) S\rho_1\varpi\rho_1 + (\rho_1^2 + e_1^2) S\rho\varpi\rho = 0.$$

If we eliminate ρ^2 or ρ_1^2 from these equations, the resultant obviously becomes divisible by $S\rho\varpi\rho$ or $S\rho_1\varpi\rho_1$, and we at once obtain the equation of one of the focal conics.

10. In passing it may be well to notice that equation (13) may be written in the simpler form

$$S.\rho\beta\rho\varpi\beta + S\rho\varpi\rho = S\beta\varpi^2\beta.$$

Also it is easy to see that if we put

$$\theta = \rho S\beta\rho - (\varpi + \rho^2)\beta$$

we have (11) in the form

$$S\beta\theta = 0,$$

and by the help of this (13) becomes

$$\theta^2 = S\rho\varpi\rho.$$

This gives another elegant mode of attacking the problem.

11. Another valuable transformation of (5'') is obtained by considering the linear and vector function, χ suppose, by which β , γ , δ are derived from the system β' , $U\gamma'$, $U\delta'$. For then we have obviously

$$\rho = x\chi\beta' + \chi\varpi^{\frac{1}{2}}\chi\beta'. \quad . \quad . \quad . \quad . \quad (5''')$$

This represents any central axis, and the corresponding form of the MINDING condition is

$$S.\gamma'\chi\varpi^{-\frac{1}{2}}\delta' = S.\delta'\chi\varpi^{-\frac{1}{2}}\gamma' \quad . \quad . \quad . \quad . \quad (7'').$$

Most of the preceding formulæ may be looked upon as results of the elimination of the function χ from these equations. This forms probably the most important feature of such investigations, so far at least as the quaternion calculus is concerned.

I employed the equation (5''') as the basis of an investigation, one or two of whose results were communicated last session to the Society.* I will now give the main features of that investigation.

12. It is evident from (5''') that the vector-perpendicular from the origin on the central axis parallel to $\chi\beta'$ is expressed by

$$\tau = \chi\varpi^\dagger\chi\beta'.$$

But there is an infinite number of values of χ for which $U\tau$ is a given versor. Hence the problem ;—to find the maximum and minimum value of $T\tau$, when $U\tau$ is given—*i.e.*, to *find the surface bounding the region which is filled with the feet of perpendiculars on central axes.*

We have

$$T\tau^2 = -S.\chi\beta'\varpi\chi\beta',$$

$$0 = T\tau S.\chi\beta'U\tau.$$

Hence

$$0 = S.\dot{\chi}\beta'\varpi\chi\beta',$$

$$0 = S.\dot{\chi}\beta'U\tau.$$

But as $T\beta'$ is constant

$$0 = S.\dot{\chi}\beta'\chi\beta'.$$

These three equations give at sight

$$(\varpi + u)\chi\beta' = u'U\tau,$$

where u, u' are unknown scalars. Operate by $S.\chi\beta'$ and we have

$$-T^2\tau - u = 0,$$

so that

$$S\tau(\varpi + \tau^2)^{-1}\tau = 0.$$

This differs from the equation of FRESNEL'S wave-surface only in having $\varpi + \tau^2$ instead of $\varpi + \tau^{-2}$ (*i.e.*, $T\tau$ for $\frac{1}{T\tau}$), and denotes therefore the reciprocal

* Proc. Roy. Soc. Edin., 1879, p. 200.

of that surface. In the statical problem, however, we have

$$\varpi\beta' = 0,$$

and thus the corresponding wave-surface has zero for one of its parameters.

[If this restriction be not imposed, the locus of the point

$$\tau = \chi\phi\chi\beta',$$

where ϕ is now any given linear and vector function whatever, will be found, by a process precisely similar to that just given, to be

$$S.(\tau - \phi'\beta')(\phi'\phi + \tau^2)^{-1}(\tau - \phi'\beta') = 0,$$

where ϕ' is the conjugate of ϕ . This, however, has nothing to do with MINDING's Theorem.]

13. As the reader may not feel secure of results derived by the differentiation of a vector function operator, it may be well to obtain the result of last section by a more usual process.

We obviously have by (5'')

$$\tau = \gamma S\gamma'\beta + \delta S\delta'\beta,$$

or (as in (11))

$$\tau^2 = S\beta\varpi\beta.$$

But also

$$S\beta U\tau = 0.$$

$$\beta^2 = -1.$$

To make $T\tau$ a maximum with these conditions, we have

$$\left. \begin{aligned} S\dot{\beta}\varpi\beta &= 0 \\ S\dot{\beta}U\tau &= 0 \\ S\dot{\beta}\beta &= 0 \end{aligned} \right\}$$

and, by elimination of β and $\dot{\beta}$ among these equations, we have as before

$$S\tau(\varpi + \tau^2)^{-1}\tau = 0.$$

The first of the undifferentiated equations is that of an elliptic cylinder of variable magnitude but constant form and position, the second a diametral plane, and the third the unit sphere. Obviously there is one maximum and one minimum value of $T\tau$. These occur when the variable ellipse given by the first and second equation *touches* the fixed circle given by the second and third.

It may do so internally or externally, and consequently the resulting equation gives two values of $T\tau$ for each value of $U\tau$.

14. This is, in fact, in quaternions identical with the second process employed by Professor CHRYSTAL. For, by writing τ for $\rho + \beta S\beta\rho$ in (11) it becomes

$$\tau^2 = S\beta\tau\beta,$$

and in the same way (13) becomes

$$\tau^4 - S\tau\tau\tau = -S\beta\tau^2\beta.$$

These, translated into Cartesian scalars, are CHRYSTAL's equations (8) and (9) (*Second Method, anté*, p. 523). They may be obtained directly by a process similar to that in section 8 above. CHRYSTAL's first method is, of course, included in the solutions afforded by the use of χ .

I may remark, in conclusion, that the process of section 4, leading to an equation like (10) above, seems to be the most natural method of applying quaternions to questions connected with congruencies.