

ASTRONOMISCHE NACHRICHTEN.

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Disquisitiones continuatae circa theoriam perturbationum corporum coelestium,
auctore P. A. Hansen.

In commentationibus circa theoriam perturbationum quas publici juris feci, in illa puta quae in huius libri Nr. 166 sqq. typis excusa est, et in illa quam anno 1830 academia Regia scientiarum Berolinensis praemio ornare dignata est, terminos quosdam neglexi minutissimos, qui revera, in planetarum saltem veterum motibus vix aliquam vim habent. Ratio cur hos omiserim est, quod censueram, magnas ambages producturam esse horum terminorum considerationem, et nullos fere fructus inde redundaturos. Quum vero in planetarum novorum lunaeque motibus termini hi fortasse non omnino rejici possint, quumque problematis solutio quo accuratior et subtilior est, eo magis ipsam eam doctrinae partem, quam debet illustrare, juvet atque promoveat: operam impendi, ut ne quid, quod quodammodo desiderari possit, deesset.

Termini vero de quibus hic agitur ii sunt, qui in formulis differentialibus elementorum variabilium per $tg \frac{1}{2} i$ multiplicati sunt *), porro ii qui in evolutis quotientibus differentialibus quantitatis Ω appellatae, illis similes per potestates ipsarum p et q multiplicati prodeunt **), denique ii qui in δP et δQ ejusdem generis sunt ***), et eodem fonte demanans quantitas quaedam, hic ϕ , $-\phi$ nominata, quae a motibus simul existentibus ambarum orbitarum in spatio pendet. Quos terminos omnes primo aspectu maxime implicitos, partim mutuo se tollere, partim ad paucos terminos simplicissimos reduci posse, his pagellis demonstrabo.

Ut per terminos illos omissos rigor geometricus quam minime turbaretur, necesse erat latitudo planetae perturbati ad planum orbitae, quod in electa temporis epocha locum habuerit referretur, hinc ope formularum trigonometricarum ad latitudinem supra eclipticam fixam, et hinc ad latitudinem supra eclipticam mobilem transgrediendum fuit. Tali enim modo calculis institutis, praevideri potuit, majorem

terminorum omissorum partem ad ordinem tertium respectu massarum referendam esse, id quod infra lectoribus huius commentationis clarius ante oculos ponetur. E contrario, per methodum quam hic sequar statim latitudinem supra eclipticam mobilem, aut supra planum quodvis computare, itaque ambages illas evitare licet. Quamquam reductio illa latitudinis ab orbitae, temporis selectae epochae respondentis plano ad planum eclipticae mobilis pauca tantum proferat, reductio tamen ad planum aequatoris mobilis, quae plurimis titulis illa praeferenda esset, haud insensibilis foret *).

Subsidia quibus terminos illos antea neglectos in calculum vocabo, introductio quantitatis huius $\omega + f \cos i d\theta$ loco longitudinis perihelii π praebet. Qua quantitate jam ill. la Grange nota, si severum iudicium exigis, hucusque nemo usus est, vel potius nemo omnem utilitatem, quam praebere potest inde hausit.

1.

Ad elementa variabilia spectata orbitae corporis cujusunque, quod secundum legem attractionis Newtonianam movetur, determinanda, formulae differentiales, ut notum, inserviunt hae

$$\begin{aligned}\frac{da}{dt} &= k, \mu \frac{2}{na} \left(\frac{d\Omega}{dc} \right) \\ \frac{dc}{dt} &= -k, \mu \frac{2}{na} \left(\frac{d\Omega}{da} \right) - k, \mu \frac{1-e^2}{na^2e} \left(\frac{d\Omega}{de} \right) \\ \frac{d\omega}{dt} &= k, \mu \frac{\sqrt{1-e^2}}{na^2e} \left(\frac{d\Omega}{de} \right) - k, \mu \frac{\cotgi}{na^2\sqrt{1-e^2}} \left(\frac{d\Omega}{di} \right) \\ \frac{de}{dt} &= k, \mu \frac{1-e^2}{na^2e} \left(\frac{d\Omega}{dc} \right) - k, \mu \frac{\sqrt{1-e^2}}{na^2e} \left(\frac{d\Omega}{d\omega} \right)\end{aligned}$$

*) Optandum esset, ut tabulae nostrae planetarum motum heliocentricum exhibentes loco reductionis longitudinis ad eclipticam et latitudinis supra idem planum reductionem longitudinis heliocentricae ad aequatorem mobilem et declinationem ab eodem plano continerent; hincinde calculus ascensionis rectae et declinationis geocentricae ad modum contraheretur.

*) V. Astr. Nachr. Nr. 166. p. 423 Aeqq. (2)

**) V. Untersuchung über die gegenseitigen Störungen des Jupiters und Saturns pag. 38.

***) Ibid. pag. 88.

$$\frac{di}{dt} = k_1 \mu \frac{\cot g i}{na^2 \sqrt{(1-e^2)}} \left(\frac{d\Omega}{d\omega} \right) - k_1 \mu \frac{\operatorname{cosec} i}{na^2 \sqrt{(1-e^2)}} \left(\frac{d\Omega}{d\theta} \right)$$

$$\frac{d\theta}{dt} = k_1 \mu \frac{\operatorname{cosec} i}{na^2 \sqrt{(1-e^2)}} \left(\frac{d\Omega}{di} \right)$$

ubi planetae cujus motum consideramus designant:

- a semiaxem majorem;
- ae excentricitatem;
- c anomaliam mediam in certo determinatoque tamquam epocham electo temporis momento;
- ω angulum inter perihelium et nodum ascendentem orbitae cum plano coordinatarum xy fixo ceterum ad arbitrium in spatio sito;
- i inclinationem plani orbitae ad idem planum;
- θ longitudinem praedicti nodi ascendentis;
- n motum medium sidereum unitate temporis t elapsam;
- m massam;
- k , mensuram vis attractivae corporis primarii vel centralis, circa quod motus relativi per aequationes praecedentes investigabuntur.

Praeterea positum est

$$\mu = 1 + m$$

$$\Omega = \frac{m'}{\mu} \left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \right) + \text{etc.}$$

ubi sunt:

$$\Delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

x, y, z coordinatae orthogoniae in spatio ad lubitum directae,

et quantitates, quibus tractus superne affixus est, ad corpus perturbans spectant.

2.

Denotantibus ν longitudinem veram in orbita, f anomaliam veram et π longitudinem perihelii, habemus, quoties perturbationes non adsunt

$$\nu = f + \pi$$

habemus quoque

$$\nu = f + \omega + \theta$$

Quae aequationes formulis notis, quae computationi longitudinis latitudinisque geocentricae inserviunt, superstructae sunt. Ratione habita perturbationum, eadem igitur aequationes valere debent, quare generaliter pono

$$\pi = \omega + \theta$$

Suppositio vero haec terminos illos per $tg \frac{1}{2} i$ multiplicatos gignit. Jam ad eos tollendos accipio esse

$$\pi, = \omega + f \cos i d\theta$$

et porro

$$\nu, = f + \pi,$$

unde prodit

$$\nu = \nu, + \int (1 - \cos i) d\theta \dots \dots \dots (1)$$

Introductio quantitatis $f \cos i d\theta$ duobus modis diversis considerari potest. Supponere potes, quantitatem hanc introductam artificium analyticum esse, quod solutione problematis peracta eo se commendat, quod ad finem propositum revera perducatur; significationem quoque geometricam ipsius ν , facile invenies, et hinc inde quantitatis illius introductionem aptam judicabis, quo termini illi tollantur *).

*) Quantitas ν certe longitudinem veram in orbita designat, sive perturbationes adsunt, sive non; nam ν summam anomaliam verae et longitudinis perihelii semper denotat. Si perturbationes desunt, vel potius si per vires perturbantes planum ipsum orbitae non movetur, ν est arcus, qui a puncto quodam fixo in eodem plano initium capit, sin planum orbitae in spatio movetur, de puncto fixo in eo sermo esse nequit, quare planum quoddam fixum eligi oportet, ut punctum in eo fixum detur, ad quod longitudines in orbita referantur. Quo in casu longitudo perihelii, itaque longitudo in orbita duobus arcibus in eodem plano non sitis constat, sive, quod idem est, initium ducit a puncto in orbita mobili, quod ab intersectione orbitae cum plano fixo retrorsum tantum distat, quantum punctum fixum directione eadem ab eadem intersectione abest. Longitudo igitur vera in orbita ab inclinatione longitudineque nodi orbitae cum isto plano fixo pendet.

Jam sint in figura juxta posita AB planum fixum sive planum ipsarum xy ; A punctum initiale longitudinum; CE planum orbitae tempore t ; C corporis locus, tum longitudo vera in orbita est

$$\nu = CE + \theta \dots \dots \dots (2)$$

quia $AE = \theta$. Temporis momento infinite parvo dt elapso, corpus ad punctum D transgressum esse suppono, ita ut DB sit

planum orbitae tempore $t + dt$. Jam nunc longitudo vera in orbita est

$$\nu + d\nu = DB + BA = DC + CB + BE + \theta$$

Demisso perpendiculari BF a puncto B in CE , obtinetur $EF = d\theta \cos i$ (quia $CEB = i$ atque $EB = d\theta$) et $CF = CB$, quoniam angulus ECB infinite parvus est. Hinc aequatio praecedens transit in hanc

$$\nu + d\nu = DC + CE - d\theta \cos i + d\theta + \theta$$

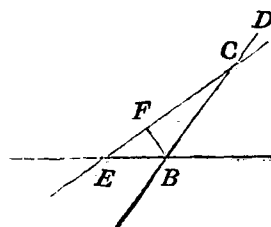
quae, subducta aequatione (2) praebet

$$d\nu = DC + (1 - \cos i) d\theta$$

qua aequatione cum differentiali aequationis (1) comparata, evadit

$$DC = d\nu,$$

Hinc sequitur, $d\nu$, angulum inter duos radios vectores ad tempora consequutiva t et $t + dt$ pertinentes interceptum denotare.



3.

Introducamus in aequationibus art. 1 quantitatem π , loco ω . Quem in finem habemus $d\pi = d\omega + \cos i d\theta$, et hinc loco aequationis valorem ipsius $\frac{d\omega}{dt}$ exhibentis hanc

$$\frac{d\pi}{dt} = k, \mu \frac{\sqrt{1-e^2}}{na^2e} \left(\frac{d\Omega}{de} \right).$$

Quantitas Ω hucusque pro functione ipsarum a, c, e, ω, i et θ habita est, nunc vero tamquam functio ipsarum a, c, e, π, i et θ considerari debet. Quem in finem ponimus,

$$\begin{aligned} d\Omega &= \left(\frac{d\Omega}{d\omega} \right) d\omega + \left(\frac{d\Omega}{d\theta} \right) d\theta + \text{etc.} \\ &= \left(\frac{d\Omega}{d\pi} \right) d\pi + \left(\frac{d\Omega}{d\theta} \right) d\theta + \text{etc.} \end{aligned}$$

ex qua, substituta aequatione $d\pi = d\omega + d\theta \cos i$ elicitur

$$\begin{aligned} \left(\frac{d\Omega}{d\omega} \right) &= \left(\frac{d\Omega}{d\pi} \right) \\ \left(\frac{d\Omega}{d\theta} \right) &= \left(\frac{d\Omega}{d\pi} \right) \cos i \end{aligned}$$

ubi lineae superpositae functionem ipsarum π , etc. indicent. Prodeunt igitur

$$\begin{aligned} \frac{de}{dt} &= k, \mu \frac{1-e^2}{na^2e} \left(\frac{d\Omega}{dc} \right) - k, \mu \frac{\sqrt{1-e^2}}{na^2e} \left(\frac{d\Omega}{d\pi} \right) \\ \frac{di}{dt} &= -k, \mu \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}} \left(\frac{d\Omega}{d\theta} \right) \end{aligned}$$

manentibus reliquis aequationibus art. 1 immutatis.

$$0 = -k, \mu \frac{\cotg i}{na^2\sqrt{1-e^2}} \left[\left(\frac{d\Omega}{d\theta} \right) \sin(\nu-\theta) + \left(\frac{d\Omega}{di} \right) \sin i \cos(\nu-\theta) \right] dt + \sin i \cos(\nu-\theta) \Delta \nu,$$

Sed infra demonstrabitur esse

$$\left(\frac{d\Omega}{d\theta} \right) = - \left(\frac{d\Omega}{di} \right) \cotg(\nu-\theta) \sin i$$

quare aequatio praecedens abit in

$$\Delta \nu = 0$$

Q. E. D.

5.

Si jam accipimus, ϕ ex elementis a, c, e , et quantitate indeterminata τ ita compositam esse, ut anomalia vera iisdem elementis et tempore t constat, et si ponimus

$$\lambda = \phi + \pi,$$

methodus quam in dissertatione mea priori de hoc argumento exposui, ne particula quidem neglecta, ad easdem aequationes differentiales pro ζ et $l\rho$, quas ibidem dedi, deducit, id quod quisque, computatione facili peracta, comperiet. Quibus itaque aequationibus *) quantitas λ , et

*) Sc. Astr. Nachr. Nr. 167 aeqq. (13) et (14) aut si mavis aeqq. (13) et (16).

4.

Jam satis notum est, aequationibus pro elementis variabilibus suppositionem inhaerere, quod coordinatae sive perturbatae sive inperturbatae eadem forma gaudent, id est quod differentialem coordinatarum ita sumta, ut elementa sola variabilia spectentur, cifrae aequalia fiunt, si valores differentialium ex aequationibus art. 1, vel ex quibuscunque earum transformationibus desumpti substituuntur. Licet proprietas haec de coordinatis a plano ipso orbitae independentibus proprie tantum valeat, tamen facile demonstrari potest, eam quoque pro quantitate ν , locum habere.

Sit s sinus latitudinis corporis supra planum ipsarum xy , tum habetur

$$s = \sin i \sin(\nu-\theta)$$

quae aequatio, introducta ν , in hanc abit

$$s = \sin i \sin(\nu - f \cos i d\theta)$$

Quum s ad planum fixum ab ipsa orbita independens referatur, certe differentiale ejus respectu elementorum evanescere debet. Sed denotato tali differentiali per praefixum signum Δ , obtinetur

$$\Delta s = 0 = \cos i \sin(\nu-\theta) di - \sin i \cos i \cos(\nu-\theta) d\theta + \sin i \cos(\nu-\theta) \Delta \nu,$$

atque hinc, substitutis pro di et $d\theta$ valoribus eorum ex artt. 1 et 3, elicitur

proinde, mutato τ in t , quantitas ν , cum rigore geometrico datur, atque hinc ope aequationis (1) ad longitudinem veram in orbita pervenitur.

6.

His ita absolutis, convertamus nos ad aequationes, per quas longitudo ad planum fixum reducta et latitudo supra idem planum supputandae sunt. Hae, secundum articulos praecedentes, sunt

$$s = \sin i \sin(\nu - f \cos i d\theta) \dots \dots \dots (3^*)$$

$$\left. \begin{aligned} \frac{di}{dt} &= -an \frac{\operatorname{cosec} i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\theta} \right) \\ \frac{d\theta}{dt} &= an \frac{\operatorname{cosec} i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{di} \right) \end{aligned} \right\} \dots \dots \dots (3)$$

(quoniam $a^3 n^2 = k, \mu$) et illa

$$\lg(l-\theta) = \cos i \lg(\nu-\theta)$$

ubi l longitudinem reductam denotat. Quae aequatio facili opera in sequentem transformatur, ubi simul ν per ν , eliminata est,

$$\sin[l - f(1 - \cos i) d\theta - \nu] = -tg^{\frac{1}{2}} i \sin[l - f(1 - \cos i) d\theta + \nu - 2f \cos i d\theta]$$

Jam quantitates duas p et q in formulis introducamus, quae ab i et θ tali modo pendeant

$$(4) \dots \dots \dots \begin{cases} p = \sin i \sin [f \cos i d\theta] \\ q = \sin i \cos [f \cos i d\theta] \end{cases}$$

Quae aequationes dant

$$tg^{\frac{1}{2}} i = \frac{p^2 + q^2}{[1 + \sqrt{(1 - p^2 - q^2)}]^2}$$

$$\sin [2f \cos i d\theta] = \frac{2pq}{p^2 + q^2}, \quad \cos [2f \cos i d\theta] = \frac{q^2 - p^2}{p^2 + q^2}$$

quibus praecedentes abeunt in has

$$(5) \dots \dots \dots s = q \sin \nu, - p \cos \nu,$$

$$\sin [l - f(1 - \cos i) d\theta - \nu] = \frac{2pq \cos [l - f(1 - \cos i) d\theta + \nu] + (p^2 - q^2) \sin [l - f(1 - \cos i) d\theta + \nu]}{[1 + \sqrt{(1 - p^2 - q^2)}]^2}$$

quarum alteram in hanc facile transferre licet:

$$(6) \dots \dots \dots l = \nu + f(1 - \cos i) d\theta + \text{arc. tg} \left[\frac{2pq \cos 2\nu + (p^2 - q^2) \sin 2\nu}{[1 + \sqrt{(1 - p^2 - q^2)}]^2 + 2pq \sin 2\nu - (p^2 - q^2) \cos 2\nu} \right]$$

Aequationes (4) differentiatiae praebent

$$(7) \dots \dots \dots \begin{cases} dp = \cos i \sin [f \cos i d\theta] di + \sin i \cos i \cos [f \cos i d\theta] d\theta \\ dq = \cos i \cos [f \cos i d\theta] di - \sin i \cos i \sin [f \cos i d\theta] d\theta \end{cases}$$

e quibus eliminando elicitur

$$(8) \dots \dots \dots \begin{cases} di = \frac{\sin [f \cos i d\theta]}{\cos i} dp + \frac{\cos [f \cos i d\theta]}{\cos i} dq \\ d\theta = \frac{\cos [f \cos i d\theta]}{\sin i \cos i} dp - \frac{\sin [f \cos i d\theta]}{\sin i \cos i} dq \end{cases}$$

Spectata vero Ω tum tamquam functione ipsarum i et θ , tum tamquam functione ipsarum p et q , differentiale ipsius Ω in utraque forma exhibitum praebet adjumento aequationum praecedentium has

$$(9) \dots \dots \dots \begin{cases} \left(\frac{d\Omega}{dp} \right) = \left(\frac{d\Omega}{di} \right) \frac{\sin [f \cos i d\theta]}{\cos i} + \left(\frac{d\Omega}{d\theta} \right) \frac{\cos [f \cos i d\theta]}{\sin i \cos i} \\ \left(\frac{d\Omega}{dq} \right) = \left(\frac{d\Omega}{di} \right) \frac{\cos [f \cos i d\theta]}{\cos i} - \left(\frac{d\Omega}{d\theta} \right) \frac{\sin [f \cos i d\theta]}{\sin i \cos i} \end{cases}$$

Si vero valores ipsarum $\frac{di}{dt}$ et $\frac{d\theta}{dt}$ ex (3) desumpti in (7) substituuntur, obtinentur

$$\frac{dp}{dt} = \frac{an}{\sqrt{(1-e^2)}} \cos i \cos [f \cos i d\theta] \left(\frac{d\Omega}{di} \right) - \frac{an}{\sqrt{(1-e^2)}} \frac{\cos i}{\sin i} \sin [f \cos i d\theta] \left(\frac{d\Omega}{d\theta} \right)$$

$$\frac{dq}{dt} = -\frac{an}{\sqrt{(1-e^2)}} \cos i \sin [f \cos i d\theta] \left(\frac{d\Omega}{di} \right) - \frac{an}{\sqrt{(1-e^2)}} \frac{\cos i}{\sin i} \cos [f \cos i d\theta] \left(\frac{d\Omega}{d\theta} \right)$$

quae ope aequationum (9) transeunt in has

$$(10) \dots \dots \dots \begin{cases} \frac{dp}{dt} = \frac{an}{\sqrt{(1-e^2)}} \cos^2 i \left(\frac{d\Omega}{dq} \right) \\ \frac{dq}{dt} = -\frac{an}{\sqrt{(1-e^2)}} \cos^2 i \left(\frac{d\Omega}{dp} \right) \end{cases}$$

7.

Aequationes (3*), (6) et (10), si factorem $\cos^2 i$ in (10) nec non terminum $f(1 - \cos i) d\theta$ in (6) excipis, et aequationes illae quas antea adhibui ejusdem formae sunt. Effectus itaque terminorum per $tg^{\frac{1}{2}} i$ multiplicatorum, quos neglexeram, ad quantitates $\cos^2 i$ et $f(1 - \cos i) d\theta$ re-

ductus est. Hinc sequitur, dummodo ad terminos in $f(1 - \cos i) d\theta$ per tempus ipsum multiplicatos respiciatur, errorem qui committitur, dum termini hi negliguntur, ordinis tertii respectu virium perturbantium esse, quoties planum ipsarum xy ita eligeatur, ut cum plano orbitae quod temporis electae epochae respondet, congruat, et proinde reductio longitudinis et latitudinis ad eclipticam mobilem ope formularum trigonometricarum absque integrationibus perficitur. Facillime statim reperitur errorem a factore $\cos^2 i$ ortum ordinis tertii evadere, idem respectu $f(1 - \cos i) d\theta$ sequenti modo demonstrabitur. Identica est aequatio haec

$$(1 - \cos i) d\theta = \frac{\sin i}{1 + \cos i} d\theta$$

quae ope alterius aequationis (8) et aequationum (4) abit in hanc

$$f(1 - \cos i) d\theta = \int \frac{q dp - p dq}{[1 + \sqrt{(1 - p^2 - q^2)}] \sqrt{(1 - p^2 - q^2)}}$$

Si nunc accipitur planum fixum esse orbitam temporis epochae respondentem, constantes arbitrariae integratis aequationibus (10) adiciendae, cifrae aequales sunt, et terminis periodicis neglectis poni potest

$$p = \alpha t + \beta t^2 + \dots$$

$$q = \alpha' t + \beta' t^2 + \dots$$

Quibus valoribus in aequatione praecedenti substitutis, invenitur;

$$f(1 - \cos i) d\theta = \frac{\alpha\beta - \alpha'\beta'}{6} t^3 + \text{etc.}$$

quam tertii ordinis respectu massarum esse quisque videt.

8.

Quum vero adjumento formularum hic evolutarum absque accuracionis detrimento planum fixum quodcunque eligere liceat, e re erit quotientes differentiales ipsius Ω , manente i arbitraria, evolvere. Quae etiam evolutio ad cognitionem terminorum in art. 10 theoriae meae Jovis atque Saturni omissorum perducet.

Priusquam evolutionem hanc aggredior, annoto, in casu ubi i arbitraria est, constantes integratis aequationibus (10) adiciendas cifrae non aequari, sed valores eos ipsarum p et q iis attribuendos esse, qui subductis perturbationibus in temporis epocha locum habeant. Denotatis itaque pro hoc tempore inclinatione et longitudine nodi ascendentis orbitae cum plano fixo per (i) et (θ) , nec non valore integralis $f(1 - \cos i) d\theta$ per c , habetur pro eodem tempore

$$f \cos i d\theta = (\theta) - c$$

Si itaque accipies

$$(p) = \sin(i) \sin[(\theta) - c]$$

$$(q) = \sin(i) \cos[(\theta) - c]$$

constantes illae (p) et (q) sunt.

Ad $\left(\frac{d\Omega}{dp}\right)$ et $\left(\frac{d\Omega}{dq}\right)$ evolvendas quantitatem Ω tam-

$$\begin{aligned} \left(\frac{d\Omega}{di}\right) &= \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \left[\sin i' \cos i \sin u' - \cos(\theta' - \theta) \cos i' \sin i \sin u' - \sin(\theta' - \theta) \sin i \cos u' \right] rr' \sin u \\ \left(\frac{d\Omega}{d\theta}\right) &= \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \left[\cos(\theta' - \theta) \cos i' \sin u' + \sin(\theta' - \theta) \cos u' \right. \\ &\quad \left. + \sin(\theta' - \theta) \cos i' \cos i \sin u' \operatorname{tg} u - \cos(\theta' - \theta) \cos i \cos u' \operatorname{tg} u \right] rr' \cos u \\ \left(\frac{d\Omega}{d\omega}\right) &= \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \left[\cos(\theta' - \theta) \cos i' \cos i \sin u' + \sin(\theta' - \theta) \cos i \cos u' \right. \\ &\quad \left. + \sin(\theta' - \theta) \cos i' \sin u' \operatorname{tg} u - \cos(\theta' - \theta) \cos u' \operatorname{tg} u + \sin i' \sin i \sin u' \right] rr' \cos u \end{aligned}$$

quam functionem ipsarum ω, i, θ etc. exhibebo, quo fit ut coordinatis forma haec attribui possit

$$x' = r' \sin(\theta - \theta') \cos i' \sin u' + r' \cos(\theta - \theta') \cos u'$$

$$y' = r' \cos(\theta - \theta') \cos i' \sin u' - r' \sin(\theta - \theta') \cos u'$$

$$z' = r' \sin i' \sin u'$$

$$x = r \cos u$$

$$y = r \sin u \cos i$$

$$z = r \sin u \sin i$$

ubi r et r' radios vectores denotant, et

$$u = f + \omega$$

$$u' = f' + \omega'$$

sunt. Quibus coordinatarum valoribus in

$$\Delta^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

substitutis, evadit

$$\begin{aligned} \Delta^2 &= r^2 + r'^2 - 2rr' \cos(\theta' - \theta) \cos u' \cos u \\ &\quad - 2rr' \cos(\theta' - \theta) \cos i \cos i' \sin u' \sin u \\ &\quad + 2rr' \sin(\theta' - \theta) \cos i' \sin u' \cos u \\ &\quad - 2rr' \sin(\theta' - \theta) \cos i \cos u' \sin u \\ &\quad - 2rr' \sin i \sin i' \sin u' \sin u \end{aligned}$$

e qua differentiando elicitur

$$\begin{aligned} \Delta \left(\frac{d\Delta}{di} \right) &= rr' \cos(\theta' - \theta) \sin i \cos i' \sin u' \sin u \\ &\quad + rr' \sin(\theta' - \theta) \sin i \cos u' \sin u \\ &\quad - rr' \cos i \sin i' \sin u' \sin u \end{aligned}$$

$$\begin{aligned} \Delta \left(\frac{d\Delta}{d\theta} \right) &= -rr' \sin(\theta' - \theta) \cos u' \cos u \\ &\quad - rr' \sin(\theta' - \theta) \cos i \cos i' \sin u' \sin u \\ &\quad - rr' \cos(\theta' - \theta) \cos i' \sin u' \cos u \\ &\quad + rr' \cos(\theta' - \theta) \cos i \cos u' \sin u \end{aligned}$$

$$\begin{aligned} \Delta \left(\frac{d\Delta}{d\omega} \right) &= rr' \cos(\theta' - \theta) \cos u' \sin u \\ &\quad - rr' \cos(\theta' - \theta) \cos i \cos i' \sin u' \cos u \\ &\quad - rr' \sin(\theta' - \theta) \cos i' \sin u' \sin u \\ &\quad - rr' \sin(\theta' - \theta) \cos i \cos u' \cos u \\ &\quad - rr' \sin i \sin i' \sin u' \cos u \end{aligned}$$

Quum vero sit

$$xx' + yy' + zz' = \frac{r^2 + r'^2}{2} - \frac{1}{2} \Delta^2$$

et

$$\Omega = \frac{m'}{\mu} \left[\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3} \right]$$

e quotientibus differentialibus praecedentibus facile obtinentur hae

Sed ex art. 3 sequitur esse

$$\left(\frac{d\Omega}{d\theta}\right) = \left(\frac{d\Omega}{d\theta}\right) - \cos i \left(\frac{d\Omega}{d\omega}\right)$$

hinc, substitutis valoribus quotientium differentialium modo

$$U = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) [\sin i' \cos i \sin(\nu' - \theta') - \cos(\theta' - \theta) \cos i' \sin i \sin(\nu' - \theta') - \sin(\theta' - \theta) \sin i \cos(\nu' - \theta')] r r'$$

positum est, et loco u et u' valores earum $\nu - \theta$ et $\nu' - \theta'$ scripti sunt. Per eandem quantitatem U etiam $\left(\frac{d\Omega}{di}\right)$ commodè exprimi potest, et statim invenitur

$$\left(\frac{d\Omega}{di}\right) = U \sin(\nu - \theta)$$

$$(11) \dots \dots \dots \begin{cases} \left(\frac{d\Omega}{dp}\right) = -\frac{U}{\cos i} \cos[\nu - f(1 - \cos i) d\theta] = -\frac{U}{\cos i} \cos \nu, \\ \left(\frac{d\Omega}{dq}\right) = \frac{U}{\cos i} \sin[\nu - f(1 - \cos i) d\theta] = \frac{U}{\cos i} \sin \nu, \end{cases}$$

9.

Quantitas Ω et quotientes ejus differentiales adhuc consideratae sunt tamquam functiones ipsarum i, θ, i' et θ' , sive quod idem est, tamquam functiones ipsarum p, q, p' et q' , neque aliter considerari debuerunt. E re tamen erit, loco illarum quantitatuum inclinationem reciprocam longitudinesque nodorum huic respondentes introducere. Licet statim quantitas Δ per has quantitates expressa adscribi possit, tamen rei accommodatissimum est, ut transformatio ab exordio perficiatur. Quem in finem definitiones sequentes notandae sunt. Nodus ascendens orbitae m' in orbita m est intersectio ea ambarum orbitalium, quam corpora, siquidem iis eadem longitudo semper attribuitur, transgressa erant, quoties latitudo corporis m' supra planum fixum latitudine ipsius m supra idem planum major (sensu algebraico) evasit. Longitudo nodi ascendens orbitae m' in orbita m est arcus, qui ab intersectione modo definita proficiens in orbitae m plano motu contrario sive retro ad punctum ex-

inventis, obtinetur

$$\left(\frac{d\Omega}{d\theta}\right) = -U \sin i \cos(\nu - \theta)$$

ubi brevitatè caussa

Quae expressiones ante omnia aequationem hanc

$$\left(\frac{d\Omega}{d\theta}\right) = -\left(\frac{d\Omega}{di}\right) \sin i \cotg(\nu - \theta)$$

praebent, qua jam in art. 4 usus sum. Substitutis vero his valoribus ipsarum $\left(\frac{d\Omega}{di}\right)$ et $\left(\frac{d\Omega}{d\theta}\right)$ in (9), evadunt

currit, quod a nodo ascendenti orbitae m in plano fixo retrorsum tantum abest, quantum punctum initiale longitudinum in plano fixo ab eodem nodo retrorsum distat; istius nodi longitudinem per Θ denotabo. Longitudo nodi descendens orbitae m in orbita m' est arcus qui ex eadem intersectione ambarum orbitalium in plano orbitae m' regreditur ad punctum, quod a nodo ascendenti orbitae m' in plano fixo retrorsum tantum distat, quantum idem punctum initiale, de quo modo loquebar, ab eodem orbitae m' nodo in plano fixo retrorsum remotum est; istius nodi longitudinem per Θ , designabo. Praeterea sit inclinatio reciproca orbitalium per I denotata.

His praemissis, facile reperitur, in triangulo sphaerico ab ambabus orbitis et plano fixo formato esse latera $\Theta - \theta$, $\Theta - \theta'$ et $\theta' - \theta$, angulosque iis resp. oppositos $180^\circ - i$, i et I . Ex relationibus vero quas trigonometria sphaerica inter illas trianguli partes suppeditat, sequentes ad propositum meum maxime accommodatas affero.

- (a) $\sin I \sin(\Theta, -\theta')$ = $\sin i \sin(\theta' - \theta)$
- (b) $\sin I \cos(\Theta, -\theta')$ = $\cos i \sin i' - \sin i \cos i' \cos(\theta' - \theta)$
- (c) $\cos I$ = $\cos i \cos i' + \sin i \sin i' \cos(\theta' - \theta)$
- (d) $\sin(\Theta - \theta) \sin i$ = $\sin(\Theta, -\theta') \sin i'$
- (e) $\sin(\Theta - \theta) \cos i$ = $\cos(\Theta, -\theta') \sin(\theta' - \theta) + \sin(\Theta, -\theta') \cos(\theta' - \theta) \cos i'$
- (f) $\cos(\Theta - \theta)$ = $\cos(\Theta, -\theta') \cos(\theta' - \theta) - \sin(\Theta, -\theta') \sin(\theta' - \theta) \cos i'$
- (g) $\sin(\Theta - \theta) \cos I$ = $\cos(\theta' - \theta) \sin(\Theta, -\theta) + \sin(\theta' - \theta) \cos(\Theta, -\theta') \cos i'$
- (h) $\cos(\Theta, -\theta')$ = $\cos(\Theta - \theta) \cos(\theta' - \theta) + \sin(\Theta - \theta) \sin(\theta' - \theta) \cos i$

Si nunc in expressione ipsum Δ^2 art. praec. loco u et u' valores earum hi $\nu - \theta$ et $(\nu' - \Theta) + (\Theta, -\theta')$

substituuntur, atque sinus et cosinus alterius arcus secundum $\nu' - \Theta$, et $\Theta, -\theta'$ resolvuntur, statim prodit

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos(\nu' - \Theta) \left\{ \begin{array}{l} \cos(\nu - \theta) [\cos(\Theta, -\theta') \cos(\theta' - \theta) - \sin(\Theta, -\theta') \sin(\theta' - \theta) \cos i] \\ + \sin(\nu - \theta) [(\cos(\Theta, -\theta') \sin(\theta' - \theta) + \sin(\Theta, -\theta') \cos(\theta' - \theta) \cos i) \cos i + \sin(\Theta, -\theta') \sin i \sin i] \end{array} \right\} \\ + 2rr' \sin(\nu' - \Theta) \left\{ \begin{array}{l} \cos(\nu - \theta) [\cos(\theta' - \theta) \sin(\Theta, -\theta') + \sin(\theta' - \theta) \cos(\Theta, -\theta') \cos i] \\ - \sin(\nu - \theta) [\cos(\Theta, -\theta') (\sin i \sin i + \cos i \cos i \cos(\theta' - \theta)) - \sin(\Theta, -\theta') \sin(\theta' - \theta) \cos i] \end{array} \right\}$$

Cujus expressionis termini, si coefficientem ipsius $\sin(\nu' - \Theta)$ excipis, ope relationum (d) usque ad (g) immediate contrahi possunt, hunc vero terminum sequenti modo ad formam simplicissimam redigere licet. Multiplicato $\sin(\Theta, -\theta') \sin(\theta' - \theta) \cos i$ per $\cos^2 i + \sin^2 i$, terminus de quo loquor sic exprimi potest

$$\sin i [\sin i \cos(\Theta, -\theta') - \sin i' \sin(\Theta, -\theta') \cdot \cos i \sin(\theta' - \theta)] \\ + \cos i \cos i' [\cos(\Theta, -\theta') \cos(\theta' - \theta) - \sin(\Theta, -\theta') \sin(\theta' - \theta) \cos i] \\ \text{quae expressio adjumento relationum (d) et (f) in hanc transit}$$

$$\sin i \sin i' [\cos(\Theta, -\theta') - \sin(\Theta - \theta) \sin(\theta' - \theta) \cos i] \\ + \cos i \cos i' \cos(\Theta - \theta)$$

Ratione vero habita relationis (h), haec abit in

$$[\cos i \cos i' + \sin i \sin i' \cos(\theta' - \theta)] \cos(\Theta - \theta)$$

quae ope relationis (c) in formam simplicissimam immediate transmutatur.

Itaque, si relationes (c) usque ad (g) substitueris, quantitates i' et θ' eliminatae erunt, et i atque θ sua sponte evanescent. Unde nanciscemur

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos(\nu' - \Theta) \cos(\nu - \Theta) \\ - 2rr' \cos I \sin(\nu' - \Theta) \sin(\nu - \Theta)$$

$$U = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \left\{ \begin{array}{l} \sin(\nu' - \Theta) [(\cos i \sin i' - \sin i \cos i' \cos(\theta' - \theta)) \cos(\Theta, -\theta') + \sin i \sin(\theta' - \theta) \sin(\Theta, -\theta')] \\ \cos(\nu' - \Theta) [(\cos i \sin i' - \sin i \cos i' \cos(\theta' - \theta)) \sin(\Theta, -\theta') - \sin i \sin(\theta' - \theta) \cos(\Theta, -\theta')] \end{array} \right\}$$

quae ope relationum (a) et (b) una cum praecedentibus aequationibus statim transmutatur in

$$U = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin(\nu' - \Phi) \sin I$$

Quae aequationes, si factorem $\cos i$ et differentiam inter Φ , et ϕ excipis, cum illis congruunt, quas in art. 10 theoriae meae Jovis atque Saturni, posito $i = 0$, inveni, termini igitur illic neglecti ad factorem hunc differentiamque

$$\left(\frac{d\Omega}{dI} \right) = - \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin I \sin(\nu' - \Phi) \sin(\nu - \Phi)$$

$$\left(\frac{d\Omega}{d\Phi} \right) = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' [\cos(\nu' - \Phi) \sin(\nu - \Phi) - \cos I \sin(\nu' - \Phi) \cos(\nu - \Phi)]$$

$$\left(\frac{d\Omega}{d\Phi'} \right) = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' [\sin(\nu' - \Phi) \cos(\nu - \Phi) - \cos I \cos(\nu' - \Phi) \sin(\nu - \Phi)]$$

quae expressio a vulgari ipsius Δ^2 expressione eo tantum discrepat, quod Θ , et ϕ identicae non sunt. Quoties plana orbitalium in spatio non moventur, dissimilitudo haec nullam vim habet, quoties vero moventur res secus se habet, quia tum $\Theta, -\phi$ functio temporis est.

Jam habuimus aequationem hanc

$$\nu = \nu' + f(1 - \cos i) d\theta$$

cui similis est haec, quae ad corpus perturbans pertinet

$$\nu' = \nu'_i + f(1 - \cos i') d\theta'$$

quibuscum ν et ν' in expressione praecedenti eliminare licet, unde obtinetur

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos(\nu' - \Phi) \cos(\nu - \Phi) \\ - 2rr' \cos I \sin(\nu' - \Phi) \sin(\nu - \Phi)$$

ubi brevitatis gratia

$$\Phi = \Theta - f(1 - \cos i) d\theta$$

$$\Phi' = \Theta' - f(1 - \cos i') d\theta'$$

scriptum est.

Resolvantur in valore ipsius U in art. praec. dato sinus et cosinus ipsius $\nu' - \theta'$ secundum $\nu' - \Theta$, et $\Theta, -\theta'$, hinc evadit

10.

Substitutis valoribus ipsarum $\left(\frac{d\Omega}{dp} \right)$ et $\left(\frac{d\Omega}{dq} \right)$ ex (11) in (10), ope expressionis praecedentis ipsius U obtinebitur

$$\frac{dp}{dt} = \frac{m'}{\mu} \frac{an}{\sqrt{(1-e^2)}} \cos i \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin I \sin(\nu' - \Phi) \sin \nu,$$

$$\frac{dq}{dt} = \frac{m'}{\mu} \frac{an}{\sqrt{(1-e^2)}} \cos i \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin I \sin(\nu' - \Phi) \cos \nu,$$

hanc reducti sunt, nec in ullo casu effectum alium proferre possunt.

Adjumento valoris ipsius Δ^2 in art. praec. inventi facile reperiuntur hae

Multiplicata prima harum aequationum per $-\cos \varphi$, secunda per $\cotg I \sin \varphi$, tertia per $\operatorname{cosec} I \sin \varphi$, additisque productis, invenitur

$$-\left(\frac{d\Omega}{dI}\right) \cos \varphi + \left[\left(\frac{d\Omega}{d\varphi}\right) \cotg I + \left(\frac{d\Omega}{d\varphi}\right) \operatorname{cosec} I \right] \sin \varphi \\ = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin I \sin (\nu' - \varphi) \sin \nu,$$

Multiplicata prima earundem per $\sin \varphi$, secunda per

$$(12) \dots \dots \dots \begin{cases} \frac{dp}{dt} = -\frac{an \cos i}{\sqrt{1-e^2}} \left[\left(\frac{d\Omega}{dI}\right) \cos \varphi - \left[\left(\frac{d\Omega}{d\varphi}\right) \cotg I + \left(\frac{d\Omega}{d\varphi}\right) \operatorname{cosec} I \right] \sin \varphi \right] \\ \frac{dq}{dt} = \frac{an \cos i}{\sqrt{1-e^2}} \left[\left(\frac{d\Omega}{dI}\right) \sin \varphi + \left[\left(\frac{d\Omega}{d\varphi}\right) \cotg I + \left(\frac{d\Omega}{d\varphi}\right) \operatorname{cosec} I \right] \cos \varphi \right] \end{cases}$$

11

Jam nunc quantitates duas P et Q in formulis intro-
ducam, quae his aequationibus determinandae sint

$$(13) \dots \dots \dots \begin{cases} dP = \cos \frac{1}{2} I \sin \frac{1}{2} (\varphi + \varphi) dI + \sin \frac{1}{2} I \cos \frac{1}{2} (\varphi + \varphi) d.(\varphi + \varphi) \\ dQ = \cos \frac{1}{2} I \cos \frac{1}{2} (\varphi + \varphi) dI - \sin \frac{1}{2} I \sin \frac{1}{2} (\varphi + \varphi) d.(\varphi + \varphi) \end{cases}$$

nec non has

$$(14) \dots \dots \dots \begin{cases} dI = \frac{\sin \frac{1}{2} (\varphi + \varphi)}{\cos \frac{1}{2} I} dP + \frac{\cos \frac{1}{2} (\varphi + \varphi)}{\cos \frac{1}{2} I} dQ \\ d.(\varphi + \varphi) = \frac{\cos \frac{1}{2} (\varphi + \varphi)}{\sin \frac{1}{2} I} dP - \frac{\sin \frac{1}{2} (\varphi + \varphi)}{\sin \frac{1}{2} I} dQ \end{cases}$$

Habita Ω tum pro functione ipsarum φ , et φ , tum pro

$$(15) \dots \dots \dots \begin{cases} \left(\frac{d\Omega}{dI}\right) = \left(\frac{d\Omega}{dP}\right) \cos \frac{1}{2} I \sin \frac{1}{2} (\varphi + \varphi) + \left(\frac{d\Omega}{dQ}\right) \cos \frac{1}{2} I \cos \frac{1}{2} (\varphi + \varphi) \\ \left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) = 2 \left(\frac{d\Omega}{dP}\right) \sin \frac{1}{2} I \cos \frac{1}{2} (\varphi + \varphi) - 2 \left(\frac{d\Omega}{dQ}\right) \sin \frac{1}{2} I \sin \frac{1}{2} (\varphi + \varphi) \end{cases}$$

atque

$$(16) \dots \dots \dots \begin{cases} \left(\frac{d\Omega}{dP}\right) = \left(\frac{d\Omega}{dI}\right) \frac{\sin \frac{1}{2} (\varphi + \varphi)}{\cos \frac{1}{2} I} + \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\cos \frac{1}{2} (\varphi + \varphi)}{\sin \frac{1}{2} I} \\ \left(\frac{d\Omega}{dQ}\right) = \left(\frac{d\Omega}{dI}\right) \frac{\cos \frac{1}{2} (\varphi + \varphi)}{\cos \frac{1}{2} I} - \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\sin \frac{1}{2} (\varphi + \varphi)}{\sin \frac{1}{2} I} \end{cases}$$

Si perpendimus, esse $\left(\frac{d\Omega}{d\nu}\right) = -\left(\frac{d\Omega}{d\varphi}\right)$ quoniam Ω est functio ipsius $\nu, -\varphi$, atque si aequationes praecedentes in (12) substituerimus, obtinebimus

$$(17) \dots \dots \dots \begin{cases} \frac{dp}{dt} = -\frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ}\right) \cos \frac{1}{2} I - \left[\frac{1}{2} \left(\frac{d\Omega}{d\nu}\right) + \frac{Q}{4} \left(\frac{d\Omega}{dP}\right) - \frac{P}{4} \left(\frac{d\Omega}{dQ}\right) \right] \frac{P}{\cos \frac{1}{2} I} \right\} \cos \frac{1}{2} (\varphi, -\varphi) \\ \quad - \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP}\right) \cos \frac{1}{2} I + \left[\frac{1}{2} \left(\frac{d\Omega}{d\nu}\right) + \frac{Q}{4} \left(\frac{d\Omega}{dP}\right) - \frac{P}{4} \left(\frac{d\Omega}{dQ}\right) \right] \frac{Q}{\cos \frac{1}{2} I} \right\} \sin \frac{1}{2} (\varphi, -\varphi) \\ \frac{dq}{dt} = \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP}\right) \cos \frac{1}{2} I + \left[\frac{1}{2} \left(\frac{d\Omega}{d\nu}\right) + \frac{Q}{4} \left(\frac{d\Omega}{dP}\right) - \frac{P}{4} \left(\frac{d\Omega}{dQ}\right) \right] \frac{Q}{\cos \frac{1}{2} I} \right\} \cos \frac{1}{2} (\varphi, -\varphi) \\ \quad - \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ}\right) \cos \frac{1}{2} I - \left[\frac{1}{2} \left(\frac{d\Omega}{d\nu}\right) + \frac{Q}{4} \left(\frac{d\Omega}{dP}\right) - \frac{P}{4} \left(\frac{d\Omega}{dQ}\right) \right] \frac{P}{\cos \frac{1}{2} I} \right\} \sin \frac{1}{2} (\varphi, -\varphi) \end{cases}$$

12.

Quum punctum initiale longitudinum plane arbitrium sit, ab uno quoque arcu arcum arbitrium subtrahere licet.

$\cotg I \cos \varphi$, tertia per $\operatorname{cosec} I \cos \varphi$, additisque productis, obtinetur

$$\left(\frac{d\Omega}{dI}\right) \sin \varphi + \left[\left(\frac{d\Omega}{d\varphi}\right) \cotg I + \left(\frac{d\Omega}{d\varphi}\right) \operatorname{cosec} I \right] \cos \varphi \\ = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr' \sin I \sin (\nu' - \varphi) \cos \nu,$$

aequationes igitur praecedentes ipsas $\frac{dp}{dt}$ et $\frac{dq}{dt}$ praebentes fiunt

$$P = 2 \sin \frac{1}{2} I \sin \frac{1}{2} (\varphi + \varphi)$$

$$Q = 2 \sin \frac{1}{2} I \cos \frac{1}{2} (\varphi + \varphi)$$

Ex his aequationibus nanciscimur has

functione ipsarum $(\varphi, +\varphi)$ et $(\varphi, -\varphi)$, adipiscimur

$$\left(\frac{d\Omega}{d.(\varphi, +\varphi)}\right) = \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right]$$

$$\left(\frac{d\Omega}{d.(\varphi, -\varphi)}\right) = \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) - \left(\frac{d\Omega}{d\varphi}\right) \right]$$

Porro, habita Ω tum pro functione ipsarum I et $\varphi, +\varphi$, tum pro functione ipsarum P et Q inveniuntur

Quum vero Ω et quotientes ejus differentiales omnes, exceptis his $\left(\frac{d\Omega}{dP}\right)$ et $\left(\frac{d\Omega}{dQ}\right)$, a puncto illo initiali indepen-

dentes sint, nullum inde lucrum redundaret, si a longitudinibus quae in istis quantitatibus occurrunt, arcum quemdam subtraheremus. Si igitur hoc artificio uti volumus, reliquum est, ut arcus propositus ab arcubus in $\left(\frac{d\Omega}{dP}\right)$ et $\left(\frac{d\Omega}{dQ}\right)$ extra $\left(\frac{d\Omega}{dI}\right)$ et $\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right)$ jacentibus nec

$$\begin{aligned} \left(\frac{d\Omega}{dP}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\sin \frac{1}{2}(\varphi, -\varphi)}{\cos \frac{1}{2}I} + \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\cos \frac{1}{2}(\varphi, -\varphi)}{\sin \frac{1}{2}I} \\ \left(\frac{d\Omega}{dQ}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\cos \frac{1}{2}(\varphi, -\varphi)}{\cos \frac{1}{2}I} - \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\sin \frac{1}{2}(\varphi, -\varphi)}{\sin \frac{1}{2}I} \end{aligned} \quad (18)$$

quarum aequationum ope (17) transeunt in has

$$\begin{aligned} \frac{dp}{dt} &= -\frac{an \cos i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{dI}\right) \\ \frac{dq}{dt} &= \frac{an \cos i}{\sqrt{(1-e^2)}} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{1}{\sin I} + \frac{an \cos i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{d\nu}\right) \operatorname{tg} \frac{1}{2}I \end{aligned} \quad (19)$$

Quum $\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right)$ eadem quantitas sit, quam alibi per $\left(\frac{d\Omega}{d\Theta}\right)$ denotavi, aequationes (19), facto $i = 0$, cum iis, quas in theoriae meae Jovis atque Saturni pag. 42 invenieram, congruunt. Constantes vero integratis aequationibus praecedentibus adjiciendae, manifesto sunt

$$\begin{aligned} (p) &= \sin(i) \sin[(\theta) - c - \varphi] \\ (q) &= \sin(i) \cos[(\theta) - c - \varphi] \end{aligned}$$

et sinus latitudinis hac aequatione datus est

$$s = q \sin(\nu, -\varphi) - p \cos(\nu, -\varphi)$$

$$\begin{aligned} \left|\frac{dp}{dt}\right| &= -\frac{an \cos i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{dI}\right) \cos \frac{1}{2}(\varphi, -\varphi) \\ &\quad - \frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{1}{\sin I} + \left(\frac{d\Omega}{d\nu}\right) \operatorname{tg} \frac{1}{2}I \right\} \sin \frac{1}{2}(\varphi, -\varphi) \\ \left|\frac{dq}{dt}\right| &= \frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{1}{\sin I} + \left(\frac{d\Omega}{d\nu}\right) \operatorname{tg} \frac{1}{2}I \right\} \cos \frac{1}{2}(\varphi, -\varphi) \\ &\quad - \frac{an \cos i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{dI}\right) \sin \frac{1}{2}(\varphi, -\varphi) \end{aligned}$$

Quibus aequationibus cum (19) comparatis, facile invenitur esse

$$(20) \dots \begin{cases} \left|\frac{dp}{dt}\right| = \frac{dp}{dt} \cos \frac{1}{2}(\varphi, -\varphi) - \frac{dq}{dt} \sin \frac{1}{2}(\varphi, -\varphi) \\ \left|\frac{dq}{dt}\right| = \frac{dp}{dt} \sin \frac{1}{2}(\varphi, -\varphi) + \frac{dq}{dt} \cos \frac{1}{2}(\varphi, -\varphi) \end{cases}$$

nec non

$$(21) \dots \begin{cases} \frac{dp}{dt} = \left|\frac{dp}{dt}\right| \cos \frac{1}{2}(\varphi, -\varphi) + \left|\frac{dq}{dt}\right| \sin \frac{1}{2}(\varphi, -\varphi) \\ \frac{dq}{dt} = -\left|\frac{dp}{dt}\right| \sin \frac{1}{2}(\varphi, -\varphi) + \left|\frac{dq}{dt}\right| \cos \frac{1}{2}(\varphi, -\varphi) \end{cases}$$

Generaliter, designantibus $\left[\frac{dp}{dt}\right]$ et $\left[\frac{dq}{dt}\right]$ valores harum quantitatum, postquam in (17) arcus quidam α subtractus

non ab arcubus alias occurrentibus deducatur. Propositum est in formulis art. praec. arcum φ subtrahere, quo facto sunt

$$\begin{aligned} P &= 2 \sin \frac{1}{2}I \sin \frac{1}{2}(\varphi, -\varphi) \\ Q &= 2 \sin \frac{1}{2}I \cos \frac{1}{2}(\varphi, -\varphi) \end{aligned}$$

atque aequationes (16) in has abeunt

$$\begin{aligned} \left(\frac{d\Omega}{dP}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\sin \frac{1}{2}(\varphi, -\varphi)}{\cos \frac{1}{2}I} + \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\cos \frac{1}{2}(\varphi, -\varphi)}{\sin \frac{1}{2}I} \\ \left(\frac{d\Omega}{dQ}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\cos \frac{1}{2}(\varphi, -\varphi)}{\cos \frac{1}{2}I} - \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{\sin \frac{1}{2}(\varphi, -\varphi)}{\sin \frac{1}{2}I} \end{aligned} \quad (18)$$

Simili modo expressio pro l supra data transmutatur. Jam nunc subtrahatur arcus $\frac{1}{2}(\varphi, +\varphi)$ in aequationibus art. praec., hinc

$$\begin{aligned} P &= 0 \\ Q &= 2 \sin \frac{1}{2}I \\ \left(\frac{d\Omega}{dP}\right) &= \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi}\right) + \left(\frac{d\Omega}{d\varphi}\right) \right] \frac{1}{\sin \frac{1}{2}I} \\ \left(\frac{d\Omega}{dQ}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{1}{\cos \frac{1}{2}I} \end{aligned}$$

ac si nunc differentialia ipsarum p et q , quo ab illis modo evolutis discernantur, lineis duabus includuntur, obtinemus

est, conservatisque signis his $\frac{dp}{dt}$ et $\frac{dq}{dt}$ pro iis in quibus arcus φ subtractus erat, aequationes (19) et (17), vel potius, (19) et (12) facili computandi ratione praebent

$$\begin{aligned} \left[\frac{dp}{dt}\right] &= \frac{dp}{dt} \cos(\varphi - \alpha) + \frac{dq}{dt} \sin(\varphi - \alpha) \\ \left[\frac{dq}{dt}\right] &= -\frac{dp}{dt} \sin(\varphi - \alpha) + \frac{dq}{dt} \cos(\varphi - \alpha) \end{aligned} \quad (22)$$

nec non

$$\begin{aligned} \frac{dp}{dt} &= \left[\frac{dp}{dt}\right] \cos(\varphi - \alpha) - \left[\frac{dq}{dt}\right] \sin(\varphi - \alpha) \\ \frac{dq}{dt} &= \left[\frac{dp}{dt}\right] \sin(\varphi - \alpha) + \left[\frac{dq}{dt}\right] \cos(\varphi - \alpha) \end{aligned} \quad (23)$$

in his aequationibus jam praecedentes quasi casus specialis continentur.

13.

Ex re est, loco φ longitudinem nodi ascendentis orbitae m' in m eam, quae temporis epocha locum habuit, subtrahere. Qua longitudine per (Θ) denotata, habemus pro hoc tempore secundum artt. 8 et 9 $\varphi = (\Theta) - c$, in aequationibus itaque (22) poni debet $\alpha = (\Theta) = \varphi + c$, unde elicitur

$$(24) \dots \left\{ \begin{aligned} \left[\frac{dp}{dt} \right] &= \frac{dp}{dt} \cos c - \frac{dq}{dt} \sin c \\ \left[\frac{dq}{dt} \right] &= \frac{dq}{dt} \cos c + \frac{dp}{dt} \sin c \end{aligned} \right.$$

Sed considerando c tamquam quantitatem primi ordinis respectu massarum, negligendoque in p et q terminos secundi ordinis, aequationes praecedentes praebent

$$(25) \dots \left[\frac{dp}{dt} \right] = \frac{dp}{dt}, \quad \left[\frac{dq}{dt} \right] = \frac{dq}{dt}$$

$$(25) \dots l = \nu + \int \frac{q dp - p dq}{[1 + \sqrt{(1-p^2-q^2)}] \sqrt{(1-p^2-q^2)}} + \text{arc. tg.} \left[\frac{(p^2-q^2) \sin 2(\nu, -(\Theta)) + 2pq \cos 2(\nu, -(\Theta))}{[1 + \sqrt{(1-p^2-q^2)}]^2 - (p^2-q^2) \cos 2(\nu, -(\Theta)) + 2pq \sin 2(\nu, -(\Theta))} \right]$$

14.

Ante omnia perscrutandum est, quomodo valores ipsarum P , Q et φ , $-\varphi$, si ad earum perturbationes respicitur, compositi sunt. Quem in finem expressiones ipsarum dI , $d\varphi$, et $d\varphi$ investigandae sunt. Praeter relationes ad triangulum inter planum fixum et ambas orbitas spectantes, quae in art. 9 allatae sunt, sequentes appono

$$(i) \dots \sin I \sin (\Theta - \theta) = \sin i' \sin (\theta' - \theta)$$

$$(k) \dots \sin I \cos (\Theta - \theta) = -\cos i' \sin i + \sin i' \cos i \cos (\theta' - \theta)$$

$$(l) \dots \sin (\Theta, -\theta') \cos i' = -\cos (\Theta - \theta) \sin (\theta' - \theta) + \sin (\Theta - \theta) \cos (\theta' - \theta) \cos i$$

$$(m) \dots \sin (\Theta, -\theta') \cos I = \sin (\Theta - \theta) \cos (\theta' - \theta) - \cos (\Theta - \theta) \sin (\theta' - \theta) \cos i$$

$$d\Theta + d\Theta - d\theta' - d\theta = \frac{\cos i - \cos i'}{1 - \cos I} (d\theta' - d\theta) - \frac{\sin I}{1 - \cos I} \sin (\Theta, -\theta') d i' + \frac{\sin I}{1 - \cos I} \sin (\Theta - \theta) d i$$

$$d\Theta - d\Theta - d\theta' + d\theta = -\frac{\cos i + \cos i'}{1 + \cos I} (d\theta' - d\theta) + \frac{\sin I}{1 + \cos I} \sin (\Theta, -\theta') d i' + \frac{\sin I}{1 + \cos I} \sin (\Theta - \theta) d i$$

Substitutis his aequationibus in

$$d(\varphi, +\varphi) = d\Theta + d\Theta - d\theta' - d\theta + \cos i' d\theta' + \cos i d\theta$$

$$d(\varphi, -\varphi) = d\Theta - d\Theta - d\theta' + d\theta + \cos i' d\theta' - \cos i d\theta$$

adjumento relationum (n) et (o), inveniuntur

$$d(\varphi, +\varphi) = \frac{\sin I}{1 - \cos I} \sin i' \cos (\Theta, -\theta') d\theta' - \frac{\sin I}{1 - \cos I} \sin (\Theta, -\theta') d i' - \frac{\sin I}{1 - \cos I} \sin i \cos (\Theta - \theta) d\theta + \frac{\sin I}{1 - \cos I} \sin (\Theta - \theta) d i$$

$$d(\varphi, -\varphi) = \frac{\sin I}{1 + \cos I} \sin i' \cos (\Theta, -\theta') d\theta' - \frac{\sin I}{1 + \cos I} \sin (\Theta, -\theta') d i' + \frac{\sin I}{1 + \cos I} \sin i \cos (\Theta - \theta) d\theta - \frac{\sin I}{1 + \cos I} \sin (\Theta - \theta) d i$$

aequationes igitur (19) immutatae valent, quoties angulum (Θ) subtrahere propositum est, siquidem non nisi ad perturbationes primi ordinis respectu massarum respiciatur. Constantes vero arbitrariae post integrationem aequationibus (19) addendae in hoc casu valore mutantur, et in has abeunt

$$(p) = \sin (i) \sin [(\theta) - c - (\Theta)]$$

$$(q) = \sin (i) \cos [(\theta) - c - (\Theta)] \dots (24^*)$$

Nunc sinus latitudinis valorem hunc habet

$$s = q \sin (\nu, -(\Theta)) - p \cos (\nu, -(\Theta))$$

et expressio (6) longitudinem reductam exhibens, si praeterea valor integralis $\int (1 - \cos i) d\theta$ ex art. 7 substituitur, abit in

$$(n) \dots \cos i' = \cos I \cos i - \sin I \sin i \cos (\Theta - \theta)$$

$$(o) \dots \cos i = \cos I \cos i' + \sin I \sin i' \cos (\Theta, -\theta')$$

Jam, differentiatia relatione (o) art. 9, ope relationum (a) (b) et (k) statim prodit

$$dI = \cos (\Theta, -\theta') d i' - \cos (\Theta - \theta) d i + \sin i' \sin (\Theta, -\theta') d\theta' - \sin i \sin (\Theta - \theta) d\theta$$

Differentiatia vero relatione (k), ope (l), (m) et (a) nanciscimur $d\Theta, -d\theta' = \cos I (d\Theta - d\theta) - \cos i' (d\theta' - d\theta) + \sin I \sin (\Theta - \theta) d i$. Eodem modo relatio (f) una cum (e), (g) et (i) praebet $d\Theta - d\theta = \cos I (d\Theta, -d\theta') + \cos i' (d\theta' - d\theta) - \sin I \sin (\Theta, -\theta') d i'$.

Ex his duabus aequationibus facili opera evadunt hae

Quibus in aequationibus nec non in illa pro dI quantitates dp et dq loco $d i$ et $d\theta$, atque dp' et dq' , quae illis analogae ad corpus perturbans spectant, loco $d i'$ et $d\theta'$ introducendae sunt, id quod fit per aequationes (8). Hinc nanciscimur has

$$dI = \sin \varphi, \frac{dp'}{\cos i'} + \cos \varphi, \frac{dq'}{\cos i'}$$

$$- \sin \varphi, \frac{dp}{\cos i} - \cos \varphi, \frac{dq}{\cos i}$$

$$d(\varphi, +\varphi) = \frac{1 + \cos I}{\sin I} \cos \varphi, \frac{dp'}{\cos i'} - \frac{1 + \cos I}{\sin I} \sin \varphi, \frac{dq'}{\cos i'}$$

$$- \frac{1 + \cos I}{\sin I} \cos \varphi, \frac{dp}{\cos i} + \frac{1 + \cos I}{\sin I} \sin \varphi, \frac{dq}{\cos i}$$

$$d(\varphi, -\varphi) = \frac{\sin I}{1+\cos I} \cos \varphi, \frac{dp'}{\cos i'} - \frac{\sin I}{1+\cos I} \sin \varphi, \frac{dq'}{\cos i'} \\ + \frac{\sin I}{1+\cos I} \cos \varphi, \frac{dp}{\cos i} - \frac{\sin I}{1+\cos I} \sin \varphi, \frac{dq}{\cos i}$$

e quibus ope (13) dP et dQ brevi calculo tales evadunt

$$dP = \cos \frac{1}{2} I \cos \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{dp'}{\cos i'} - \frac{dp}{\cos i} \right\} - \cos \frac{1}{2} I \sin \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{dq'}{\cos i'} + \frac{dq}{\cos i} \right\} \\ dQ = \cos \frac{1}{2} I \cos \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{dq'}{\cos i'} - \frac{dq}{\cos i} \right\} + \cos \frac{1}{2} I \sin \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{dp'}{\cos i'} + \frac{dp}{\cos i} \right\}$$

Quae aequationes nec non illa pro $d(\varphi, -\varphi)$, si valores ipsarum dp et dq ex (17) atque valores ipsarum dp' et dq' ex aequationibus ad corpus perturbans spectantibus et ipsis (17) analogis desumpti substituti fuerint; per integrationem ipsas P , Q , et $\varphi, -\varphi$ praebeunt, quae tum computationi perturbationum secundi altiorumque ordinum longitudinis, latitudinis et radii vectoris inservire poterunt. Sed quoties nonnisi perturbationes secundi ordinis harum coordinatarum computandae sunt, in P , Q , et $\varphi, -\varphi$ termini

primi ordinis tantum requiruntur, itaque sufficit, ut loco dp' , dp , dq' et dq integralia earum terminos primi ordinis tantum continentia, omissis constantibus arbitrariis, substituantur, quo statim valores debiti ipsarum P , Q et $\varphi, -\varphi$, quos δP , δQ et $\delta(\varphi, -\varphi)$ denotabo, ex aequationibus praecedentibus evadant. Jam supponendo, in computatis dp' , dp , dq' et dq arcum $\frac{1}{2}(\varphi, +\varphi)$ subductum esse, valores ipsarum δP , δQ et $\delta(\varphi, -\varphi)$ ope praecedentium aequationum supputandi ita se habent

$$\delta P = \cos \frac{1}{2} I \cos \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta p'|}{\cos i'} - \frac{|\delta p|}{\cos i} \right\} - \cos \frac{1}{2} I \sin \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta q'|}{\cos i'} + \frac{|\delta q|}{\cos i} \right\} \\ \delta Q = \cos \frac{1}{2} I \cos \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta q'|}{\cos i'} - \frac{|\delta q|}{\cos i} \right\} + \cos \frac{1}{2} I \sin \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta p'|}{\cos i'} + \frac{|\delta p|}{\cos i} \right\} \\ \delta(\varphi, -\varphi) = \operatorname{tg} \frac{1}{2} I \cos \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta p'|}{\cos i'} + \frac{|\delta p|}{\cos i} \right\} - \operatorname{tg} \frac{1}{2} I \sin \frac{1}{2}(\varphi, -\varphi) \left\{ \frac{|\delta q'|}{\cos i'} - \frac{|\delta q|}{\cos i} \right\}$$

Quum vero, ut art. 12 docet, manifestum sit, per subtractionem arcus φ , vel, id quod in perturbationibus primi ordinis idem est, per subtractionem arcus (\odot) evasisse formulas simplicissimas pro computandis p et q : suppono id revera factum esse. Eadem ratione in computatis p' et q' accipio longitudinem nodi ascendentis orbitae m in m' , hoc est $180^\circ + (\odot)$, vel $180^\circ + \varphi$, subductam, sive $\delta p'$ et $\delta q'$ ope formularum ipsis (19) plane analogarum computatas esse. Quocirca necesse est, hae quantitates loco $|\delta p|$, $|\delta q|$, $|\delta p'|$ et $|\delta q'|$ in aequationes praecedentes introducantur.

Ut $|\delta p|$ et $|\delta q|$ eliminentur jam inserviunt aequationes (20), $|\delta p'|$ vero et $|\delta q'|$ per aequationes (22) inveniuntur, si in his arcus φ' sive $180^\circ + \varphi$, loco φ , arcus $\frac{1}{2}(\varphi, +\varphi)$ loco α et $\delta p'$ atque $\delta q'$ resp. loco $\frac{dp}{d\epsilon}$ atque $\frac{dq}{d\epsilon}$ substituti fuerint. Quo facto evadunt

$$|\delta p'| = -\delta p' \cos \frac{1}{2}(\varphi, -\varphi) - \delta q' \sin \frac{1}{2}(\varphi, -\varphi) \\ |\delta q'| = \delta p' \sin \frac{1}{2}(\varphi, -\varphi) - \delta q' \cos \frac{1}{2}(\varphi, -\varphi)$$

Quarum aequationum atque (20) ope expressiones supra datae transeunt in simplicissimas has

$$\delta P = -\cos \frac{1}{2} I \left\{ \frac{\delta p'}{\cos i'} + \frac{\delta p}{\cos i} \right\} \\ \delta Q = -\cos \frac{1}{2} I \left\{ \frac{\delta q'}{\cos i'} + \frac{\delta q}{\cos i} \right\} \dots \dots \dots (26) \\ \delta(\varphi, -\varphi) = -\operatorname{tg} \frac{1}{2} I \left\{ \frac{\delta p'}{\cos i'} - \frac{\delta p}{\cos i} \right\}$$

quae terminos primi ordinis respectu massarum omnes qui in P , Q et $\varphi, -\varphi$ existunt, continent. Licet e deductione modo peracta significatio signorum in his aequationibus adhibitorum lectori jam pateat, tamen breviter repeto, δp et δq nec non, mutatis mutandis $\delta p'$ et $\delta q'$ esse integralia aequationum (19) sine constantibus arbitrariis, atque, his substitutis, δP , δQ et $\delta(\varphi, -\varphi)$ ita considerandas esse, ac si arcus $\frac{1}{2}(\varphi, +\varphi)$ subtractus esset. Non dubium est, quin expressiones hae omnium simplicissimae sint. Praeterea conveniunt cum illis quibus in theoria Jovis et Saturni usus sum, (quoniam $\frac{\delta p'}{\cos i'}$, $\frac{\delta p}{\cos i}$ etc. eadem sunt, quas illic p' , p etc. denotaveram) termini igitur primi ordinis respectu massarum in δP et δQ , l. c. pag. 88 neglecti, hoc loco ad factorem $\cos \frac{1}{2} I$ reducti sunt; attamen quantitas $\delta(\varphi, -\varphi)$ illic plane neglecta est.

15.

Computandis perturbationibus secundi ordinis respectu massarum in longitudine atque in radio vectore quantitas T , quam in theoria Jovis atque Saturni definivi, inservit. Quae quantitas considerari debet tanquam functio ipsarum λ, ν, ν' etc. ρ, r, r' etc. $P, Q, \phi, -\phi$ etc. *); quantitates vero λ , et ρ functiones sunt ipsarum $n\zeta$ et $l\rho$; ν , et r ipsarum nz et lr ; ν' , et r' ipsarum $n'z'$ et $l'r'$ et sic porro. Quum vero λ, ν, ν' , etc. cum illis loco citato λ, ν, ν' , etc. denotatis plane congruant, eam tantum ipsius T partem, quae ex P, Q et $\phi, -\phi$ nascitur hoc loco evolvam, nec ad partem eam quam reliquae quantitates praebent respiciam, quippe quae in theoria Jovis atque Saturni jam accurate evoluta sit. Habemus itaque

$$\delta T = \left(\frac{dT}{dP}\right)\delta P + \left(\frac{dT}{dQ}\right)\delta Q + \left(\frac{dT}{d(\phi, -\phi)}\right)\delta(\phi, -\phi)$$

De computandis $\left(\frac{dT}{dP}\right)$ et $\left(\frac{dT}{dQ}\right)$ omnia valent, quae in libro saepe memorato attuli, dummodo quotientes differentiales ipsius Ω , a quibus pendent, ita computentur, ut formulae hoc loco evolutae postulant, et idem valet de computanda $\left(\frac{dT}{d(\phi, -\phi)}\right)$. Quum in $\delta P, \delta Q$ et $\delta(\phi, -\phi)$ ex aequationibus (26) depromendis arcus $\frac{1}{2}(\phi, +\phi)$ subtractus sit, idem in quotientibus differentialibus ipsius Ω quibus hic utemur subtrahi debet. Fit igitur per aequationes (16)

$$\left(\frac{d\Omega}{dP}\right) = \left[\left(\frac{d\Omega}{d\phi,}\right) + \left(\frac{d\Omega}{d\phi}\right)\right] \frac{1}{2 \sin \frac{1}{2} I}$$

$$\left(\frac{d\Omega}{dQ}\right) = \left(\frac{d\Omega}{dI}\right) \frac{1}{\cos \frac{1}{2} I}$$

nec non

$$r \left(\frac{d^2 \Omega}{dr dP}\right) = \left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) + r \left(\frac{d^2 \Omega}{dr d\phi}\right)\right] \frac{1}{2 \sin \frac{1}{2} I}$$

$$r \left(\frac{d^2 \Omega}{dr dQ}\right) = r \left(\frac{d^2 \Omega}{dr dI}\right) \frac{1}{\cos \frac{1}{2} I}$$

quae computationi ipsarum $\left(\frac{dT}{dP}\right)$ et $\left(\frac{dT}{dQ}\right)$ inserviunt, ut l. c. sub oculos cadit. Ad computationem ipsius $\left(\frac{dT}{d(\phi, -\phi)}\right)$

quantitatibus $\left[\left(\frac{d\Omega}{d\phi,}\right) - \left(\frac{d\Omega}{d\phi}\right)\right]$
atque $\left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) - r \left(\frac{d^2 \Omega}{dr d\phi}\right)\right]$ utemur, quae sequenti

*) Hoc etc. signum ad quantitates ipsis P, Q , et $\phi, -\phi$ analogas referendum est, quae aderunt quoties plura corpora perturbantia existunt.

modo obtinebuntur. Quum sit $\left(\frac{d\Omega}{d\phi}\right) = -\left(\frac{d\Omega}{d\phi,}\right)$, erit

$$\frac{dS}{dt} = -\frac{an}{V(1-e^2)} \left(\frac{d\Omega}{d\phi}\right)$$

ubi S eandem quantitatem denotat, qua alibi in theoria perturbationum usus sum. Hinc statim prodit

$$\frac{1}{2} \left[\left(\frac{d\Omega}{d\phi,}\right) - \left(\frac{d\Omega}{d\phi}\right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{d\Omega}{d\phi,}\right) + \left(\frac{d\Omega}{d\phi}\right) \right] + \left(\frac{dS}{dt}\right) \frac{V(1-e^2)}{an}$$

Porro, e quantitate in libro saepe memorato V appellata evadit $r \frac{d^2 S}{dr dt}$, mutato r in t . Sed e praecedentibus habetur

$$r \frac{d^2 S}{dr dt} = -\frac{an}{V(1-e^2)} r \left(\frac{d^2 \Omega}{dr d\phi}\right)$$

erit igitur

$$\frac{1}{2} \left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) - r \left(\frac{d^2 \Omega}{dr d\phi}\right) \right]$$

$$= \frac{1}{2} \left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) + r \left(\frac{d^2 \Omega}{dr d\phi}\right) \right] + \bar{V} \frac{V(1-e^2)}{an}$$

ubi linea ipsi V superposita indicat, r in t mutandum esse.

Quantitates $\left[\left(\frac{d\Omega}{d\phi,}\right) + \left(\frac{d\Omega}{d\phi}\right)\right]$ et $\left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) + r \left(\frac{d^2 \Omega}{dr d\phi}\right)\right]$, quae ceteroquin usurpantur, eadem sunt, quas alibi $\left(\frac{d\Omega}{d\phi}\right)$ et $r \left(\frac{d^2 \Omega}{dr d\phi}\right)$ denotavi, ut per art. 10 facile probatur.

Praeterea quantitates illae sequenti modo exhiberi possunt. Altera aequatio (19) facile probat esse

$$\frac{an}{V(1-e^2)} \left[\left(\frac{d\Omega}{d\phi,}\right) - \left(\frac{d\Omega}{d\phi}\right) \right] = \frac{\sin I}{\cos i} \frac{dq}{dt} + \frac{dS}{dt} (1 + \cos I)$$

et simili modo

$$\frac{an}{V(1-e^2)} \left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) - r \left(\frac{d^2 \Omega}{dr d\phi}\right) \right] = \frac{\sin I}{\cos i} r \frac{d^2 q}{dr dt} + \bar{V} (1 + \cos I)$$

Quae aequationes probant, approximative poni posse, quoties I parvus est

$$\frac{an}{V(1-e^2)} \left[\left(\frac{d\Omega}{d\phi,}\right) - \left(\frac{d\Omega}{d\phi}\right) \right] = 2 \frac{dS}{dt} = 2 \bar{T}$$

$$\frac{an}{V(1-e^2)} \left[r \left(\frac{d^2 \Omega}{dr d\phi,}\right) - r \left(\frac{d^2 \Omega}{dr d\phi}\right) \right] = 2 \bar{V}$$

sc. quia $\frac{dS}{dt} = \bar{T}$. Jam ad $\left(\frac{dT}{d(\phi, -\phi)}\right)$ obtinendam

ponatur in formulis quae in theoria Jovis atque Saturni datae

$$\text{ipsam } V \text{ praebent, } \frac{1}{2} \left[\left(\frac{d\Omega}{d\varphi} \right) - \left(\frac{d\Omega}{d\varphi} \right) \right] \text{ et.....}$$

$$\frac{1}{2} \left[r \left(\frac{d^2\Omega}{dr d\varphi} \right) - r \left(\frac{d^2\Omega}{dr d\varphi} \right) \right] \text{ resp. loco } r \left(\frac{d\Omega}{dr} \right) \text{ et...}$$

$$\left[r^2 \left(\frac{d^2\Omega}{dr^2} \right) + r \left(\frac{d\Omega}{dr} \right) \right].$$

Facili calculo instituto reperi hos terminos in motu Jovis atque Saturni nullam vim habere.

$$\frac{dp_i}{dt} = -\frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ} \right) \cos \frac{1}{2} I - \left[\frac{Q}{4} \left(\frac{d\Omega}{dP} \right) - \frac{P}{4} \left(\frac{d\Omega}{dQ} \right) \right] \frac{P}{\cos \frac{1}{2} I} \right\}$$

$$\frac{dq_i}{dt} = \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP} \right) \cos \frac{1}{2} I + \left[\frac{Q}{4} \left(\frac{d\Omega}{dP} \right) - \frac{P}{4} \left(\frac{d\Omega}{dQ} \right) \right] \frac{Q}{\cos \frac{1}{2} I} \right\}$$

ex (17) inveniuntur hae

$$\frac{dp}{dt} = \left\{ \frac{dp_i}{dt} + \frac{1}{2} \frac{dS}{dt} \frac{P \cos i}{\cos \frac{1}{2} I} \right\} \cos \frac{1}{2} (\varphi, -\varphi)$$

$$- \left\{ \frac{dq_i}{dt} + \frac{1}{2} \frac{dS}{dt} \frac{Q \cos i}{\cos \frac{1}{2} I} \right\} \sin \frac{1}{2} (\varphi, -\varphi)$$

$$\frac{dq}{dt} = \left\{ \frac{dq_i}{dt} + \frac{1}{2} \frac{dS}{dt} \frac{Q \cos i}{\cos \frac{1}{2} I} \right\} \cos \frac{1}{2} (\varphi, -\varphi)$$

$$+ \left\{ \frac{dp_i}{dt} + \frac{1}{2} \frac{dS}{dt} \frac{P \cos i}{\cos \frac{1}{2} I} \right\} \sin \frac{1}{2} (\varphi, -\varphi)$$

Si in his aequationibus quantitatibus P , Q , $\varphi, -\varphi$ et i resp. incrementa haec δP , δQ , $\delta(\varphi, -\varphi)$ et δi attribuuntur facile eliciuntur, quae perturbationes secundi ordinis praebent, aequationes hae

$$\left| \frac{dp}{dt} \right| = (A \cos \frac{1}{2} (\varphi, -\varphi) - B \sin \frac{1}{2} (\varphi, -\varphi)) \delta P$$

$$+ (C \cos \frac{1}{2} (\varphi, -\varphi) - D \sin \frac{1}{2} (\varphi, -\varphi)) \delta Q$$

$$+ (E \cos \frac{1}{2} (\varphi, -\varphi) - F \sin \frac{1}{2} (\varphi, -\varphi)) \delta(\varphi, -\varphi)$$

$$+ \left(\frac{P \cos i}{2 \cos \frac{1}{2} I} \cos \frac{1}{2} (\varphi, -\varphi) - \frac{Q \cos i}{2 \cos \frac{1}{2} I} \sin \frac{1}{2} (\varphi, -\varphi) \right) \frac{d^2 S}{dt^2}$$

$$- \left(\frac{dp_i}{dt} \cos \frac{1}{2} (\varphi, -\varphi) - \frac{dq_i}{dt} \sin \frac{1}{2} (\varphi, -\varphi) \right) \frac{\sin i \delta i}{\cos i}$$

$$\left| \frac{dq}{dt} \right| = (B \cos \frac{1}{2} (\varphi, -\varphi) + A \sin \frac{1}{2} (\varphi, -\varphi)) \delta P$$

$$+ (D \cos \frac{1}{2} (\varphi, -\varphi) + C \sin \frac{1}{2} (\varphi, -\varphi)) \delta Q$$

$$+ (F \cos \frac{1}{2} (\varphi, -\varphi) + E \sin \frac{1}{2} (\varphi, -\varphi)) \delta(\varphi, -\varphi)$$

$$+ \left(\frac{Q \cos i}{2 \cos \frac{1}{2} I} \cos \frac{1}{2} (\varphi, -\varphi) + \frac{P \cos i}{2 \cos \frac{1}{2} I} \sin \frac{1}{2} (\varphi, -\varphi) \right) \frac{d^2 S}{dt^2}$$

$$- \left(\frac{dq_i}{dt} \cos \frac{1}{2} (\varphi, -\varphi) + \frac{dp_i}{dt} \sin \frac{1}{2} (\varphi, -\varphi) \right) \frac{\sin i \delta i}{\cos i}$$

$$\frac{dp}{dt} = A \delta P + C \delta Q + E \delta(\varphi, -\varphi) - \frac{dp_i}{dt} \frac{\sin i \delta i}{\cos i}$$

$$\frac{dq}{dt} = B \delta P + D \delta Q + F \delta(\varphi, -\varphi) - \frac{dq_i}{dt} \frac{\sin i \delta i}{\cos i} + \frac{d^2 S}{dt^2} \cdot \text{tg} \frac{1}{2} I \cos i$$

Sed in art. 13 in supputandis perturbationibus primi ordinis ipsarum p et q supposui arcum (\odot) loco φ subductum

16.

Quum in ipsarum p et q perturbationibus secundi ordinis respectu massarum termini ab ipsis nz , $n'z'$ lr , lr' orientes iidem sint, quos in libro saepe memorato determinavi, hoc loco non nisi terminos quos P , Q et $\varphi, -\varphi$ subministrant, considerabo.

Positis aequationibus his

ubi brevitatis causa

$$A = \left(\frac{d^2 p_i}{dP dt} \right) + \frac{1}{2} \frac{dS}{dt} \frac{d \cdot \frac{P}{\cos \frac{1}{2} I}}{dP} \cos i$$

$$B = \left(\frac{d^2 q_i}{dP dt} \right) + \frac{1}{2} \frac{dS}{dt} \frac{d \cdot \frac{Q}{\cos \frac{1}{2} I}}{dP} \cos i$$

$$C = \left(\frac{d^2 p_i}{dQ dt} \right) + \frac{1}{2} \frac{dS}{dt} \frac{d \cdot \frac{P}{\cos \frac{1}{2} I}}{dQ} \cos i$$

$$D = \left(\frac{d^2 q_i}{dQ dt} \right) + \frac{1}{2} \frac{dS}{dt} \frac{d \cdot \frac{Q}{\cos \frac{1}{2} I}}{dQ} \cos i$$

$$E = \left(\frac{d^2 p_i}{d(\varphi, -\varphi) dt} \right) - \frac{1}{2} \frac{dq_i}{dt} - \frac{1}{4} \frac{dS}{dt} \frac{Q \cos i}{\cos \frac{1}{2} I}$$

$$F = \left(\frac{d^2 q_i}{d(\varphi, -\varphi) dt} \right) + \frac{1}{2} \frac{dp_i}{dt} + \frac{1}{4} \frac{dS}{dt} \frac{P \cos i}{\cos \frac{1}{2} I}$$

posui, et $\frac{1}{2} p$ atque $\frac{1}{2} q$ terminos primi ordinis in p atque q , nec non $\frac{2}{2} S$ terminos secundi ordinis in S denotant. Quum in harum aequationum coefficientibus ubique valores ii quantitatum quas involvunt substitui debeant, qui pro temporis epocha locum habeant, per aequationes (21) statim a $\left| \frac{dp}{dt} \right|$ et $\left| \frac{dq}{dt} \right|$ ad $\frac{dp}{dt}$ et $\frac{dq}{dt}$ ubi arcus φ subtractus est transgredi licet. Hoc modo evadunt simpliciores hae

esse, quare necesse est, hoc loco idem subtrahatur. Evidens vero est, hac transmutatione expressiones praecedentes non

mutari, quoties c est quantitas primi ordinis, terminique tertii altiorumque ordinum negliguntur. At aequationes (24) duos continent terminos secundi ordinis, nempe

$$(27) \dots \left\{ \begin{aligned} \frac{dp}{dt} &= A\delta P + C\delta Q + E\delta(\varphi, -\varphi) - \frac{dp}{dt} \frac{\sin i \delta i}{\cos i} - \frac{dq}{dt} c \\ \frac{dq}{dt} &= B\delta P + D\delta Q + F\delta(\varphi, -\varphi) - \frac{dq}{dt} \frac{\sin i \delta i}{\cos i} + \frac{dp}{dt} c + \frac{dS}{dt} \operatorname{tg} \frac{1}{2} I \cos i \end{aligned} \right.$$

perturbationes secundi ordinis omnes continent. Termini primi ordinis, ut jam dixi, ope (19) computentur, nec non et constantes integralibus addendae, et s , et l ope formularum earundem obtinentur, quas in fine art. 13 dedi.

$-\frac{dq}{dt} c$ et $\frac{dp}{dt} c$, quas omittere hoc loco non licet. Quibus ad expressiones praecedentes additis, aequationes hae

17.

Ad evolvendos coefficientes A, B etc. aequationes (14) subtracto $\frac{1}{2}(\varphi, +\varphi)$, praebent

$$\frac{dI}{dP} = 0, \quad \frac{dI}{dQ} = \frac{1}{\cos \frac{1}{2} I}, \quad \frac{d(\varphi, +\varphi)}{dP} = \frac{1}{\sin \frac{1}{2} I}, \quad \frac{d(\varphi, +\varphi)}{dQ} = 0$$

itaque

$$\frac{d \cdot \frac{P}{\cos \frac{1}{2} I}}{dP} = \frac{1}{\cos \frac{1}{2} I}, \quad \frac{d \cdot \frac{P}{\cos \frac{1}{2} I}}{dQ} = 0, \quad \frac{d \cdot \frac{Q}{\cos \frac{1}{2} I}}{dP} = 0, \quad \frac{d \cdot \frac{Q}{\cos \frac{1}{2} I}}{dQ} = \frac{1}{\cos^2 \frac{1}{2} I}$$

Si jam differentiationes in expressionibus ipsarum A, B , etc. art. praec. indicatae perficiuntur, evadunt

$$\begin{aligned} A &= -\frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dP dQ} \right) \cos \frac{1}{2} I - \frac{1}{2} \left(\frac{d\Omega}{dP} \right) \operatorname{tg} \frac{1}{2} I \right\} + \frac{1}{2} \frac{dS}{dt} \cdot \frac{\cos i}{\cos^2 \frac{1}{2} I} \\ B &= -\frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dQ^2} \right) \cos \frac{1}{2} I - \frac{1}{2} \left(\frac{d\Omega}{dQ} \right) \operatorname{tg} \frac{1}{2} I \right\} \\ C &= \frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dP^2} \right) \frac{1}{\cos \frac{1}{2} I} - \frac{1}{2} \left(\frac{d\Omega}{dP} \right) \operatorname{tg} \frac{1}{2} I \right\} \\ D &= \frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dP dQ} \right) \frac{1}{\cos \frac{1}{2} I} + \frac{1}{2} \left(\frac{d\Omega}{dP} \right) \frac{\operatorname{tg} \frac{1}{2} I}{\cos^2 \frac{1}{2} I} \right\} + \frac{1}{2} \frac{dS}{dt} \cdot \frac{\cos i}{\cos^3 \frac{1}{2} I} \\ E &= -\frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dQ d(\varphi, -\varphi)} \right) \cos \frac{1}{2} I + \frac{1}{2} \left(\frac{d\Omega}{dP} \right) \frac{1}{\cos \frac{1}{2} I} \right\} - \frac{1}{2} \frac{dS}{dt} \cdot \operatorname{tg} \frac{1}{2} I \cos i \\ F &= \frac{an \cos i}{\sqrt{(1-e^2)}} \left\{ \left(\frac{d^2 \Omega}{dP d(\varphi, -\varphi)} \right) \frac{1}{\cos \frac{1}{2} I} - \frac{1}{2} \left(\frac{d\Omega}{dQ} \right) \cos \frac{1}{2} I \right\} \end{aligned}$$

ubi in quotientibus differentialibus arcus $\frac{1}{2}(\varphi, +\varphi)$ subtrahendus est. Sub hac vero conditione aequationes (16) differentiatiae praebent

$$\begin{aligned} \left(\frac{d^2 \Omega}{dP^2} \right) &= \left[\left(\frac{d^2 \Omega}{d\varphi,^2} \right) + 2 \left(\frac{d^2 \Omega}{d\varphi, d\varphi} \right) + \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] \frac{1}{4 \sin^2 \frac{1}{2} I} + \left(\frac{d\Omega}{dI} \right) \frac{1}{2 \sin \frac{1}{2} I \cos \frac{1}{2} I} \\ \left(\frac{d^2 \Omega}{dP dQ} \right) &= \left[\left(\frac{d^2 \Omega}{dI d\varphi,} \right) + \left(\frac{d^2 \Omega}{dI d\varphi} \right) \right] \frac{1}{2 \sin \frac{1}{2} I \cos \frac{1}{2} I} - \left[\left(\frac{d\Omega}{d\varphi,} \right) + \left(\frac{d\Omega}{d\varphi} \right) \right] \frac{1}{4 \sin^2 \frac{1}{2} I} \\ \left(\frac{d^2 \Omega}{dQ^2} \right) &= \left(\frac{d^2 \Omega}{dI^2} \right) \frac{1}{\cos^2 \frac{1}{2} I} + \left(\frac{d\Omega}{dI} \right) \frac{\sin \frac{1}{2} I}{2 \cos^3 \frac{1}{2} I} \\ \left(\frac{d^2 \Omega}{dP d(\varphi, -\varphi)} \right) &= \left[\left(\frac{d^2 \Omega}{d\varphi,^2} \right) - \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] \frac{1}{4 \sin \frac{1}{2} I} \\ \left(\frac{d^2 \Omega}{dQ d(\varphi, -\varphi)} \right) &= \left[\left(\frac{d^2 \Omega}{dI d\varphi,} \right) - \left(\frac{d^2 \Omega}{dI d\varphi} \right) \right] \frac{1}{2 \cos \frac{1}{2} I} \\ \left(\frac{d\Omega}{dP} \right) &= \left[\left(\frac{d\Omega}{d\varphi,} \right) + \left(\frac{d\Omega}{d\varphi} \right) \right] \frac{1}{2 \sin \frac{1}{2} I} \\ \left(\frac{d\Omega}{dQ} \right) &= \left(\frac{d\Omega}{dI} \right) \frac{1}{\cos \frac{1}{2} I} \end{aligned}$$

Porro habetur

Quibus substitutis obtinentur

$$\begin{aligned}
 A &= -\frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left[\left(\frac{d^2 \Omega}{dI d\varphi} \right) + \left(\frac{d^2 \Omega}{dI d\varphi} \right) \right] \frac{1}{2 \sin \frac{1}{2} I} - \left[\left(\frac{d\Omega}{d\varphi} \right) + \left(\frac{d\Omega}{d\varphi} \right) \right] \frac{1}{4 \sin^2 \frac{1}{2} I \cos \frac{1}{2} I} \right\} + \frac{1}{2} \left(\frac{dS}{dt} \right) \frac{\cos i}{\cos \frac{1}{2} I} \\
 B &= \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left[\left(\frac{d^2 \Omega}{d\varphi^2} \right) + 2 \left(\frac{d^2 \Omega}{d\varphi d\varphi} \right) + \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] \frac{1}{4 \sin^2 \frac{1}{2} I \cos \frac{1}{2} I} + \left(\frac{d\Omega}{dI} \right) \frac{1}{2 \sin \frac{1}{2} I} \right\} \\
 C &= -\frac{an \cos i}{\sqrt{1-e^2}} \left(\frac{d^2 \Omega}{dI^2} \right) \frac{1}{\cos \frac{1}{2} I} \\
 D &= \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left[\left(\frac{d^2 \Omega}{dI d\varphi} \right) + \left(\frac{d^2 \Omega}{dI d\varphi} \right) \right] \frac{1}{2 \sin \frac{1}{2} I \cos^2 \frac{1}{2} I} - \left[\left(\frac{d\Omega}{d\varphi} \right) + \left(\frac{d\Omega}{d\varphi} \right) \right] \frac{\cos I}{4 \sin^2 \frac{1}{2} I \cos^3 \frac{1}{2} I} \right\} + \frac{1}{2} \left(\frac{dS}{dt} \right) \frac{\cos i}{\cos^3 \frac{1}{2} I} \\
 E &= -\frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left[\left(\frac{d^2 \Omega}{d\varphi dI} \right) - \left(\frac{d^2 \Omega}{d\varphi dI} \right) \right] + \left[\left(\frac{d\Omega}{d\varphi} \right) + \left(\frac{d\Omega}{d\varphi} \right) \right] \frac{1}{4 \sin \frac{1}{2} I \cos \frac{1}{2} I} \right\} - \frac{1}{2} \left(\frac{dS}{dt} \right) \operatorname{tg} \frac{1}{2} I \cos i \\
 F &= \frac{an \cos i}{\sqrt{1-e^2}} \left\{ \left[\left(\frac{d^2 \Omega}{d\varphi^2} \right) - \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] \frac{1}{4 \sin \frac{1}{2} I \cos \frac{1}{2} I} - \frac{1}{2} \left(\frac{d\Omega}{dI} \right) \right\}
 \end{aligned}$$

Hoc loco $\left(\frac{d^2 \Omega}{d\varphi^2} \right) + 2 \left(\frac{d^2 \Omega}{d\varphi d\varphi} \right) + \left(\frac{d^2 \Omega}{d\varphi^2} \right)$ et $\left(\frac{d^2 \Omega}{dI d\varphi} \right) + \left(\frac{d^2 \Omega}{dI d\varphi} \right)$ eadem quantitates sunt, quas alibi per $\left(\frac{d^2 \Omega}{d\varphi^2} \right)$ et $\left(\frac{d^2 \Omega}{dI d\varphi} \right)$ denotavi. Quantitates vero $\left(\frac{d^2 \Omega}{d\varphi^2} \right) - \left(\frac{d^2 \Omega}{d\varphi^2} \right)$ et $\left(\frac{d^2 \Omega}{dI d\varphi} \right) - \left(\frac{d^2 \Omega}{dI d\varphi} \right)$ similiter modo computari possunt, ut quantitates analogae art. 15. Nempè habetur

$$\begin{aligned}
 \frac{an}{\sqrt{1-e^2}} \left[\left(\frac{d^2 \Omega}{d\varphi^2} \right) - \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] &= \frac{an}{\sqrt{1-e^2}} \left[\left(\frac{d^2 \Omega}{d\varphi^2} \right) + 2 \left(\frac{d^2 \Omega}{d\varphi d\varphi} \right) + \left(\frac{d^2 \Omega}{d\varphi^2} \right) \right] + 4 \left(\frac{dT}{dP} \right) \sin \frac{1}{2} I \\
 \frac{an}{\sqrt{1-e^2}} \left[\left(\frac{d^2 \Omega}{d\varphi dI} \right) - \left(\frac{d^2 \Omega}{d\varphi dI} \right) \right] &= \frac{an}{\sqrt{1-e^2}} \left[\left(\frac{d^2 \Omega}{d\varphi dI} \right) + \left(\frac{d^2 \Omega}{d\varphi dI} \right) \right] + 2 \left(\frac{dT}{dQ} \right) \cos \frac{1}{2} I
 \end{aligned}$$

ubi, ut ante, linea superscripta significat, τ in t mutandum esse. Ceterum formulas praecedentes accurate inspicientem non fugiet, quantitatem $\varphi, -\varphi$ in p et q terminos proferre non posse, quin saltem per tertiam potestatem inclinationis reciprocae multiplicati sint.

Restat ut δi evolvatur. Quum ad computandas perturbationes secundi ordinis in p et q non nisi termini primi ordinis ipsius δi requirantur, prior aequatio (8), substitutis integralibus aequationum (19) loco dp et dq , omissisque constantibus arbitrariis, valorem quaesitum praebet. Habemus igitur

$$\delta i = \sin[(\theta) - c - (\Theta)] \frac{\delta p}{\cos i} + \cos[(\theta) - c - (\Theta)] \frac{\delta q}{\cos i}$$

sive

$$\sin i \delta i = (p) \frac{\delta p}{\cos i} + (q) \frac{\delta q}{\cos i}$$

ubi, reiectis terminis secundi ordinis, poni potest

$$\begin{aligned}
 (p) &= \sin(i) \sin[(\theta) - (\Theta)] \\
 (q) &= \sin(i) \cos[(\theta) - (\Theta)]
 \end{aligned}$$

Quos terminos in theoria Jovis atque Saturni alio modo in calculum vocavi, sed methodus hic proposita illa simplicior videtur.

18.

In praecedentibus longitudes latitudinesque ad planum quodvis fixum sunt relatae, reliquum igitur est, ut ostendatur quomodo ad planum mobile referri possint.

Longitudo perihelii, quae ipsi ν inest, longitudo nodi et inclinatio orbitae per integrationem aequationum differentialium exhibitarum ita exprimuntur, ut functiones temporis et valorum illorum sint, quos accipiunt quoties vires perturbantes evanescent, sive, quod hoc loco idem est, quoties $t = 0$ est. Jam casus duos distinguere possumus. Aut animo concipi potest, valores hos, puta (π) , (θ) et (i) immediate dari, aut per longitudinem nodi θ_0 et inclinationem i_0 plani fixi ad aliud planum exprimi. De casu priore nil dicendum est, casus vero alter ad solutionem problematis propositi perducet. Quo in casu orbita corporis perturbati ad planum refertur, cujus inclinatio ad tertium quoddam planum i_0 est, et cujus nodus ascendens in hoc longitudinem θ_0 habet. Tum itaque habetur $(\pi) = \varphi(\theta_0, i_0)$, $(\theta) = \psi(\theta_0, i_0)$, $(i) = \chi(\theta_0, i_0)$ denotantibus φ , ψ et χ functiones quasdam. Sed profecto res una et eadem est, sive (π) , (θ) et (i) , sive $\varphi(\theta_0, i_0)$, $\psi(\theta_0, i_0)$ et $\chi(\theta_0, i_0)$ in aequationibus differentialibus ipsarum ζ , p et q substi-

tuuntur, dummodo functiones hae a tempore independentes sint; in utroque enim casu eisdem integralium valores redundaturos esse, manifestum est.

Licet nobis accipere, inter θ_0 et i_0 aequationem aliquam locum habere, dummodo tempus non contineat, qua re conclusiones nostrae nullo modo turbantur. Quam vero aequationem, introducta quantitate quadam arbitraria τ , in duas distribuere licet, quarum op θ_0 et i_0 ex functionibus illis eliminari possunt, ita ut evadant $(\pi) = \phi'\tau$, $(\theta) = \psi'\tau$ et $(i) = \chi'\tau$, ubi ϕ' , ψ' et χ' functiones alias designant. Tali modo longitudo latitudoque corporis perturbati ad planum relatae erunt, cujus situs in spatio propter arbitriam quantitatem τ indeterminatus est, quod planum igitur situ mutatur, prout τ ita sive aliter accipitur. Sed post integrationes peractas nihil impedit, quo minus tempus ipsum aut functionem quamlibet temporis loco τ ponamus, itaque orbita ad planum relata erit, quod cum tempore mobile est.

19.

En eorum quae modo exposui applicationem. Planum mobile eclipticae ad eclipticam certo determinatoque tempore respondentem, aut planum mobile aequatoris ad aequatorem certo tempore respondentem refertur ope aequationum talium

$$\sin i_0 \sin(\theta_0 - (\Theta)) = p_0 = \alpha_0 t + \beta_0 t^2 + \dots$$

$$\sin i_0 \cos(\theta_0 - (\Theta)) = q_0 = \alpha'_0 t + \beta'_0 t^2 + \dots$$

ubi i_0 inclinatio plani mobilis ad fixum, et θ_0 longitudo nodi assendentis illius plani in hoc est, $\alpha_0, \beta_0 \dots$, vero et $\alpha'_0, \beta'_0 \dots$ coefficientes numericos denotant quorum α_0 et α'_0 primi, β_0 vero et β'_0 secundi ordinis respectu massarum

$$(28) \dots \dots \dots \begin{cases} \sin(i) \sin(\theta - \theta_0) = \sin[i] \sin([\theta] - \theta_0) \\ \sin(i) \cos(\theta - \theta_0) = -\cos[i] \sin i_0 + \sin[i] \cos i_0 \cos([\theta] - \theta_0) \\ \sin(i) \sin \Delta = \sin i_0 \sin([\theta] - \theta_0) \end{cases}$$

quibus (i) , (θ) et Δ in functione ipsarum i_0 et θ_0 exprimuntur. Si orbitae corporis perturbati planum temporis epochae respondens ad tertium planum refertur, argumentum latitudinis quodlibet per $u + \Delta$ expressum est, denotante u arcum quendam a loco corporis pendentem, longitudo itaque in orbita aequalis est huic $u + \Delta + [\theta]$. Sin eadem orbita ad secundum planum refertur, argumentum latitudinis eodem

$$(29) \dots \dots \dots (\pi) = [\pi] + (\theta) - [\theta] - \Delta$$

Positis

$$(30) \dots \dots \dots \begin{cases} (p), = \sin(i) \sin([\theta] - (\Theta)) \\ (q), = \sin(i) \cos([\theta] - (\Theta)) \\ [p] = \sin[i] \sin([\theta] - (\Theta)) \\ [q] = \sin[i] \cos([\theta] - (\Theta)) \end{cases}$$

sunt. Eliminato t inter aequationes praecedentes, aequationem inter i_0 et θ_0 adipiscimur, quae a tempore independens est. Casus igitur quem in art. praec. consideravi locum habet. Jam ante omnia oportet, (π) , (i) et (θ) per i_0 et θ_0 exprimantur, id quod fit ope longitudinis perihelii $[\pi]$, inclinationis $[i]$ et longitudinis nodi $[\theta]$ orbitae respectu ejusdem plani ad quod i_0 et θ_0 relatae sunt. Tum aequatio inter i_0 et θ_0 , de qua modo locutus sum, adjumento quantitatis arbitrariae τ in duas distribui debet, quod quum infinite variis modis fieri possit, omnium simplicissimum erit, aequationes praecedentes, a quibus proficiscebamur posito τ loco t , restituere. Ergo ponatur

$$p_0 = \alpha_0 \tau + \beta_0 \tau^2 + \dots$$

$$q_0 = \alpha'_0 \tau + \beta'_0 \tau^2 + \dots$$

Hinc itaque nanciscemur $(\pi) = \phi'(\tau)$, $(i) = \psi'(\tau)$ et $(\theta) = \chi'(\tau)$; quibus functionibus loco (π) , (i) et (θ) in formulis differentialibus pro p , q et l substitutis et post integrationes peractas τ et t mutato, quantitates p , q et l ad eclipticam mobilem vel ad aequatorem mobilem relatae erunt, prout $\alpha_0, \beta_0 \dots \alpha'_0, \beta'_0$ ceteraeque constantes ad illud vel ad hoc planum spectant.

Consideremus triangulum sphaericum, ab orbita corporis perturbati temporis epochae respondente, a plano quod per i_0 et θ_0 datum est, et a plano ad quod i_0 et θ_0 referuntur, formatum. Jam denotavi orbitae inclinationem longitudinemque nodi ad hoc planum per $[i]$ et $[\theta]$ nec non longitudinem perihelii per $[\pi]$. Quo in triangulo latera duo sunt $(\theta) - \theta_0$ et $[\theta] - \theta_0$, tertium vero latus Δ appellabo; angulique his lateribus resp. oppositi sunt $180^\circ - [i]$, (i) et i_0 . Trigonometria vero sphaerica praebet relationes has

tempore est u , longitudo igitur in orbita $= u + (\theta)$. Eadem vero longitudes etiam exprimuntur resp. per $\phi + [\pi]$ et $\phi + (\pi)$ denotante ϕ anomaliam veram. Ergo

$$u + \Delta + [\theta] = \phi + [\pi]$$

$$u + (\theta) = \phi + (\pi)$$

unde concluditur

primae duae aequationes (28) praebent, nullo termino neglecto, ut in Astr. Nachr. Nr. 168 exposui,

$$\left. \begin{aligned} (p)_i &= [p] - p_0 \cos[i] - p_0 \frac{[p]p_0 + [q]q_0}{1 + \sqrt{(1 - p_0^2 - q_0^2)}} \\ (q)_i &= [q] - q_0 \cos[i] - q_0 \frac{[p]p_0 + [q]q_0}{1 + \sqrt{(1 - p_0^2 - q_0^2)}} \end{aligned} \right\} \dots\dots\dots (31)$$

Habetur quoque

$$\sin^2(i) = (p)_i^2 + (q)_i^2$$

quae aequatio, substitutis valoribus ipsarum $(p)_i$ et $(q)_i$ ex praecedentibus aequationibus desumptis, neglectisque potestibus tertiis altioribusque ipsarum p_0 et q_0 , subministrat:

$$\sin^2(i) = \sin^2[i] - 2 \cos[i] ([p]p_0 + [q]q_0) - ([p]p_0 + [q]q_0)^2 + \cos^2[i] (p_0^2 + q_0^2)$$

hinc, radice extracta, prodit

$$\begin{aligned} \sin(i) &= \sin[i] - \cotg[i] ([p]p_0 + [q]q_0) - \frac{1}{2} \frac{([p]p_0 + [q]q_0)^2}{\sin^3[i]} + \frac{1}{2} \frac{\cos^2[i]}{\sin[i]} (p_0^2 + q_0^2) \\ \cos(i) &= \cos[i] + ([p]p_0 + [q]q_0) - \frac{1}{2} \cos[i] (p_0^2 + q_0^2) \end{aligned}$$

Quae aequationes post multiplicationem per $\cos[i]$ atque $\sin[i]$ et subtractionem praebent

$$(i) - [i] = \dots \frac{[p]p_0 + [q]q_0}{\sin[i]} - \frac{1}{2} \frac{\cos[i]}{\sin^3[i]} ([p]p_0 + [q]q_0)^2 + \frac{1}{2} \cotg[i] (p_0^2 + q_0^2) \dots\dots\dots (32)$$

Praeterea prior aequatio praecedens facile praebet

$$\frac{1}{\sin(i)} = \frac{1}{\sin[i]} + \frac{\cos[i]}{\sin^3[i]} ([p]p_0 + [q]q_0) + \frac{1}{2} \frac{1 + 2 \cos^2[i]}{\sin^5[i]} ([p]p_0 + [q]q_0)^2 - \frac{1}{2} \frac{\cos^2[i]}{\sin^3[i]} (p_0^2 + q_0^2)$$

Positis in dextra parte aequationum harum

$$\begin{aligned} \sin[(\theta) - (\Theta)] &= \frac{(p)_i}{\sin(i)} \\ \cos[(\theta) - (\Theta)] &= \frac{(q)_i}{\sin(i)} \end{aligned}$$

loco $(p)_i$, $(q)_i$ et $\sin(i)$ valoribus modo evolutis, duae aequationes oriuntur, e quibus simili modo ut $(i) - [i]$, evadit

$$(\theta) - [\theta] = \frac{\cos[i]}{\sin^2[i]} ([p]q_0 - [q]p_0) + \frac{1}{2} \frac{1 + \cotg^2[i]}{\sin^2[i]} ([p][q]q_0^2 + ([p]^2 - [q]^2)p_0q_0 - [p][q]p_0^2) \dots\dots\dots (33)$$

Tertia aequatio (28) statim subministrat hanc

$$\Delta = \frac{[p]q_0 - [q]p_0}{\sin[i] \sin(i)}$$

quae, substituto valore ipsius $\frac{1}{\sin(i)}$, praebet

$$\Delta = \frac{[p]q_0 - [q]p_0}{\sin^2[i]} + \frac{\cos[i]}{\sin^4[i]} ([p][q]q_0^2 + ([p]^2 - [q]^2)p_0q_0 - [p][q]p_0^2) \dots\dots\dots (33^*)$$

ex aequationibus istis pro Δ et $(\theta) - [\theta]$ per subtractionem evadit

$$(\theta) - [\theta] - \Delta = - \frac{[p]q_0 - [q]p_0}{1 + \cos[i]} + \frac{[p][q]q_0^2 + ([p]^2 - [q]^2)p_0q_0 - [p][q]p_0^2}{2(1 + \cos[i])^2} \dots\dots\dots (34)$$

Quum valor integralis $\int (1 - \cos i) d\theta$ temporis epochae respondens per c denotatus sit *), habemus pro eodem tempore secundum art. 2

$$(\pi_i) = (\pi) - c$$

Jam si orbita corporis perturbati ad planum quoddam fixum

refertur, fit $\nu = \nu$, quoties $\iota = 0$, itaque $(\pi_i) = (\pi)$ et $c = 0$; sin vero orbita eadem ad planum quoddam mobile refertur, non modo $\iota = 0$ sed etiam $\tau = 0$ poni debet, ut fiat $\nu = \nu$, quare adipiscimur

$$(\pi_i) = [\pi]$$

quia hoc in casu $[\pi]$ ipsi ν tali modo inest, quali ν , ex (π_i)

*) V. art. 8.

composita est. Aequationes igitur praecedentes probant generaliter haberi

$$[\pi] = (\pi) - c$$

$$c = \frac{[q]p_0 - [p]q_0}{1 + \cos[i]} + \frac{[p][q]q_0^2 + ([p]^2 - [q]^2)p_0q_0 - [p][q]p_0^2}{2(1 + \cos[i])^2}$$

Haec omnia sunt, quae ad longitudinis latitudinisque reductionem ad planum mobile requiruntur. Aequationes omnes modo evolutae respectu ipsarum p_0 et q_0 usque ad potestates tertias accuratae sunt, id quod in earum applicatione ad systema nostrum solare semper sufficit, respectu vero ipsarum $[p]$ et $[q]$ sive ipsius $[i]$ rigor geometricus laesus non est.

In computandis valoribus numericis quotientium differentialium ipsius Ω semper $c = 0$ ponere et loco ϕ , et ϕ resp. has (Θ) et (Θ) substituere licebit, dummodo quantitates hae ex $[i]$, $[\theta]$, $[i']$ et $[\theta']$ computatae fuerint, quod nunquam fieri non potest.

20.

Antequam ultimam correctionem qua ad computandas longitudes opus est tractabo, evolutionem aequationum art. praec. ulteriorem afferam, quo earum applicatio facilius reddatur. Substitutis loco p_0 et q_0 valoribus earum per τ expressis, facile invenitur

$$\cos(i) = \cos[i] + \cos[i'] A\tau$$

$$c = D\tau + D'\tau^2$$

ubi

$$A = \frac{[p]\alpha_0 + [q]\alpha'_0}{\cos[i]}; \quad D = \frac{[q]\alpha_0 - [p]\alpha'_0}{1 + \cos[i]}$$

$$D' = \frac{[q]\beta_0 - [p]\beta'_0}{1 + \cos[i]} - \frac{([p]\alpha_0 + [q]\alpha'_0)([q]\alpha_0 - [p]\alpha'_0)}{2(1 + \cos[i])^2}$$

$$(36) \dots \dots \dots \begin{cases} p = [p] + B\tau + \alpha t + B'\tau^2 + \beta t^2 + (A\alpha - D\alpha')\tau t \\ \quad + (i, i')_s \sin(i g + i' g') + (i, i')_c \cos(i g + i' g') \\ \quad + (A(i, i')_s - D[i, i']_s)\tau \sin(i g + i' g') + (A(i, i')_c - D[i, i']_c)\tau \cos(i g + i' g') \\ q = [q] + C\tau + \alpha' t + C'\tau^2 + \beta' t^2 + (A\alpha' + D\alpha)\tau t \\ \quad + [i, i']_s \sin(i g + i' g') + [i, i']_c \cos(i g + i' g') \\ \quad + (A[i, i']_s + D(i, i')_s)\tau \sin(i g + i' g') + (A[i, i']_c + D(i, i')_c)\tau \cos(i g + i' g') \end{cases}$$

ubi restat, ut τ in t mutetur *).

Quantum ad terminos periodicos, in theoria Jovis atque Saturni demonstravi, eos quam proxime in latitudine prolutos esse hos

*) Superfluum fortasse erit animadvertere, in his formulis i et i' inclinationes non designare, sed numeros integros esse, signa vero $(i, i')_s$, $(i, i')_c$, $[i, i']_s$ et $[i, i']_c$ functiones ipsarum i et i' denotare, quarum indoles ex praecedentibus sponte pateat.

sive

$$c = (\pi) - [\pi] = (\theta) - [\theta] - \Delta$$

Itaque, adjumento expressionis (34) nanciscimur

Constantes arbitrariae integralibus aequationum (19) et (27) addendae, et ex (24*) desumendae, sunt

$$(p) = (p), \cos c - (q), \sin c$$

$$(q) = (q), \cos c + (p), \sin c$$

quae itaque, ope valorum ipsarum (p) , (q) , et c art. praec. abeunt in has

$$(p) = [p] + B\tau + B'\tau^2$$

$$(q) = [q] + C\tau + C'\tau^2$$

ubi

$$B = -\alpha_0 + [p] \frac{[p]\alpha_0 + [q]\alpha'_0}{1 + \cos[i]}$$

$$B' = -\beta_0 + [p] \frac{[p]\beta_0 + [q]\beta'_0}{1 + \cos[i]} - \frac{1}{2}[p](\alpha_0^2 + \alpha'_0{}^2)$$

$$C = -\alpha'_0 + [q] \frac{[p]\alpha_0 + [q]\alpha'_0}{1 + \cos[i]}$$

$$C' = -\beta'_0 + [q] \frac{[p]\beta_0 + [q]\beta'_0}{1 + \cos[i]} - \frac{1}{2}[q](\alpha_0^2 + \alpha'_0{}^2)$$

Jam accipio, ex integratis aequationibus (19) et (27), substituto $\cos[i]$ loco $\cos(i)$, omissisque terminis per c multiplicatis nec non constantibus arbitrariis, prodiesse

$$\alpha t + \beta t^2 + (i, i')_s \sin(i g + i' g') + (i, i')_c \cos(i g + i' g') \dots \text{prop}$$

$$\text{et } \alpha' t + \beta' t^2 + [i, i']_s \sin(i g + i' g') + [i, i']_c \cos(i g + i' g') \dots \text{pro } q$$

Hinc, terminis illis restitutis, evadit

$$\delta s = -\frac{1}{2} \left\{ (i, i')_s - [i, i']_c \right\} \sin[(i+1)g + i' g' + \omega]$$

$$-\frac{1}{2} \left\{ (i, i')_c + [i, i']_s \right\} \cos[(i+1)g + i' g' + \omega]$$

$$-\frac{1}{2} \left\{ (i, i')_s + [i, i']_c \right\} \sin[(i-1)g + i' g' - \omega]$$

$$-\frac{1}{2} \left\{ (i, i')_c - [i, i']_s \right\} \cos[(i-1)g + i' g' - \omega]$$

quorum ceteroquin duo plerumque minatissimi sunt. Termini igitur periodici in (36) per τ multiplicati praebent in latitudine hos

$$\begin{aligned}
& -\frac{1}{2} \left\{ A((i, i')_s - [i, i']_c) - D((i, i')_c + [i, i']_s) \right\} \tau \sin[(i+1)g + i'g' + \omega] \\
& -\frac{1}{2} \left\{ A((i, i')_c + [i, i']_s) + D((i, i')_s - [i, i']_c) \right\} \tau \cos[(i+1)g + i'g' + \omega] \\
& -\frac{1}{2} \left\{ A((i, i')_s + [i, i']_c) + D((i, i')_c - [i, i']_s) \right\} \tau \sin[(i-1)g + i'g' - \omega] \\
& -\frac{1}{2} \left\{ A((i, i')_c - [i, i']_s) - D((i, i')_s + [i, i']_c) \right\} \tau \cos[(i-1)g + i'g' - \omega]
\end{aligned}$$

Unde sequitur, si terminus quicunque periodicus ipsius s , ratione non habita ipsius τ , denotatur per

$$\begin{aligned}
& E \sin(ig + i'g' + \omega) + F \cos(ig + i'g' + \omega) \} \dots \dots \dots (37) \\
& + G \sin(ig + i'g' - \omega) + H \cos(ig + i'g' - \omega)
\end{aligned}$$

accessuros esse, quam proxime, terminos hos

$$\begin{aligned}
& (AE - DF) \tau \sin(ig + i'g' + \omega) + (AF + DE) \tau \cos(ig + i'g' + \omega) \} \dots \dots \dots (38) \\
& + (AG + DH) \tau \sin(ig + i'g' - \omega) + (AH - DG) \tau \cos(ig + i'g' - \omega)
\end{aligned}$$

21.

Reductioni longitudinis ad planum fixum aut mobile aequatio (25) inservit, quae tamen satis implicita

videtur. Differentiata vero, dum ν , constans ponatur, expressionem praebet persimplicem hanc

$$d(l - \nu) = \frac{1}{2} \frac{q dp - p dq + (p dp - q dq) \sin 2(\nu, -(\Theta)) + (q dp + p dq) \cos 2(\nu, -(\Theta))}{[1 - \frac{1}{2}(p^2 + q^2) + \frac{1}{2}(q^2 - p^2) \cos 2(\nu, -(\Theta)) + pq \sin 2(\nu, -(\Theta))] \sqrt{1 - p^2 - q^2}} \dots \dots \dots (39)$$

quae praeterea ad theorematum duo insignia viam munit.

Quum reductio differentialis ad hanc formam haud simplex appellari possit, in gratiam lectorum praecipua ejus momenta afferre, e re esse putavi. Quem in finem ante omnia aequationem appono identicam hanc

$$(p^2 + q^2) = (1 + \sqrt{1 - p^2 - q^2}) (1 - \sqrt{1 - p^2 - q^2})$$

qua pluries utar. Tum, posito

$$x = \frac{2pq \cos \alpha - (q^2 - p^2) \sin \alpha}{(1 + \sqrt{1 - p^2 - q^2})^2 + 2pq \sin \alpha + (q^2 - p^2) \cos \alpha}$$

ubi brevitatis causa

$$\alpha = 2(\nu, -(\Theta)) \text{ et } \sqrt{1 - p^2 - q^2} = \sqrt{1 - p^2 - q^2}$$

scripsi, aequatio (25) statim praebet

$$d(l - \nu) = \frac{q dp - p dq}{(1 + \sqrt{1 - p^2 - q^2}) \sqrt{1 - p^2 - q^2}} + \frac{dx}{1 + x^2}$$

E valore vero ipsius x modo allato emergit

$$1 + x^2 = \frac{(1 + \sqrt{1 - p^2 - q^2})^4 + 4(1 + \sqrt{1 - p^2 - q^2})^2 pq \sin \alpha + 2(1 + \sqrt{1 - p^2 - q^2})^2 (q^2 - p^2) \cos \alpha + (q^2 - p^2)^2 + 4p^2 q^2}{D^2}$$

designante D denominatorem ipsius x . Quum vero sit

$$(q^2 - p^2)^2 + 4p^2 q^2 = (q^2 + p^2)^2 = (1 + \sqrt{1 - p^2 - q^2})^2 (1 - \sqrt{1 - p^2 - q^2})^2$$

aequatio praecedens facile transmutatur in hanc

$$1 + x^2 = \frac{4(1 + \sqrt{1 - p^2 - q^2})^2 [1 - \frac{1}{2}(p^2 + q^2) + pq \sin \alpha + \frac{1}{2}(q^2 - p^2) \cos \alpha]}{D^2}$$

Differentiando vero expressionem ipsius x , dum α quasi constans tractetur, prodit

$$dx = 2 \frac{\left\{ (p dq + q dp) (1 + \sqrt{1 - p^2 - q^2})^2 \cos \alpha - (q dq - p dp) (1 + \sqrt{1 - p^2 - q^2})^2 \sin \alpha - 2pq (q dq - p dp) \right.}{D^2}$$

Sed identica est

$$\begin{aligned}
-2pq (q dq - p dp) + (q^2 - p^2) (p dq + q dp) &= (p^2 + q^2) (q dp - p dq) \\
&= (1 + \sqrt{1 - p^2 - q^2}) (1 - \sqrt{1 - p^2 - q^2}) (q dp - p dq)
\end{aligned}$$

quare

$$dx = 2 \frac{\left\{ (p dq + q dp) (1 + \sqrt{1 - p^2 - q^2}) \sqrt{1 - p^2 - q^2} \cos \alpha - (q dq - p dp) (1 + \sqrt{1 - p^2 - q^2}) \sqrt{1 - p^2 - q^2} \sin \alpha \right.}{D^2 \sqrt{1 - p^2 - q^2}}$$

Divisa hac formula per valorem ipsius $1+x^2$ modo evolutum additaque quantitate $\frac{qdp - pdq}{(1+\sqrt{V})\sqrt{V}}$, postquam ad eundem denominatorem reducta est, invenitur

$$d(l-\nu) = \frac{\frac{1}{2} \left\{ \begin{aligned} &[(pdq + qdp)(1+\sqrt{V})\sqrt{V} + 2pq(pdp + qdq) + (q^2 - p^2)(qdp - pdq)] \cos \alpha \\ &- [(q dq - p dp)(1+\sqrt{V})\sqrt{V} + 2pq(qdp - pdq) + (q^2 - p^2)(pdp + qdq)] \sin \alpha \\ &+ (qdp - pdq)[(1-\sqrt{V})\sqrt{V} + 2 - (p^2 + q^2)] \end{aligned} \right\}}{[1 - \frac{1}{2}(p^2 + q^2) + pq \sin \alpha + \frac{1}{2}(q^2 - p^2) \cos \alpha] (1+\sqrt{V})\sqrt{V}}$$

Sed habetur identice

$$\begin{aligned} 2pq(pdp + qdq) + (q^2 - p^2)(qdp - pdq) &= (pdq + qdp)(p^2 + q^2) \\ &= (pdq + qdp)(1+\sqrt{V})(1-\sqrt{V}) \\ -2pq(qdp - pdq) + (q^2 - p^2)(pdp + qdq) &= (q dq - p dp)(p^2 + q^2) \\ &= (q dq - p dp)(1+\sqrt{V})(1-\sqrt{V}) \\ 2 - (p^2 + q^2) &= 2 - (1+\sqrt{V})(1-\sqrt{V}) = 1 + \sqrt{V}^2 \end{aligned}$$

Quibus valoribus in formula praecedenti substitutis, statim fere expressio (39) emergit.

Jam numerator huius expressionis, substitutis.....

$$2[d p \cos(\nu, -(\Theta)) - d q \sin(\nu, -(\Theta))] [q \cos(\nu, -(\Theta)) + p \sin(\nu, -(\Theta))]$$

Sed

$$d q \sin(\nu, -(\Theta)) - d p \cos(\nu, -(\Theta)) = d s$$

Substitutis his valoribus in (39), eliminatisque p et q ope expressiones earum per i et $f \cos i d \theta$, elicitur

$$(40) \dots \dots \dots l = \nu + R + c - d s \frac{t g(i) \cos(\nu, -(\theta))}{1 - \sin^2(i) \sin^2(\nu, -(\theta))}$$

ubi $R + c$ constans integrali adjecta est. Quantitas R per hanc aequationem datur

$$t g R = - \frac{t g^{\frac{1}{2}}(i) \sin 2(\nu, -(\theta))}{1 + t g^{\frac{1}{2}}(i) \cos 2(\nu, -(\theta))}$$

valor vero ipsius c in art. praec. datus est. Quoties planum ad quod longitudines reducuntur fixum est, habetur $c = 0$, et ipsarum (i) et (θ) valores numerici certi determinatique sunt. Aequationes igitur praecedentes hoc continent

Theorema I.

„Perturbationes primi ordinis respectu massarum reductionis longitudinum in orbita ad planum quodlibet fixum, ne minimis quidem neglectis, producto constant, perturbationum sinus latitudinis in quantitatem, quae solam, variabilem ν , continet.“

$$(41) \dots \dots \dots d' R = \frac{1}{2} \frac{(q) d'(p) - (p) d'(q) + ((p) d'(p) - (q) d'(q)) \sin 2(\nu, -(\Theta)) + ((q) d'(p) + (p) d'(q)) \cos 2(\nu, -(\Theta))}{[1 - \frac{1}{2}((p)^2 + (q)^2) + \frac{1}{2}((q)^2 - (p)^2) \cos 2(\nu, -(\Theta)) + (p)(q) \sin 2(\nu, -(\Theta))] \sqrt{1 - (p)^2 - (q)^2}} + d' c'$$

ubi d' differentiale respectu ipsius τ designat. Quae expressio, quum simili modo transformari possit, ut ex-

$$(42) \dots \dots \dots R = R' + c' - d'(s) \frac{t g[i] \cos(\nu, -[\theta])}{1 - \sin^2[i] \sin^2(\nu, -[\theta])}$$

$\cos^2(\nu, -(\Theta)) - \sin^2(\nu, -(\Theta))$ loco $\cos 2(\nu, -(\Theta))$ et $2 \sin(\nu, -(\Theta)) \cos(\nu, -(\Theta))$ loco $\sin 2(\nu, -(\Theta))$, perfacile sub forma hac redigitur

$$d(l-\nu) = -d s \frac{t g i \cos(\nu, -f \cos i d \theta)}{1 - \sin^2 i \sin^2(\nu, -f \cos i d \theta)}$$

Quae aequatio, si neglecto quadrato vis perturbantis integratur, praebet

$$d(l-\nu) = -d s \frac{t g(i) \cos(\nu, -(\theta))}{1 - \sin^2(i) \sin^2(\nu, -(\theta))}$$

Quod theorema facile ad planum reductionis mobile extenditur. Hoc in casu (i) et (θ) functiones ipsarum $[i]$, $[\theta]$ et indeterminatae τ sunt, ut supra ostendi, illos itaque quantitates secundum τ differentiare nobis licet. Sed R , postquam quantitas haec

$$\int \frac{(q) d'(p) - (p) d'(q)}{(1 + \sqrt{1 - (p)^2 - (q)^2}) \sqrt{1 - (p)^2 - (q)^2}}$$

ei addita est, et valor ipsius $l - \nu$, ex aequatione (25) desumptus ejusdem formae sunt, differentiatam itaque R , expressio ipsi (39) plane similis necessarie prodit, in qua vero differentialia ad ipsum τ referenda sunt. Posito itaque

$$c' = - \int \frac{(q) d'(p) - (p) d'(q)}{(1 + \sqrt{1 - (p)^2 - (q)^2}) \sqrt{1 - (p)^2 - (q)^2}}$$

habetur

pressio (39), praebet, quoties quadratum virium perturbantium negligitur,

ubi $\delta'(s)$ eam sinus latitudinis perturbationum partem designat, quae functio ipsius τ per constantes arbitrarias (p) et (q) in sinu latitudinis verae introducta est, R' con-

stantem integrationi additam et per hanc aequationem computandam

$$tg R' = - \frac{tg^{\frac{2}{3}} [\dot{\epsilon}] \sin 2(\nu, - [\theta])}{1 + tg^{\frac{2}{3}} [\dot{\epsilon}] \cos 2(\nu, - [\theta])} \dots \dots \dots (43)$$

denotat, et c' formula hac datur

$$c' = - \frac{[q] \delta'(p) - [p] \delta'(q)}{\cos [\dot{\epsilon}] (1 + \cos [\dot{\epsilon}])} \dots \dots \dots (44)$$

Quum quadratum virium perturbantium hoc loco neglexerim, in factore ipsius δs in (40) $[\dot{\epsilon}]$ et $[\theta]$ resp. loco ($\dot{\epsilon}$) et (θ) ponere licitum est, unde obtinebis, si valorem ipsius R ex (42) substitueris

$$l = \nu + R' + c' + c - (\delta s + \delta'(s)) \frac{tg [\dot{\epsilon}] \cos (\nu, - [\theta])}{1 - \sin^2 [\dot{\epsilon}] \sin^2 (\nu, - [\theta])}$$

ubi, ut rem disertis verbis definiam, habetur

$$\delta s + \delta'(s) = s - \sin [\dot{\epsilon}] \sin (\nu, - [\theta]).$$

Quae aequationes hoc praebent

Theorema II.

„Perturbationes primi ordinis respectu massarum reductionis longitudinum in orbita ad planum quodlibet mobile, ne minimis quidem neglectis, ut supra, producto constant perturbationum sinus latitudinis in quantatem, quae solam variabilem ν , continet, cui vero producto hoc in casu quantitas $c + c'$ addenda est.

Facile vero perspicitur, theorema primum in theoremate secundo quasi casum specialem contineri. — Perturbationes primi ordinis respectu massarum in hac reductione plerumque sufficiunt, itaque per hoc theorema facillime in calculum vocari possunt, quando tamen acciderit, ut ad perturbationes secundi ordinis respicere oporteret, formula (39) commode usurpatur. Quo in casu substituantur valores ipsarum p, q, dp et dq , quales per aequationes (36) exhibiti sunt, post peractam integrationem, ubi ν , tamquam constans spectanda erat, mutetur τ in t et substituitur loco ν , valor ejus verus perturbatus. Constans integrali addenda semper est $= R + c$, ubi R ex integrata expressione (41) obtinetur, cui tamquam constans arbitraria R' ex (43) desumenda addi debet.

22.

Ne quid desit, de puncto initiali longitudinum insequentibus agere mihi propositum est. Longitudines ad planum mobile relatae, quales in praecedentibus derivatae sunt, initium ducunt a puncto fixo in plano fixo ad quod tum orbita corporis perturbati ope $[p]$ et $[q]$, tum planum mobile ope p_0 et q_0 relatum est, quod punctum fixum

secundum morem suetum nihil aliud est, nisi aequinoctium vernale illi tempori respondens, quo planum mobile cum plano fixo coincidit, vel quod idem est, aequinoctium vernale temporis epochae respondens. Item, punctum initiale dici potest id, quod in plano mobili ab intersectione huius plani cum isto plano fixo retrorsum tantum distat, quantum illud aequinoctium vernale ab eadem intersectione directione eadem remotum est. Ergo, quum astronomi longitudes cuivis tempori respondentes ab aequinoctio vernali ejusdem temporis numerent, longitudinibus hic l appellatis, etiamsi praecessio lunisolaris et nutatio non existerent, correctione adhuc opus est. Quae tamen correctio non eadem est, sive longitudes ad eclipticam mobilem, sive ad aequatorem mobilem reducturus es, quare hos casus duos seorsim tractari nos oportet.

23.

Initium indagandae correctionis faciamus, ubi longitudes ad eclipticam mobilem reducendae sunt. Quo in casu calculi elementa ex triangulo ab ecliptica fixa temporis epochae respondente, ab ecliptica mobili tempore t et ab aequatore ejusdem temporis formato petenda sunt. Designantibus ψ arcum quem, tempore t elapso, intersectio aequatoris retrocedens in fixa ecliptica percurrit (sive ψ summam praecessionis lunisolaris et nutationis longitudinis tempore t), h longitudinem nodi ascendentis eclipticae mobilis in fixa, erit latus trianguli nostri ab ecliptica fixa praebitum $\psi + h$; jam transfer arcum h ab eodem nodo in trianguli latus quod eclipticae mobilis pars est, et nomina residuum huius lateris ψ , tum latus hoc integrum erit $\psi + h$ et simul ψ , correctio quaesita longitudinum ad eclipticam mobilem reductarum, (sive ψ , summa praecessionis generalis et nutationis longitudinis), tertium trianguli nostri latus, quod aequatoris mobilis pars est, λ nominatum sit. Jam inclinatione eclipticae fixae cum mobili ϕ , obliquitate aequatoris mobilis erga eclipticam fixam ε atque obliquitate erga eclipticam mobilem ε , designatis, erunt anguli lateribus $\psi + h$, $\psi + h$ et λ resp. oppositi $180^\circ - \varepsilon$, ε et ϕ . Per trigonometriam igitur sphaericam elicitur

$$\begin{aligned}\sin \varepsilon, \sin (\psi, +h) &= \sin \varepsilon \sin (\psi + h) \\ \sin \varepsilon, \cos (\psi, +h) &= \cos \varepsilon \sin \varphi + \sin \varepsilon \cos \varphi \cos (\psi + h) \\ \sin \varepsilon, \sin \lambda &= \sin \varphi \sin (\psi + h)\end{aligned}$$

Comparatis his aequationibus cum (28), emergit, illas ex his evasuras esse, quoties $(\theta) - \theta_0$ in $\psi, +h$; $[\theta] - \theta_0$ in $\psi + h$; Δ in λ ; i_0 in φ ; $[i]$ in $180^\circ - \varepsilon$ et (i) in $180^\circ - \varepsilon$, mutantur. Quantum ad (i) aequationes illae proprie quidem in suspenso relinquunt, utrum in ε , an in $180^\circ - \varepsilon$,

$$(45) \dots \dots \dots \begin{cases} \varepsilon, &= \varepsilon + \sin \varphi \sin (\psi + h) + \frac{1}{2} \sin^2 \varphi \cotg \varepsilon \sin^2 (\psi + h) \\ \psi, &= \psi - \cotg \varepsilon \sin \varphi \sin (\psi + h) + (\frac{1}{2} + \cotg^2 \varepsilon) \sin^2 \varphi \sin (\psi + h) \cos (\psi + h) \\ \lambda &= \operatorname{cosec} \varepsilon \sin \varphi \sin (\psi + h) - \cotg \varepsilon \operatorname{cosec} \varepsilon \sin^2 \varphi \sin (\psi + h) \cos (\psi + h) \end{cases}$$

Jam accipio esse

$$\begin{aligned}\sin \varphi \sin h &= \gamma t + \delta t^2 \\ \sin \varphi \cos h &= \gamma' t + \delta' t^2\end{aligned}$$

tum secundum ill. *Poisson* habetur *)

$$\begin{aligned}\varepsilon &= k + \Delta \varepsilon + \frac{1}{2} \zeta \gamma t^2 \\ \psi &= \Delta \psi + \zeta t + \left\{ \frac{3ef}{2(1+\omega)} + \gamma' \cotg 2k \right\} \zeta t^2\end{aligned}$$

ubi temporis epochae respondententes ζ praecessio lunisolaris annua, k obliquitas eclipticae et e excentricitas orbitae terrae sunt; f vero ipsius e variationem annuam, ω rationem vis attractivae lunae ad vim solis in sphaeroidem tel-

$$\begin{aligned}\varepsilon, &= k + \Delta \varepsilon + \gamma' t + \left\{ \delta' - \frac{1}{2} \zeta \gamma + \frac{1}{2} \gamma^2 \cotg k \right\} t^2 \\ \psi, &= \Delta \psi + \left\{ \zeta - \gamma \cotg k \right\} t + \left\{ \frac{3ef\zeta}{2(1+\omega)} - \frac{\frac{1}{2} \gamma' \zeta}{\sin k \cos k} - \delta \cotg k + \gamma \gamma' \cotg^2 k + \frac{1}{2} \gamma \gamma' \right\} t^2 \\ \lambda &= \frac{\gamma}{\sin k} t + \left\{ \frac{\delta}{\sin k} + \frac{\zeta \gamma'}{\sin k} - \gamma \gamma' \frac{\cos k}{\sin^2 k} \right\} t^2\end{aligned}$$

Aequatio haec pro ψ , paullulum discrepat ab illa quam ill. *Poisson* l. c. prodidit, cuius rei causa in eo posita est, quod egregius ille geometra punctum initiale longitudinum, antequam praecessio nutatioque addita est, paullulum aliter supponit.

Posthac longitudo vera corporis perturbati ad planum

$$\begin{aligned}p_0 &= \alpha_0 t + \beta_0 t^2 = \sin \varphi \sin (h - (\Theta)) = \sin \varphi \sin h \cos (\Theta) - \sin \varphi \cos h \sin (\Theta) \\ q_0 &= \alpha'_0 t + \beta'_0 t^2 = \sin \varphi \cos (h - (\Theta)) = \sin \varphi \cos h \cos (\Theta) + \sin \varphi \sin h \sin (\Theta)\end{aligned}$$

evadunt

$$\begin{aligned}\alpha_0 t + \beta_0 t^2 &= \left\{ \gamma t + \delta t^2 \right\} \cos (\Theta) - \left\{ \gamma' t + \delta' t^2 \right\} \sin (\Theta) \\ \alpha'_0 t + \beta'_0 t^2 &= \left\{ \gamma' t + \delta' t^2 \right\} \cos (\Theta) + \left\{ \gamma t + \delta t^2 \right\} \sin (\Theta)\end{aligned}$$

mutanda sit, sed quum non dubium sit, quin $[i]$ in $180^\circ - \varepsilon$ mutari debeat, profecto (i) in ε , transire nequit, quoniam anguli hi non aliter quam illi directione eadem numerati sunt. Aequationes itaque omnes ab (28) derivatae, mutatis mutandis de praecedentibus quoque valent. Quamobrem aequationes (32), (33) et (33*), postquam loco $[p]$, p_0 etc. valores earum per $[i]$, i_0 etc. expressi substituti fuerint in has transmutabuntur.

luris **) et $\Delta \psi$ atque $\Delta \varepsilon$ resp. nutationem longitudinis atque obliquitatis eclipticae denotant.

Aequationes (45), si potestates ipsius ψ secunda altiores rejiciuntur, abeunt in has

$$\begin{aligned}\varepsilon, &= \varepsilon + \sin \varphi \cos h - \psi \sin \varphi \sin h \\ &\quad + \frac{1}{2} \cotg \varepsilon \sin^2 \varphi \sin h \cos h \\ \psi, &= \psi - \psi \cotg \varepsilon \sin \varphi \cos h - \cotg \varepsilon \sin \varphi \sin h \\ &\quad + (\frac{1}{2} + \cotg^2 \varepsilon) \sin^2 \varphi \sin h \cos h \\ \lambda &= \operatorname{cosec} \varepsilon \sin \varphi \sin h + \psi \operatorname{cosec} \varepsilon \sin \varphi \cos h \\ &\quad - \cotg \varepsilon \operatorname{cosec} \varepsilon \sin^2 \varphi \sin h \cos h\end{aligned}$$

e quibus, substitutis expressionibus praecedentibus, neglectisque productis ipsarum $\Delta \psi$, $\Delta \varepsilon$ in γ , γ' etc., elicitur

eclipticae mobilis reducta et ab aequinoctio vernali tempori t respondente, hoc est secundum morem suetum numerata erit $= l + \psi$.

Quantitates supra i_0 atque θ_0 nominatae et φ atque h resp. identicae sunt, quare, quum habeatur

*) V. *Poisson*, mémoire sur le mouvement de la terre autour de son centre de gravité (Mém. de l'ac. d. sc. Tome VII. pag. 248.)

**) Accepta e. g. massa lunae $\frac{1}{81}$ massae terrae, habetur $\omega = 2,35333$.

quae comparatae praebent

$$\left. \begin{aligned} \alpha_0 &= \gamma \cos(\Theta) - \gamma' \sin(\Theta) \\ \beta_0 &= \delta \cos(\Theta) - \delta' \sin(\Theta) \\ \alpha'_0 &= \gamma' \cos(\Theta) + \gamma \sin(\Theta) \\ \beta'_0 &= \delta' \cos(\Theta) + \delta \sin(\Theta) \end{aligned} \right\} \dots \dots \dots (46)$$

24.

Restat ut correctio definiatur, quam longitudinum ad aequatorem mobilem reductarum punctum initiale pati debet, ut in aequinoctio vernali mobili semper jaceat. Fingatur triangulum sphaericum inter eclipticam certo determinatoque tempore respondentem, aequatorem ejusdem temporis et aequatorem pro tempore indefinito t . Designantibus φ' et h' inclinationem et longitudinem nodi ascendentis aequatoris mobilis in aequatorem fixum, nec non λ' residuum trianguli nostri lateris, quod aequatoris mobilis pars est, si latus, quod aequatoris fixi pars est, subduxeris, tum erunt huius trianguli latera ψ , $\lambda' + h'$ et h' , angulique iis resp. oppositi φ' , $180^\circ - k$ et ε .

Si λ' ad longitudes additur, efficitur ut initium suum ab intersectione aequatoris mobilis cum ecliptica fixa ducant, quae

vero intersectio secundum art. praec. ab aequinoctio vernali mobili arcu λ directione inversa abest, designata igitur correctione longitudinum quaesita per $\psi_{//}$ erit

$$\psi_{//} = \lambda' - \lambda$$

Trigonometria sphaerica inter illas quantitates praebet relationes has

$$\begin{aligned} \sin \varphi' \sin h' &= \sin \varepsilon \sin \psi \\ \sin \varphi' \cos h' &= \cos \varepsilon \sin k - \sin \varepsilon \cos k \cos \psi \\ \sin \varphi' \sin(\lambda' + h') &= \sin k \sin \psi \\ \sin \varphi' \cos(\lambda' + h') &= -\cos k \sin \varepsilon + \sin k \cos \varepsilon \cos \psi \end{aligned}$$

Neglectis semper quantitibus tertii ordinis nec non quadratis et productis ipsarum $\Delta\psi$ et $\Delta\varepsilon$, conservatis vero productis nutationis in praecessionem, priores duae harum aequationum facili opera praebent

$$\left. \begin{aligned} \sin \varphi' \sin h' &= \Delta\psi \sin k + \left\{ \zeta^2 \sin k + \Delta\varepsilon \zeta \cos k \right\} t + \zeta \sin k \left\{ \frac{3ef}{2(1+\omega)} + \gamma' \cotg 2k \right\} t^2 \\ \sin \varphi' \cos h' &= -\Delta\varepsilon + \Delta\psi \zeta \sin k \cos k \cdot t + \frac{1}{2} \zeta^2 \left\{ \zeta^2 \sin k \cos k - \gamma \right\} t^2 \end{aligned} \right\} \dots \dots \dots (47)$$

Divisa secunda per primam et quarta per tertiam prodeunt

$$\begin{aligned} \cotg h' &= -\frac{\sin(\varepsilon - k)}{\sin \varepsilon \sin \psi} + \cos k \tg \frac{1}{2} \psi \\ -\cotg(\lambda' + h') &= \frac{\tg \lambda' - \cotg h'}{1 + \tg \lambda' \cotg h'} = \frac{\sin(\varepsilon - k)}{\sin k \sin \psi} + \cos \varepsilon \tg \frac{1}{2} \psi \end{aligned}$$

e quibus, eliminando h' , elicetur

$$\begin{aligned} \tg \lambda' &= (\cos k + \cos \varepsilon) \tg \frac{1}{2} \psi + \sin(\varepsilon - k) \frac{\sin \varepsilon - \sin k}{\sin \varepsilon \sin k \sin \psi} - \tg \lambda' \frac{\sin^2(\varepsilon - k)}{\sin \varepsilon \sin k \sin^2 \psi} \\ &\quad + \tg \lambda' \cos \varepsilon \cos k \tg^2 \frac{1}{2} \psi + \tg \lambda' \tg \frac{1}{2} \psi \frac{\sin^2(\varepsilon - k)}{\sin \varepsilon \sin k \sin \psi} \end{aligned}$$

Quum valor approximatus ipsius $\tg \lambda'$ per hanc aequationem $= \psi \cos k$ evadat, terminus secundus et tertius dextrae partis usque ad quantitates tertii ordinis se mutuo tollunt, qua re et quum ultimi termini quantitates tertii ordinis sese sponte praestent, emergit

$$\begin{aligned} \psi_{//} &= \Delta\psi \cos k + \left\{ \zeta \cos k - \frac{1}{2} \Delta\varepsilon \zeta \sin k - \gamma \csc k \right\} t \\ &\quad + \left\{ \frac{3ef\zeta}{2(1+\omega)} \cos k - \zeta \gamma' \sin k - \frac{\delta + \frac{1}{2}\zeta\gamma}{\sin k} + \gamma\gamma' \frac{\cos k}{\sin^2 k} \right\} t^2 \dots \dots \dots (48) \end{aligned}$$

Jam longitudo vera ad aequatorem mobilem reducta et ab aequinoctio vernali mobili numerata, hoc est, ascensio recta heliocentrica erit $= l + \psi_{//}$.

$\lambda' = \frac{1}{2}(\cos k + \cos \varepsilon) \psi$
usque ad quantitates tertii ordinis, unde substitutis in aequatione hac $\psi_{//} = \lambda' - \lambda$ valoribus ipsarum ε , ψ et λ ex art. praec. petendis evadit

Quo in casu φ' et h' conveniunt resp. cum i_0 et θ_0 sed in aequationibus (47) et (48) tum termini praestant qui a t , sive quod idem est a τ liberi sunt, tum termini qui ceteris

dissimiles per producta haec $\Delta\psi \cdot \zeta$ et $\Delta\epsilon \cdot \zeta$ multiplicati sunt. Ut termini hi in calculum vocari possint, suppono, ipsas $\alpha_0, \alpha'_0, \beta_0$ et β'_0 resp. quantitibus his $\Delta\alpha_0, \Delta\alpha'_0, \Delta\beta_0$ et $\Delta\beta'_0$, et deinde etiam has B, B', C, C', D et D' resp. quantitibus his $\Delta B, \Delta B'$, etc. tali modo augeri, ut

(p), (q) et c, quales in art. 20 datae sint, fiant

$$(p) = [p] + \Delta B + (B + \Delta B')\tau + B'\tau^2$$

$$(q) = [q] + \Delta C + (C + \Delta C')\tau + C'\tau^2$$

$$c = \Delta D + (D + \Delta D')\tau + D'\tau^2$$

Habentur itaque

$$B = -\alpha_0 + [p] \frac{[p]\alpha_0 + [q]\alpha'_0}{1 + \cos[i]}$$

$$C = -\alpha'_0 + [q] \frac{[p]\alpha_0 + [q]\alpha'_0}{1 + \cos[i]}$$

$$D = \frac{[q]\alpha_0 - [p]\alpha'_0}{1 + \cos[i]}$$

$$B' = -\beta_0 + [p] \frac{[p]\beta_0 + [q]\beta'_0}{1 + \cos[i]} - \frac{1}{2} [p] (\alpha_0^2 + \alpha'_0{}^2)$$

$$C' = -\beta'_0 + [q] \frac{[p]\beta_0 + [q]\beta'_0}{1 + \cos[i]} - \frac{1}{2} [q] (\alpha_0^2 + \alpha'_0{}^2)$$

$$D' = \frac{[q]\beta_0 - [p]\beta'_0}{1 + \cos[i]} - \frac{([q]\alpha_0 - [p]\alpha'_0)([p]\alpha_0 + [q]\alpha'_0)}{2(1 + \cos[i])^2}$$

$$\Delta B = -\Delta\alpha_0 + [p] \frac{[p]\Delta\alpha_0 + [q]\Delta\alpha'_0}{1 + \cos[i]}$$

$$\Delta C = -\Delta\alpha'_0 + [q] \frac{[p]\Delta\alpha_0 + [q]\Delta\alpha'_0}{1 + \cos[i]}$$

$$\Delta D = \frac{[q]\Delta\alpha_0 - [p]\Delta\alpha'_0}{1 + \cos[i]}$$

$$\Delta B' = -\Delta\beta_0 + [p] \frac{[p]\Delta\beta_0 + [q]\Delta\beta'_0}{1 + \cos[i]} - [p] (\alpha_0\Delta\alpha_0 + \alpha'_0\Delta\alpha'_0)$$

$$\Delta C' = -\Delta\beta'_0 + [q] \frac{[p]\Delta\beta_0 + [q]\Delta\beta'_0}{1 + \cos[i]} - [q] (\alpha_0\Delta\alpha_0 + \alpha'_0\Delta\alpha'_0)$$

$$\Delta D' = \frac{[q]\Delta\beta_0 - [p]\Delta\beta'_0}{1 + \cos[i]} - \frac{([q]\Delta\alpha_0 - [p]\Delta\alpha'_0)([p]\alpha_0 + [q]\alpha'_0) + ([p]\Delta\alpha_0 + [q]\Delta\alpha'_0)([q]\alpha_0 - [p]\alpha'_0)}{2(1 + \cos[i])^2}$$

Jam ponendo secundum aequationes (47)

$$\gamma_i = \zeta \sin k; \quad \delta_i = \zeta \sin k \left\{ \frac{3ef}{2(1+\omega)} + \gamma' \cotg 2k \right\}$$

$$\gamma'_i = 0; \quad \delta'_i = \frac{1}{2} \zeta \left\{ \zeta \sin k \cos k - \gamma \right\}$$

$$\Delta\gamma_i = \Delta\psi \sin k; \quad \Delta\delta_i = \Delta\epsilon \cdot \zeta \cos k$$

$$\Delta\gamma'_i = -\Delta\epsilon; \quad \Delta\delta'_i = \Delta\psi \cdot \zeta \sin k \cos k$$

aequationes (46) praebent, mutatis mutandis

$$\alpha_0 = \zeta \sin k \cos(\Theta)$$

$$\beta_0 = \left\{ \frac{3ef}{2(1+\omega)} + \gamma' \cotg 2k \right\} \zeta \sin k \cos(\Theta) - \frac{1}{2} \left\{ \zeta \sin k \cos k - \gamma \right\} \zeta \sin(\Theta)$$

$$\alpha'_0 = \zeta \sin k \sin(\Theta)$$

$$\beta'_0 = \left\{ \frac{3ef}{2(1+\omega)} + \gamma' \cotg 2k \right\} \zeta \sin k \sin(\Theta) + \frac{1}{2} \left\{ \zeta \sin k \cos k - \gamma \right\} \zeta \cos(\Theta)$$

$$\Delta\alpha_0 = \Delta\psi \sin k \cos(\Theta) + \Delta\epsilon \sin(\Theta)$$

$$\Delta\beta_0 = \Delta\epsilon \zeta \cos k \cos(\Theta) - \Delta\psi \zeta \sin k \cos k \sin(\Theta)$$

$$\Delta\alpha'_0 = \Delta\psi \sin k \sin(\Theta) - \Delta\epsilon \cos(\Theta)$$

$$\Delta\beta'_0 = \Delta\epsilon \zeta \cos k \sin(\Theta) + \Delta\psi \zeta \sin k \cos k \cos(\Theta)$$

unde sequentes expressiones adipiscimur, quibus in articulo subsequenti utemur.

$$\begin{aligned}
B &= -\zeta \sin k \cos [\theta] \cos ([\theta] - (\odot)) - \cos [i] \zeta \sin k \sin [\theta] \sin ([\theta] - (\odot)) \\
C &= \zeta \sin k \cos [\theta] \sin ([\theta] - (\odot)) - \cos [i] \zeta \sin k \sin [\theta] \cos ([\theta] - (\odot)) \\
D &= \frac{\sin [i]}{1 + \cos [i]} \zeta \sin k \cos [\theta] \\
\Delta B &= -\Delta \psi \sin k \cos [\theta] \cos ([\theta] - (\odot)) - \cos [i] \Delta \psi \sin k \sin [\theta] \sin ([\theta] - (\odot)) - \Delta \varepsilon \sin [\theta] \cos ([\theta] - (\odot)) \\
&\quad + \cos [i] \Delta \varepsilon \cos [\theta] \sin ([\theta] - (\odot)) \\
\Delta C &= \Delta \psi \sin k \cos [\theta] \sin ([\theta] - (\odot)) - \cos [i] \Delta \psi \sin k \sin [\theta] \cos ([\theta] - (\odot)) + \Delta \varepsilon \sin [\theta] \sin ([\theta] - (\odot)) \\
&\quad + \cos [i] \Delta \varepsilon \cos [\theta] \cos ([\theta] - (\odot)) \\
\Delta D &= \frac{\sin [i]}{1 + \cos [i]} \{ \Delta \psi \sin k \cos [\theta] + \Delta \varepsilon \sin [\theta] \} \\
\Delta B' &= \Delta \psi \zeta \sin k \cos k \sin [\theta] \cos ([\theta] - (\odot)) - \cos [i] \Delta \psi \zeta \sin k \cos k \cos [\theta] \sin ([\theta] - (\odot)) \\
&\quad - \Delta \varepsilon \zeta \cos k \cos [\theta] \cos ([\theta] - (\odot)) - \cos [i] \Delta \varepsilon \zeta \cos k \sin [\theta] \sin ([\theta] - (\odot)) - \sin [i] \Delta \psi \zeta \sin^2 k \sin ([\theta] - (\odot)) \\
\Delta C' &= -\Delta \psi \zeta \sin k \cos k \sin [\theta] \sin ([\theta] - (\odot)) - \cos [i] \Delta \psi \zeta \sin k \cos k \cos [\theta] \cos ([\theta] - (\odot)) \\
&\quad + \Delta \varepsilon \zeta \cos k \cos [\theta] \sin ([\theta] - (\odot)) - \cos [i] \Delta \varepsilon \zeta \cos k \sin [\theta] \cos ([\theta] - (\odot)) - \sin [i] \Delta \psi \zeta \sin^2 k \cos ([\theta] - (\odot)) \\
\Delta D &= \frac{\sin [i]}{1 + \cos [i]} \{ \Delta \varepsilon \zeta \cos k \cos [\theta] - \Delta \psi \zeta \sin k \cos k \sin [\theta] \} \\
&\quad - \frac{\sin^2 [i]}{2(1 + \cos [i])^2} \{ \Delta \psi \zeta \sin^2 k \sin 2[\theta] - \Delta \varepsilon \zeta \sin k \cos 2[\theta] \}
\end{aligned}$$

Ipsam A vero art. 20 quantitate ΔA augeri non opus est, quia per quantitates primi ordinis respectu massarum ubique multiplicata est.

25.

Methodus art. praec. tametsi omni quae desiderari potest praecisione gaudeat, hoc incommodi habet, quod perturbationes declinationis heliocentricae a B etc. et ΔB etc. orientes praebet, quarum coefficientes minutissimi dici nequeunt. Quum enim praecessio lunisolaris annua, sive $\zeta = 50''$ circiter sit, emergit in ipsa s sive in declinatione heliocentrica terminus circiter hic $20'' t \cos \nu$, quumque maximi ipsarum $\Delta \psi$ et $\Delta \varepsilon$ termini resp. hi $-17'' \sin \delta$ et $9'' \cos \delta$ (designante δ longitudinem nodi ascendentis orbitae lunae) sint, evadunt in ipsa s circiter hi $-7'' \sin \delta \cos \nu$ et $9'' \cos \delta \sin \nu$; praeterea necesse est, producti nutationis in praecessionem ratio separatim habeatur, ut in art. praec. feci. Quo quidem negotio non indiget, qui nutationem ascensionibus declinationibusque more vulgari applicare vult, attamen terminus $20'' t \cos \nu$, nihilominus superest, qui paucis annis a temporis epocha elapsis, admodum magnus fiet, et proinde multiplicatio per t requisita satis molesta evadet, etsi quantitas $20'' \cos \nu$, in tabula foret redacta. Quae quum ita

sint, artificium jam definiam, per quod incommodum id plane removeri poterit, quodque ad relationes elegantissimas perducet.

Introducatur quantitas indeterminata $\Delta \eta + (\eta + \Delta \eta') \tau$ ita ut sit

$$s = \sin i \sin [\nu + \Delta \eta + (\eta + \Delta \eta') \tau - (\odot)] - (\cos i d\theta + \Delta \eta + (\eta + \Delta \eta') \tau)$$

sive

$$s = q, \sin [\nu + \Delta \eta + (\eta + \Delta \eta') \tau - (\odot)] - p, \cos [\nu + \Delta \eta + (\eta + \Delta \eta') \tau - (\odot)]$$

ponendo brevitatis causa

$$\begin{aligned}
p, &= p \cos (\Delta \eta + (\eta + \Delta \eta') \tau) + q \sin (\Delta \eta + (\eta + \Delta \eta') \tau) \\
q, &= q \cos (\Delta \eta + (\eta + \Delta \eta') \tau) - p \sin (\Delta \eta + (\eta + \Delta \eta') \tau)
\end{aligned}$$

hanc quantitatē $\Delta \eta + (\eta + \Delta \eta') \tau$ talem determinare licet, qualis perturbationes eas ipsius s de quibus loquebar, quam minimas reddat.

Terminis tertii ordinis nec non quadratis et productis ipsarum $\Delta \eta$ et $\Delta \eta'$ neglectis, aequationes praecedentes praebent

$$\begin{aligned}
p, &= p + q \Delta \eta + q (\eta + \Delta \eta' - p \Delta \eta \cdot \eta) \tau - \frac{1}{2} p \eta^2 \tau^2 \\
q, &= q - p \Delta \eta - p (\eta + \Delta \eta' + q \Delta \eta \cdot \eta) \tau - \frac{1}{2} q \eta^2 \tau^2
\end{aligned}$$

Jam substituendo valores ipsarum p et q ex (36), si rationem ipsarum $\Delta B, \Delta C$ etc. art. praecedentis habueris invenies

$$\begin{aligned}
p, &= [p] + \Delta X + \{X + \Delta X' - \alpha\} \tau + \alpha t + \{B' + C\eta - \frac{1}{2} [p] \eta^2\} \tau^2 + \{A\alpha - D\alpha' + \alpha' \eta\} \tau t + \beta t^2 \\
&\quad + \text{terminis periodicis.} \\
q, &= [q] + \Delta Y + \{Y + \Delta Y' - \alpha'\} \tau + \alpha' t + \{C - B\eta - \frac{1}{2} [q] \eta^2\} \tau^2 + \{A\alpha' + D\alpha - \alpha \eta\} \tau t + \beta t^2 \\
&\quad + \text{terminis periodicis.}
\end{aligned} \dots\dots\dots (49)$$

ubi

$$(50) \dots \left\{ \begin{array}{l} \Delta X = \Delta B + [q] \Delta \eta \\ \Delta Y = \Delta C - [p] \Delta \eta \\ X = B + \alpha + [q] \eta \\ Y = C + \alpha' - [p] \eta \\ \Delta X' = \Delta B' + C \Delta \eta + \Delta Y \eta + [q] \Delta \eta' \\ \Delta Y' = \Delta C' - B \Delta \eta - \Delta X \eta - [p] \Delta \eta' \end{array} \right.$$

et perturbationes ipsius s a nutatione et praecessione orientes erunt

$$\Delta \xi \sin(\nu + \Delta \eta + (\eta + \Delta \eta')t + V) + (\xi + \Delta \xi')t \sin(\nu + \Delta \eta + (\eta + \Delta \eta')t + W)$$

ubi

$$\begin{aligned} \Delta \xi &= \sqrt{(\Delta X^2 + \Delta Y^2)}; \\ \xi + \Delta \xi' &= \sqrt{(X + \Delta X')^2 + (Y + \Delta Y')^2} \end{aligned}$$

et V atque W arcus designant, quos definire hoc loco nihil attinet. Condicio igitur ut $\Delta \xi$ et $\xi + \Delta \xi'$ quam minimi fiant, exigit ut $\Delta X^2 + \Delta Y^2$ et $(X + \Delta X')^2 + (Y + \Delta Y')^2$ quam minimi evadant, sed quum X et Y praecessionem ipsam $\Delta X'$ vero et $\Delta Y'$ non nisi producta praecessionis in nutationem contineant, perpaulo aberit valor ipsius $\xi + \Delta \xi'$

$$\Delta X = \cos[i] \left\{ \Delta \varepsilon \cos[\theta] - \Delta \psi \sin k \sin[\theta] \right\} \sin([\theta] - (\Theta))$$

$$\Delta Y = \cos[i] \left\{ \Delta \varepsilon \cos[\theta] - \Delta \psi \sin k \sin[\theta] \right\} \cos([\theta] - (\Theta))$$

$$X = \left\{ \frac{[p]\alpha + [q]\alpha'}{\sin[i]} - \cos[i] \zeta \sin k \sin[\theta] \right\} \sin([\theta] - (\Theta))$$

$$Y = \left\{ \frac{[p]\alpha + [q]\alpha'}{\sin[i]} - \cos[i] \zeta \sin k \sin[\theta] \right\} \cos([\theta] - (\Theta))$$

$$\Delta X' = \left\{ \begin{aligned} &\Delta \psi \cdot \zeta \sin k \left[\frac{\sin k}{\sin[i]} \cos^2[\theta] - \cos k \cos[i] \cos[\theta] - \sin k \sin[i] \right] \\ &+ \Delta \varepsilon \cdot \zeta \left[\frac{\sin k}{\sin[i]} \cos[\theta] - \cos k \cos[i] \right] \sin[\theta] \end{aligned} \right\} \sin([\theta] - (\Theta))$$

$$\Delta Y' = \left\{ \begin{aligned} &\Delta \psi \cdot \zeta \sin k \left[\frac{\sin k}{\sin[i]} \cos^2[\theta] - \cos k \cos[i] \cos[\theta] - \sin k \sin[i] \right] \\ &+ \Delta \varepsilon \cdot \zeta \left[\frac{\sin k}{\sin[i]} \cos[\theta] - \cos k \cos[i] \right] \sin[\theta] \end{aligned} \right\} \cos([\theta] - (\Theta))$$

et hinc valores minimi

$$\Delta \xi = \cos[i] \left\{ \Delta \varepsilon \cos[\theta] - \Delta \psi \sin k \sin[\theta] \right\}$$

$$\xi = \frac{[p]\alpha + [q]\alpha'}{\sin[i]} - \zeta \sin k \cos[i] \sin[\theta]$$

$$\begin{aligned} \Delta \xi' &= \Delta \psi \cdot \zeta \sin k \left[\frac{\sin k}{\sin[i]} \cos^2[\theta] - \cos k \cos[i] \cos[\theta] - \sin k \sin[i] \right] \\ &+ \Delta \varepsilon \cdot \zeta \left[\frac{\sin k}{\sin[i]} \cos[\theta] - \cos k \cos[i] \right] \sin[\theta] \end{aligned}$$

Quum eadem quantitas $\Delta \eta + (\eta + \Delta \eta')\tau$ in formulis pro reducenda longitudine aequo jure introduci possit, omnia quae de hac reductione exposui nunquam non valent, et accuratio ne minimum quidem laeditur, si ubique $l + \Delta \eta + (\eta + \Delta \eta')\tau$

a valore suo minimo, si $X^2 + Y^2$ et $\Delta X'^2 + \Delta Y'^2$ separatim quam minimi redduntur. Algorithmus igitur notus ad binas aequationum (50) applicatus praebet

$$\Delta \eta = \frac{[p] \Delta C - [q] \Delta B}{[p]^2 + [q]^2}$$

$$\eta = \frac{[p](C + \alpha') - [q](B + \alpha)}{[p]^2 + [q]^2}$$

$$\Delta \eta' = \frac{[p](\Delta C' - B \Delta \eta - \Delta X \eta) - [q](\Delta B' + C \Delta \eta - \Delta Y \eta)}{[p]^2 + [q]^2}$$

Substitutis in his aequationibus valoribus quantitatum quas continent ex art. praec. depromtis, inveniuntur

$$\Delta \eta = \frac{\Delta \psi \sin k \cos[\theta] + \Delta \varepsilon \sin[\theta]}{\sin[i]}$$

$$\eta = \frac{[p]\alpha' - [q]\alpha}{\sin^2[i]} + \zeta \frac{\sin k}{\sin[i]} \cos[\theta]$$

$$\begin{aligned} \Delta \eta' &= \frac{\Delta \psi \cdot \zeta \sin k}{\sin^2[i]} \left\{ \sin k \cos[i] \sin 2[\theta] - \cos k \sin[i] \sin[\theta] \right\} \\ &- \frac{\Delta \varepsilon \cdot \zeta}{\sin^2[i]} \left\{ \sin k \cos[i] \cos 2[\theta] - \cos k \sin[i] \cos[\theta] \right\} \end{aligned}$$

atque tum

loco l ; $\nu + \Delta \eta + (\eta + \Delta \eta')\tau$ loco ν ; p , loco p et q , loco q substituuntur. Formula nominatim reductam longitudinem sive ascensionem rectam heliocentricam exhibens nunc est

$$l + \psi_{III} = (\nu + \Delta\eta + (\eta + \Delta\eta')t) + \psi_{III} + R' - (\delta s + \delta'(s)) \frac{tg[i] \cos(\nu + \Delta\eta + (\eta + \Delta\eta')t - [\theta])}{1 - \sin^2[i] \sin^2(\nu + \Delta\eta + (\eta + \Delta\eta')t - [\theta])}$$

ubi

$$\psi_{III} = \psi_{II} + c + c' - \Delta\eta - (\eta + \Delta\eta')t \dots \dots \dots (51)$$

et ubi, si res postularet, loco ultimi termini integralia aequationum (39) et (41) substitui possunt, dummodo c' in expressione ipsius ψ_{III} deleatur *).

Per aequationes (49) habentur

$$\begin{aligned} \delta'p &= \Delta X + \left\{ X + \Delta X' - \alpha \right\} \tau \\ \delta'q &= \Delta Y + \left\{ Y + \Delta Y' - \alpha' \right\} \tau \end{aligned}$$

$$\begin{aligned} \psi_{III} &= \Delta\psi \left[\cos k - \frac{\cos[i]}{\sin[i]} \sin k \cos[\theta] \right] - \Delta\epsilon \frac{\cos[i]}{\sin[i]} \sin[\theta] \\ &+ \frac{[q]\alpha - [p]\alpha'}{\cos[i] \sin^2[i]} t + \zeta \left\{ \cos k - \frac{\cos[i]}{\sin[i]} \sin k \cos[\theta] \right\} t - \gamma \operatorname{cosec} k \cdot t \\ &+ \Delta\psi \cdot \zeta \sin k \left\{ \frac{\cos[i]}{\sin[i]} \cos k \sin[\theta] - \frac{1 + \cos^2[i]}{\sin^2[i]} \sin k \sin 2[\theta] \right\} t \\ &- \Delta\epsilon \cdot \zeta \left\{ \frac{\cos[i]}{\sin[i]} \cos k \cos[\theta] - \frac{1 + \cos^2[i]}{\sin^2[i]} \sin k \cos 2[\theta] + \frac{1}{2} \sin k \right\} t \end{aligned}$$

ubi quoque terminos per t^2 multiplicatos omisi, quos facile restituet quicumque hanc rem aliqua attentione dignari velit.

Aequationes praecedentes monstrant coefficientem $\xi + \Delta\xi'$ et terminum in $\Delta\xi$ per $\Delta\psi$ multiplicatum, nec non quantitatem ψ_{III} praesertim in casibus ubi nodi prope a punctis 0 et 180 graduum jacent deminutos esse, contra ea in casibus ubi nodi prope a punctis 90 et 270 graduum jacent, deminutionem hanc nullius momenti esse.

Si vero $[\theta] = 0$ et simul $[i] = k$, hoc est si corpus perturbatum in ecliptica mobili ipsa movetur, fit

$$\begin{aligned} \alpha &= (\gamma \cos(\odot) - \gamma' \sin(\odot)) \cos k \\ \alpha' &= (\gamma \sin(\odot) + \gamma' \cos(\odot)) \cos k \\ [p] &= -\sin k \sin(\odot) \\ [q] &= \sin k \cos(\odot) \end{aligned}$$

atque tum

$$\Delta\eta = \Delta\psi; \quad \eta = \zeta - \gamma \cotg k; \quad \Delta\eta' = 0$$

sive

$$\Delta\eta + (\eta + \Delta\eta')t = \psi,$$

porro

$$\Delta\xi = \Delta\epsilon \cos k; \quad \xi = \gamma' \cos k; \quad \Delta\xi' = 0$$

sive

$$\frac{\Delta\xi + (\xi + \Delta\xi')t}{\cos k} = \epsilon - k$$

deinde

$$\psi_{III} = 0.$$

*) Evidens est in expressione (43) ipsius R' etiam $\dots \dots \dots$
 $\nu + \Delta\eta + (\eta + \Delta\eta')t$ loco ν , substitui debere.

ubi tamen terminos per τ^2 multiplicatos omisi, quoniam ad finem quem hic persequor non necessarii sunt. Aequatio itaque (44) subministrat

$$c' = \frac{[q]\alpha - [p]\alpha'}{\cos[i] (1 + \cos[i])} \tau$$

et aequatio (51) substitutis valoribus quantitatum quas continet ex hoc et praec. art. petendis, abit in

Hinc jam intelligi potest, terminos illos minutissimos futuros esse, quoties inclinatio orbitae ad eclipticam parva est, sed proprietas haec sequenti modo clarius perspicitur.

Si corporis orbitae inclinatio ad eclipticam parva est, longitudines nodorum valores 90 et 270 graduum nunquam assequi possunt, sed certis arctisque limitibus, inter quos puncta 0 et 180 graduum medium obtinent locum, semper circumscriptae sunt. Quocirca e re est, valores quam maximos longitudinum nodorum, aliquot inclinationum valoribus respondentes indagare. Positis orbitae inclinatione ad eclipticam g et longitudine nodi n , triangulum sphaericum inter orbitam, eclipticam et aequatorem praebet:

$$\begin{aligned} \sin[i] &= \sin g \frac{\sin n}{\sin[\theta]} \\ \sin n \cotg[\theta] &= \sin k \cotg g + \cos k \cos n \end{aligned}$$

Sumtis $[\theta]$ et n variabilibus, altera aequatio differentiatia praebet postquam $d[\theta] = 0$ factum est, aequationem maximi hanc

$$\cotg \nu = -tg n \cos k$$

ubi ν valorem maximum ipsius $[\theta]$ denotat. Quacum ν ex aequatione hac

$$\sin n \cotg \nu = \sin k \cotg g + \cos k \cos n$$

eliminata, prodit

$$\cos n = -\cotg k tg g$$

quae igitur valorem eum ipsius n praebet, qui ipsi ν respondet. Quo invento, aequationes praecedentes ν et $[i]$ subministrant. Ope harum formularum, positis $g = 0 = 2^\circ = 4^\circ = 6^\circ = 8^\circ$

nec non $\zeta \sin k = 20''$, $\Delta\psi = -16'',8 \sin \delta$ et $\Delta\varepsilon = 9'',0 \cos \delta$ valores computavi, quos sequens tabella ostendit.

g	$[i]$	ν	Valores maximi ipsarum	
			$\xi +$	$\Delta\xi$
0	23,5	0,0	0,0	$+ 8,2 \cos \delta$
2	23,4	$+ 5,0$	1,6	$+ 8,2 \cos \delta + 0,5 \sin \delta$
4	23,2	10,1	3,2	$+ 8,1 \cos \delta + 1,1 \sin \delta$
6	22,8	15,2	4,8	$+ 8,0 \cos \delta + 1,6 \sin \delta$
8	22,1	20,5	6,5	$+ 7,7 \cos \delta + 2,1 \sin \delta$

†) Ubi tamen ad terminum $\frac{[p]x + [q]x'}{\sin [i]}$ respicere non potui, qui ceteroquin semper minutissimus est, et in paucis admodum casibus una minuta secunda major evadit.

*) unde conspicuum est, quantam introducta quantitas η in coefficientem hunc ξ , qui alias $= 20''$ evaderet, habeat vim, quoties inclinatio orbitae ad eclipticam parva est. Verum in ipso $\Delta\xi$ terminus $8'' \cos \delta$ circiter semper eminet, et nullo modo eliminari potest. Ut eluceat, qualem magnitudinem termini a productis nutationis in praecessionem prolatis habeant, positis $[i] = 22^\circ 1$, $[\theta] = \nu = 20^\circ 5$, computavi has

$$\Delta\xi \cdot t = + 0''000029 t \sin \delta + 0''000109 t \cos \delta$$

$$\Delta\eta' \cdot t = - 0,001399 t \sin \delta + 0,000696 t \cos \delta$$

$$\psi_{III} = + 0,001510 t \sin \delta - 0,000751 t \cos \delta$$

$$\psi_{III} + \Delta\eta' \cdot t = + 0,000111 t \sin \delta - 0,000055 t \cos \delta$$

Ecce igitur valores maximos harum quantitatum, quoties inclinatio orbitae ad eclipticam 8 graduum est, quae itaque revera his valoribus semper fere minores prodire debent. Quum vero maximus earum effectus in declinatione sit $= \Delta\xi' \cdot t$, et effectus earum quam proxime in ascensione

recta sit $= \psi_{III} + \Delta\eta' \cdot t$: concludimus, terminos a productis nutationis in praecessionem prolatis omnino rejici posse, id quod nobis non liceret, si quantitatem η non introduxissemus.

Ceterum facile ostendi potest, quantitatem $\Delta\eta + (\eta + \Delta\eta')t$, qualis hic determinata est, arcum orbitae corporis perturbati inter aequatorem fixum et aequatorem mobilem interceptum significare **). Jam quum arcus hic in orbitis ad eclipticam parum inclinatis summae praecessionis et nutationis circiter aequalis sit: constat terminos producta nutationis in praecessionem continentes, quos negligere fas esse demonstrabam, non nisi nullius momenti esse posse. Quantitas vero $\Delta\eta + \eta\tau$, quum post τ in t mutatum terminis constet, quorum quisque ab unica variabili pendet, perfacile in tabulis redigitur, imo ηt longitudinibus quasi praecessio statim addi potest.

Simili modo quo $\eta\tau$, quantitas haec $\eta'\tau^2$ introduci potuisset, ut coefficientens in latitudine per t^2 multiplicatus ad minimum suum valorem redigeretur, sed quum coefficientens hic sua sponta jam minutissimus est, nihil lucri inde redundaret.

Item, substitutio huius quantitatis $\Delta\eta + \eta\tau$, quae si nihil negligere in votis est ubique perfici debet, in perturbationibus periodicis tum declinationis tum reductionis longitudinum ad aequatorem plerumque superflua erit, et tum termini periodici sicut in expressionibus (36) allati sunt, immediate conservari possunt, ita ut pars illa ipsius s , quae ab his terminis provenit, sit =

$$\left\{ \left[[i, i']_s + (A[i, i']_s + D(i, i')_s) t \right] \sin (ig + i'g') + \left[[i, i']_c + (A[i, i']_c + D(i, i')_c) t \right] \cos (ig + i'g') \right\} \sin (\nu - (\Theta))$$

$$- \left\{ \left[(i, i')_s + (A(i, i')_s - D[i, i']_s) t \right] \sin (ig + i'g') + \left[(i, i')_c + (A(i, i')_c - D[i, i']_c) t \right] \cos (ig + i'g') \right\} \cos (\nu - (\Theta))$$

ubi etiam termini per t multiplicati plerumque negligi possunt.

Ad motum lunae indagandum formulas nostras applicaturus, propter celerrimum perigaei nodorumque motum dif-

ficultatibus aliquot fortasse delinebitur, quae tamen per artificia, iis quibus hoc loco usus sum, haud dissimilia removeri possunt; sed de his alias.

*) Itaque nodi orbitae lunae, qui in ecliptica totam peripheriam brevi perlustrant, in aequatore modo oscillant, et amplitudinem 25 graduum circiter tantum absolvunt.

**) Itaque sicut $\nu - [\theta]$ argumentum latitudinis respectu aequatoris fixi denotat, ita $\nu + \Delta\eta + (\eta + \Delta\eta')t - [\theta]$ argumentum latitudinis respectu aequatoris mobilis designat.