TRANSFINITE NUMBERS AND THE PRINCIPLES OF MATHEMATICS.

PART I.

One result of Georg Cantor's discovery of the transfinite cardinal and ordinal numbers has been the development of more satisfactory views on the principles of mathematics. To this end, also, the symbolic logic of Peano, Frege, and Russell[1] contributed by enabling one, for the first time, to reach precision in such subjects as the relation of logic to mathematics, and the meaning of "definition" and "existence."

In this first part, I give an account of these things, and, in the second part, I will review the modifications in logic and in our views of the principles of mathematics which progress in the theory of aggregates has necessitated. I hope to show that, just as we have been forced, especially during the nineteenth century, to a more rigorous foundation of the methods and results of mathematical analysis, so we are forced to logical investigations by that development of mathematics to which I have just referred.

In this article I wish to emphasize an aspect in the development of views on the principles of mathematics other than that of the gradual *rapprochement* of mathematics and logic and their final reconciliation owing to the good offices of the logic of relations as promulgated by De Morgan, C. S. Peirce, Schröder, Dedekind, Frege, Peano, and Russell. I wish to point out the service which the theory of transfinite numbers has done first, in drawing atten-

[1] This symbolic logic is a great advance on the older symbolic logic, of which Schröder has given an excellent account (*Vorlesungen über die Algebra der Logik,* 3 volumes, Leipsic, 1890 and subsequent years; part of the third volume is not yet published).

tion to what are known as "the contradictions of the theory of aggregates," and hence to the necessity for a remoulding of logic; and, secondly, in clearly separating cardinal and ordinal numbers in our minds, and so making us, by analogy, more precise in our distinctions of signless integers and positive integers, of the integer n and the ratio $n:1$, and so on. These distinctions have made us give up the "principle of permanence," which formerly played such a great part in mathematics, for we are compelled to admit that it consists in identifying things whose difference is clearly discernible.

Thus, the advance of mathematics has brought it nearer and nearer to logic; the extent of the validity of mathematical conceptions and methods has been examined ever more closely; and it is not difficult to see that, by this, we have attained to a more thorough knowledge, and even, by the capacity which we have gained of avoiding those pseudo-problems to which methods extended beyond their domain of validity give rise, to a *practical* advance.

I.

Cantor was led to see the necessity for introducing certain definitely infinite numbers by his mathematical researches on infinite aggregates of points situated on a finite line (using a geometrical terminology for conceptions which are, in reality, purely arithmetical); but, logically, the theory is independent of this origin, and here[2] I will give the independent grounds on which, in the *Grundlagen,* Cantor made the introduction of these numbers rest.

Among the finite integers $1, 2, \ldots, \nu, \ldots$ there is no greatest, but, although it would be contradictory to speak of a greatest finite integer μ (for there is always a greater one $\mu+1$), there is no contradiction[3] involved in introducing a *new,* non-finite number (ω), which is defined as the *first* number that follows *all* the numbers $1, 2, \ldots, \nu, \ldots$ (in their order of magnitude). The

[2] A very full historical account by me appeared in the *Archiv der Math. und Phys.* for 1906 and 1909, and the rest will appear shortly.

[3] This is the point which will be found to require for its adequate discussion, all the resources of logic (see below).

interest that attaches to the introduction of a series of such "transfinite" numbers, the first ones of which Cantor has denoted:

$$\omega,\ \omega+1,\ \omega+2,\ \ldots,\ \omega+\nu,\ \ldots\ \omega.2,\ \omega.2+1,\ \ldots\ \omega.\nu,\ \ldots\omega^2,\ldots$$
$$\omega^\nu,\ \ldots\omega^\omega,\ \ldots\omega^{\omega^\omega},\ \ldots,\ \ldots,\ \text{(N)}$$

ν being any finite integer, is, of course, to be seen from the history of those mathematical questions which necessitated the introduction of these numbers;[4] but here we are only concerned with the question whether the conception of such numbers is logically possible, that is to say, leads to no contradiction.[5] That Cantor, to most intents and purposes, showed this by his above introduction and subsequent definition of ω, is true, and, further, he successfully classified and answered the objections made by philosophers and mathematicians, from the time of Aristotle, against the actual (or *completed,* as distinguished from the "potential" or "becoming") infinite.[6] A characteristic and illuminating example of this criticism was given *à propos* of Dühring's arguments against the actual infinite (*Eigentlich-Unendlich*).[7] These arguments can, said Cantor, be reduced, either to the statement that a definite finite number, however large, can never be infinite (a statement which is a truism) or that a variable unlimitedly great finite number can not be thought of with the predicate of definiteness, and hence also not with the predicate of being (which again immediately results from the essence of variability). To conclude, as Dühring does, the non-thinkability of definitely infinite numbers is like arguing that, because there are innumerable intensities of green, there can be no red.[8]

[4] The use of transfinite numbers in important questions of mathematics has been shown, for example, by G. H. Hardy (*Proc. Lond. Math. Soc.* (2), vol. I, 1904, pp. 285-290) and myself (*Mess. of Math.*, April, 1904, pp. 166-171, and Crelle's *Journ. für Math.*, Bd. CXXVIII, 1905, pp. 169-210).

[5] Cantor (*Grundlagen einer allgemeinen Mannichfaltigkeitslehre,* Leipsic, 1883, pp. 18-20), maintained the thesis that the formation of concepts in mathematics is completely *free,* and has only to satisfy the condition of the logical consistency of these concepts with one another. Such concepts then have "existence" (in mathematics). Cf. below on the question whether "freedom from contradiction" is necessary or sufficient for the "existence" of a concept.

[6] *Grundlagen,* pp. 9-18, 43-46; *Zur Lehre vom Transfiniten,* Halle a. S., 1890 (reprint of Cantor's articles in the *Zeitschr. f. Phil. u. philos. Kritik,* Bde. LXXXVIII, XCI, and XCII, 1885-1887).

[7] See *Grundlagen,* pp. 44-45.

[8] The arguments against the infinite in mathematics have also been discussed exhaustively by Couturat (*De l'infini mathématique,* Paris, 1896, pp.

The logically exact investigation as to the existence of numbers defined by an infinite process (as ω is by the finite numbers, or an irrational number by the rationals) was begun by Russell, and I return to the question in the next section.

The series of the transfinite numbers was, now, shown by Cantor to fall into certain divisions, which he called "number-classes"; which are characterized by the property that, if α and β are any numbers of the same class, all the numbers (from 1 on) preceding α can be brought (in a different order, of course) into a correspondence,[9] which is one-one, with all those preceding β; and inversely Cantor expressed this by saying that the first class of numbers had the same "power" as the second, or that one, and only one, "power" belonged to each "number-class."

Thus, in addition to the series of finite and transfinite (ordinal) numbers, there is a series of finite and transfinite powers; for finite aggregates the conceptions of power and (ordinal) number appear to coincide,[10] and such an aggregate has always the same number, however it may be arranged; but a given infinite aggregate, though no re-arrangement can alter its power, since this attribute is, by the definition, independent of order, can have various (ordinal) numbers,—in fact, any number of a certain class,—according to the way in which it is arranged.

But, even when an aggregate is "simply ordered" (that is to say, when an "order" is given to the terms of an aggregate such that, if a and b be any two terms, a either *precedes* or *follows* b in virtue of some relation, not necessarily in order of space or time), it need not have an ordinal number. In fact, Cantor's ordinal numbers only apply to certain kinds of ordered aggregates, which he called "well-ordered," and which are characterized by the property that any selection of terms has, in the order of the original series, an element of lowest rank. Thus, the series

441-503) and by Russell (*The Principles of Mathematics,* vol. I, Cambridge, 1903, pp. 355-362).

[9] See below.

[10] However, strictly speaking they do not *coincide.* The point is the same as the one about signless integers (classes) and positive integers (relations) referred to below.

$$a_1, a_2, \ldots, a_\nu, \ldots; b_1, b_2$$

is well-ordered, but not the series

$$b_1, b_2; \ldots, a_\nu, \ldots, a_2, a_1,$$

where ν is any finite number and the dots indicate that *all* the a_ν's, where ν is finite, occur in the order shown. Accordingly, Cantor generalized and renamed his fundamental concepts in the theory of transfinite numbers as follows:[11]

"By an 'aggregate' or 'manifold' (*Menge*), we understand any collection by the mind (*Zusammenfassung*) M of definite well-distinguished objects *m* of our intuition or of our thought (which are called the 'elements' of M) to a whole.

"Every aggregate M has a definite 'power,' which we also call 'cardinal number.'

"We call 'power' (*Mächtigkeit*) or 'cardinal number' of M the general concept which, by means of our active faculty of thought, is obtained from the aggregate M by abstracting from the nature (*Beschaffenheit*) of its different elements *m* and from the order in which they are given."

Cantor proved that, in order that two aggregates, M and N, should have the same cardinal number, it is necessary and sufficient that they should be "equivalent"[12] (*äquivalent*), that is to say, that there should be a one-one correspondence between the elements *m* and the elements *n*. The operations of addition, multiplication, and exponentiation for cardinal numbers were then defined,[13] and certain other questions of mathematical importance investigated, including a short treatment of the *finite* cardinal numbers[14] and the smallest transfinite cardinal number ($\aleph_0$).[15] But also, what concerns us intimately at present, Cantor also mentioned a series of

[11] *Math. Ann.*, Bd. XLVI (1895), pp. 481-512; Bd. XLIX (1897), pp. 207-246.

[12] Russell has used the word "similar" instead of "equivalent" and "like" instead of "similar" (Cantor's *ähnlich*, see below); while Dedekind used *ähnlich* where Cantor used *äquivalent*. At Dr. Carus's suggestion, we follow Cantor's terminology here.

[13] *Math. Ann.*, XLVI, pp. 485-488.

[14] *Ibid.*, pp. 489-492.

[15] *Ibid.*, pp. 492-495.

cardinal numbers ascending in magnitude and such that there is no cardinal number between two consecutive terms of the series:

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \cdots,\ \aleph_\nu,\ \cdots\aleph_\omega,\ \aleph_{\omega+1},\ \cdots,\ \aleph_\gamma,\ \cdots;\ \text{(A)}$$

as a subject for future investigation,[16] and implied that every transfinite cardinal number is to be found in this series. The conception ot an "ordered aggregate" was then introduced:[17]

"We call an aggregate M 'simply ordered,' if a definite 'order of precedence' (*Rangordnung*) rules its elements m, so that of any two elements m_1 and m_2 one takes the 'lower' and the other the 'higher' rank; and so that if of three elements m_1, m_2, and m_3, m_1 is lower than m_2 and m_2 lower than m_3, m_1 is always lower than m_3." Such orders are order of magnitude and order of succession in time. Evidently, we are presupposed to have the idea of such a relation in general and it is not defined.

"Every (simply-) ordered aggregate has a definite *'ordinal type,'* by which we understand the general concept which results from M, when we abstract from the nature of the elements m, but retain the order of precedence among them." That two ordered aggregates should have the same type, it is necessary and sufficient they should be "similar" (*ähnlich*); that is to say, that there should be a one-one correspondence such that the order of precedence of corresponding elements is kept.

An important case of a simply-ordered aggregate is a "well-ordered aggregate,[18] which has been characterized above. The types of well-ordered aggregates were, now, called "ordinal numbers," and thus we arrive at the series (N). Now, the cardinal numbers of the various "segments"[19] of this series (N) form the series (A), which is such that there is no cardinal number which lies, in magnitude, between two consecutive Alephs, and none less than any one (for example consider $\aleph_\omega$) which is not one of the Alephs preceding

[16] *Ibid.*, pp. 495, 484.

[17] *Ibid.*, pp. 496-498.

[18] See the article in *Math. Ann.*, Bd. XLIX.

[19] The "segment" defined by the term a of a well-ordered series is the series of all terms preceding a. Cantor used the word *Abschnitt* (*Math. Ann.*, Bd. XLIX, p. 210).

that one in (A). Further, (A) possesses the remarkable property of being similar to (N). The other investigations of Cantor on ordinal numbers are of more exclusively mathematical interest.

In the question as to the existence of the various cardinal numbers and ordinal types defined by Cantor, there was still an opportunity left for skepticism, and one of the chief objects of Russell's work[20] was so to define the numbers as to leave no doubt about their existence. We must, then, next give an account of that part of modern work on symbolic logic which is necessary for the comprehension of this object.

II.

Peano's logical calculus differs from the previous systems of algebra of logic[21] in one or both of the respects of being more convenient in symbolism and of containing more subtle distinctions between certain fundamental ideas. Thus, in the latter respect, Peano had the distinction, which was not possessed by Schröder[22]

[20] Cf. *op. cit.*, pp. ix, 111-116, 277-286, 313, 321-322, 497-498.

[21] Although Leibniz had worked out projects of an algebra of logic and a general symbolism, his work in this direction only began to be known when his manuscripts began to be published by. J. E. Erdmann in 1840. The work in this direction of Leibniz's successors—of whom the greatest was J. H. Lambert—made little impression, and it was George Boole and Augustus De Morgan, about the middle of the nineteenth century, who must be regarded as the true founders of what we now know as symbolic logic. A valuable work of an orthodox Boolian character, containing much careful historical research, is J. Venn's *Symbolic Logic*, London, 1880 (2d. ed., 1894); and the most complete works on the logic of Leibniz are: B. Russell, *A Critical Exposition of the Philosophy of Leibniz*, Cambridge, 1900; and L. Couturat, *La logique de Leibniz d'après des documents inédits*, Paris, 1901, and *Opuscules et fragments inédits de Leibniz*, Paris, 1903.

It must be mentioned that the introduction of "propositions containing variables" and of implication between them was first explicitly made by H. MacColl in 1878. Still MacColl did not observe, like Frege and Peano, that these notions made it possible to formulate all mathematical deductions in symbols—what was impossible with the traditional or Aristotelian logic—and indeed, as Russell has shown, rather confused the essential difference between these *propositional functions* and *propositions* proper. The logic of relations of De Morgan, Peirce, Schröder, Frege, Dedekind, Peano and Russell will be referred to afterwards.

[22] Cf. Schröder, *op. cit.*, Bd. II, 2. Abteilung, Leipsic, 1905, pp. 461, 597; *Verh. d. Math. Congr. in Zürich*, Leipsic, 1898, p. 154; G. Frege, *Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik, Archiv für systemat. Phil.*, I, 1895, pp. 433-456; *Grundgesetze der Arith.*, I, Jena, 1893, p. 2; Russell, *Principles*, pp. 19, 78; Couturat, *Les Principes des mathématiques*, Paris, 1905, pp. 22-21 (a German translation of this book by C. Siegel was published at Leipsic in 1908 under the title: *Die philosophischen Prinzipien der Mathematik*).

Frege's work, which began in 1879, is of a far more subtle character than or any other previous writer, between "the class (or individual) *a is a member* of the class *b*" and "the class *a is contained in* the class *b*"; the former was symbolized by Peano $a\epsilon b$, the latter by a different sign;[23] and the latter may be defined: $x\epsilon a$ implies, for every such x, $x\epsilon b$.

Again, Peano distinguished between a term (say x) of a class and the class (ιx) composed of that single term, treated the conception of the "variable" at some length, and so on.

While referring for more detailed accounts of Peano's system of writing all mathematical propositions in logical symbolism, which implies a calculus of logic, to other works,[24] we shall here notice more particularly some points in it and in Russell's work of great importance to us in our present subject.

* * *

When the propositions a, b, said Peano,[25] contain undetermined entities $x, y, \ldots,$[26] as they do in general, then the suffix $x, y, \ldots$ attached to the sign of implication between a and b makes the whole read: "a implies b, whatever $x, y, \ldots$ may be" (provided, of course, they satisfy the conditions that may have been imposed on them at the beginning), and if a and b contain two groups of undeter-

Peano's (cf. Russell, *op. cit.*, pp. 500-522), and consequently far more suited to the investigation of the principles of mathematics—for which purpose, indeed, his ideography was invented. His symbolism, however, is so cumbrous, that Russell, who, independently of Frege, arrived at many of Frege's points of view, combined Frege's ideas with Peano's symbolism (slightly modified) in his most recent work (*Amer. Journ. of Math.*, XXVIII, 1906; and XXX, 1908).

[23] Since Peano wrote *b*C*a* for "*b* contains *a*," for "*a* is contained in *b*" he used a sign which is a deformation of an inverted C.

[24] See pp. 370-378 (on the symbolism of Peano and Russell) in Whitehead's Memoir *On Cardinal Numbers* (*Amer. Journ. of Math.*, Vol. XXIV, 1902, pp. 367-394); the references to the calculus of logic in the works of Russell and Couturat, and to Peano's logic in Russell, *op. cit.*, pp. 26-32, and Couturat, *op. cit.*, pp. 5, 6, 18, 24, 27; Peano's various *Formulaires* and the volumes of his *Rivista di matematica*; and Couturat's account of the work of Peano and his school in the *Bull. des sci. math.*, 2e série, t. XXV, 1901.

[25] *Arithmetices Principia nova methodo exposita*, Turin, 1889, p. viii; *Notations de logique mathématique*, Turin, 1894, pp. 16-18, 20-22.

On the subject of the variable, propositional functions, formal implication, individual and class, see Couturat, *op. cit.*, pp. 17, 21-23.

[26] We may restrict $x, y, \ldots$ to be real or imaginary numbers, points, classes, propositions,.... For example "x and y are (real or complex) numbers" implies (whatever numbers x and y may be) $(x+y)^2 = x^2 + 2xy + y^2$.

mined entities $x, y, \ldots$ and $u, v, \ldots$, and we wish to say that $u, v, \ldots$ are such that, whatever $x, y, \ldots$ may be, a implies b, then we write as suffixes only the entities $(x, y, \ldots)$ with respect to which we make the deduction. The resulting proposition is then a condition between $u, v, \ldots$, and is independent of $x, y, \ldots$.[27]

If the value of a formula does not depend on the undetermined entity in it, just as the value of a definite integral does not depend on the variable (x) of integration, it is not necessary to explain the signification of x, as was done above.

If p is a proposition containing a variable x, we denote the class of x's which satisfy p_x by $x \ni p_x$ and read it: "the x's *such that* p_x is true." If p_x contains other variables $u, v, \ldots$ besides x, $x \ni p_x$ denotes a class which is a function of $u, v, \ldots$, but independent of x.

III.

All the propositions of pure mathematics are, according to Russell,[28] of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. Logical constants are all notions definable[29] in terms of the following: Implication, the relation of a term to a class of which it is a member (ϵ), the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely, the notion of truth. A proposition[30] is de-

[27] Thus, if x is a real number $(x \epsilon q)$, and we write $x \epsilon q$ implies, for any such v, $ax^2+bx+c=0$, the proposition is "a, b, c are such that, whatever x is, $ax^2+bx+c=0$." The implication *without* any index (which is equivalent to that with *all* the indices x, a, b, c) states the false proposition: "whatever numbers x, a, b, c are, $ax^2+bx+c=0$."

[28] *Op. cit.*, p. 3; cf. Couturat's book quoted—which may be described as a more popular exposition of Russell's work—pp. 1-6.

[29] For the meaning of this term, see below.

[30] The calculus of propositions (Russell, *op. cit.*, pp. 13-18, Couturat, *op. cit.*, pp. 8-16) must precede those of classes (Russell, *op. cit.*, pp. 18-23; Couturat, *op. cit.*, pp. 16-26) and of relations (Russell, *op. cit.*, pp. 23-26; Couturat,

finable as "that which implies itself,"[31] and must be distinguished clearly from what Peano (and Russell in the above statement) called "a proposition containing a variable," and Russell, in far preferable language, a "propositional function."[32] A proposition, we may say, is anything that is true or that is false. An expression such as "x is a man" is, therefore, not a proposition; but if we give to x any constant value whatever,[33] the expression becomes a proposition. This schematic form standing for any one of a whole class of propositions is called a "propositional function," and we may explain, but not define, this notion as follows: ϕx is a propositional function if, for every value of x, ϕx is a proposition, determinate when x is given. In this, x is called the *variable,* and we may say that a propositional function is, in general, true for some values of the variable and false for others.

When we say "x is a man implies x is mortal for all values of x," we have a genuine proposition, in which, though the letter x appears, it is absorbed in the same kind of way as the x under the integral sign in a definite integral, so that the result is no longer a function of x. In this case, x is what Peano called an "apparent variable," since the proposition does not depend upon the variable; whereas the variable was called "real" in propositional functions. Genuine propositions do not depend upon a variable or variables.[34]

op. cit., pp. 27-34), since the principles of the calculus of propositions are used in all reasoning.

On the calculus of classes, cf. the note on the theory of aggregates in Couturat, *op. cit.*, pp. 219-228.

The logic of relations, the mathematical importance of which was shown by Dedekind's application of it (Dedekind himself rediscovered much of it independently) and by Schröder's work, was, as Schröder rightly observed (*Verh. des ersten Math.-Congr. in Zürich, 1897,* Leipsic, 1898) somewhat neglected by Peano to the disadvantage of his logic. It was Russell (*Rev. de Math.,* VII) who first completed Peano's logic by a logic of relations in which the Peirce-Schröder ideas were modified so as to fit in with a logic which comprised more subtle distinctions than that of Schröder. Cf. Couturat, *op. cit.,* pp. 27-28.

Cf also Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens,* Halle a. S., 1879 (Frege's work on ideography), pp. 15-24; *Funktion und Begriff,* Jena, 1891; *Grundgesetze,* I, 1893, pp. 5-25.

[31] Russell, *op. cit.*, p. 15.

[32] *Op. cit.*, pp. 12-13, 19-20; cf. Couturat, *op. cit.*, pp. 17-18.

[33] Such as "Socrates," "Plato," "the number 2."

[34] On the notion of the "variables" (the presence of which is marked by

In pure mathematics we assert what Russell[35] called the *formal implication*[36]: "$\phi(x, y, \ldots)$ implies $\psi(x, y, \ldots)$, whatever values $x, y, \ldots$ may have"; but we do not assert either ϕ or ψ of the entities $x, y, \ldots$; whilst, in applied mathematics, results which have been shown by pure mathematics to follow from some hypothesis are actually asserted of some constant satisfying the hypothesis in question. Thus terms which were variables become constant, and a new premise is always required, namely: this particular entity satisfies the hypothesis in question.[37]

The values of x "such that" ϕx is true form a class, and Russell[38] defined a class as all the terms satisfying some propositional function. That some limitation was required in this statement was recognized by Russell himself, in consequence of the contradiction discovered by him;[39] and this limitation, which forms indeed the kernel of our investigations, will be discussed at length hereafter.

IV.

The treatment of the meanings which can be attached to the word "definition" by Peano[40] and Burali-Forti prepared the way for a thoroughly satisfactory theory.[41]

The simplest form of a definition is, in Peano's symbolism,

$$x = a \qquad \text{D}f.,$$

where x is a sign which has not, as yet, a signification, a is a group of signs having a known signification, and the sign of equality followed by "Df" — note that "$=$ Df" is one sign — indicates that we agree, for the sake of brevity, because the group

the occurrence of the words *any* or *some*, and may take any values) and "constants" in logic, see Russell, *op. cit.*, pp. 5-8, 89-94; Couturat, *op. cit.*, pp. 21-24; and Frege, "Was ist eine Funktion?" *Boltzmann-Festschrift*, 1904, pp. 656-666.

[35] *Op. cit.*, p. 5.

[36] As distinguished from the *material implication* (*op. cit.*, p. 14) between genuine propositions.

[37] Russell, *op. cit*, p. 8; cf. Couturat, *op. cit.*, pp. 4-5.

[38] *Op. cit.*, p. 20.

[39] *Op. cit.*, pp. 366-368, 101-107.

[40] *Notations*, pp. 44-51; *Les définitions mathématiques* (Bibl. du Congrès Internat. de Phil., III, "Logique et histoire des sciences," Paris, 1901, pp. 279-288).

[41] On definitions see also Frege, *Grundgesetze*, I, 43-52; II (1902), 69-80.

a denotes an important concept, to write the simple sign x instead of the group a.[42] Sometimes what we define is not a simple sign, but a group of signs, between which there are new signs, or a group of signs which have a signification separately, but such that their aggregate has not yet a signification. Then the definition follows an hypothesis (h) and has the form:

$$h \text{ implies that } x = a \qquad \text{D}f.^{43}$$

There are ideas, which we obtain by abstraction, which cannot be defined under the above form.[44] Let u be an object; by abstraction we deduce a new object ϕu; we cannot form an equality:

$$\phi u = \text{known expression} \qquad \text{D}f.,$$

for ϕu is an object of nature different from all those considered hitherto. Then we define *equality*[45] as follows:

$$h_{u,v} \text{ implies}: \phi u = \phi v . = . p_{u,v} \qquad \text{D}f.,^{46}$$

where $h_{u,v}$ is the hypothesis on the objects u and v, and $\phi u = \phi v$ is the equality which we define as meaning the same thing as $p_{u,v}$, a relation, with a known meaning, between u and v, which must satisfy the three conditions of being:

[42] Thus, Russell said (*op. cit.*, p. 429): "What distinguishes other branches of mathematics from logic is merely complication, which usually takes the form of an hypothesis that the variable belongs to some rather complicated class. Such a class will usually be denoted by a single symbol; and that the statement of the class in question is to be represented by such and such a symbol is what mathematicians call a *definition*. That is to say, a definition is no part of mathematics at all, and does not make any statement concerning the entities dealt with by mathematics, but is simply and solely a statement of a symbolic abbreviation: it is a proposition concerning symbols, not concerning what is symbolized." As regards the *philosophical* meaning of "definition," see *op. cit.*, pp. 15, 27, 111-112. Also (*op. cit.*, p. 15): "In the mathematical sense, a new propositional function is said to be defined when it is stated to be equivalent to (i. e., to imply and be implied by) a propositional function which has either been accepted as indefinable or has been defined in terms of indefinables." Cf. Couturat, *op. cit.*, pp. 10, 36-37.

[43] For example, in the definition of e^x as a power-series, the hypothesis is that x is a (real or complex) number.

[44] Peano, *Notations*, pp. 45-49.

[45] See the fifth section.

[46] Peano used dots (., :, .:., ::) to separate the parts of a proposition, and the main implication of a proposition is always that immediately preceded or followed by the greatest collection (in one place) of dots. Further, dots between propositions are a sign of joint assertion or "logical multiplication" (p and q). Thus, in the proposition (if the letters denote propositions): p implies $q . x$: implies $. r$ or s .:. implies $. t$, the part p to s is the hypothesis (analyzable into an hypothesis "p implies q, and the proposition x is asserted" and the protasis "r or s",—a "logical addition") and t is the protasis (cf. *Notations*, pp. 11-13).

1. *Reflexive*; that is to say, $\phi u = \phi u$ or $p_{u,u}$ is true, whatever u is;

2. *Symmetrical*; that is to say, $\phi u = \phi v$ implies $\phi v = \phi u$, or $p_{u,v}$ implies $p_{v,u}$;

3. *Transitive*; that is to say, $\phi u = \phi v$ and $\phi v = \phi w$ imply that $\phi u = \phi w$, or $p_{u,v}$ and $p_{v\,w}$ imply $p_{u,w}$.[47]

Among his examples, Peano gave[48] Stolz's[49] definition of a rational number. If a and b are natural numbers and b is not a multiple of a, the expression $\frac{b}{a}$ has no meaning; but we make to correspond to the couple a, b, a new object, different from all those we have considered hitherto, which we will denote by $\frac{b}{a}$, and which we define by the relation of equality, which satisfies our three conditions.

$$\frac{b}{a} = \frac{d}{c} . = . \quad ad = bc \qquad \text{D}f.^{50}$$

Again, the "upper limit of a class of rational numbers a" ($l'a$) was defined[51] by abstraction (b being also a class of rational numbers):

$l'a = l'b . =$. "If x is any rational number; then, if there are any members of a greater than x, there are members of b greater than x, and *vice versa*." D*f*.

Definitions are not, strictly speaking, necessary. Thus, each proposition on irrational numbers (the foregoing "upper limits") is a proposition on aggregates of rational numbers; each proposition on rational numbers becomes a proposition on whole numbers; and so on. A definition has no need of proof, it is merely the effect of our will to represent a group of signs by a simpler expression. We have not, for example, to prove the existence of what we define. Naturally, it is proper to define existent things in practice, but

[47] Cf. papers by Vailati and De Amicis in *Riv. di Mat.*, I, 1891.

[48] *Op. cit.*, p. 47.

[49] *Vorlesungen über allgemeine Arithmetik*, Bd. I, p. 43.

[50] In this line, the sign = has, with Peano, three different meanings: the first, equality as defined by abstraction; the second, equality by D*f*; and the third, equality between whole numbers.

[51] Peano, *op. cit.*, p. 47; cf. *Arith. Princ.*, p. 15, and *Formulario de Mathematica*, 1905, p. 105.

sometimes we define things which do not exist. Thus Euclid,[52] in order to prove that the number of primes is infinite, said: Let us put $\delta\epsilon$ = smallest common multiple of the primes; and then proved that $\delta\epsilon$ does not exist.

We cannot define everything; to define a sign *x*, we must be able to compose a sign *a* from known signs such that we have $x{=}a$ D*f*. Thus, we must know some signs already. The question, Can the object *x* be defined? is not quite correctly put; we should rather say: Can *x* be defined by means of the objects *a, b,* . . . ?, and there is a certain amount of arbitrariness[53] as to which objects we take as ultimate,—the minimum of objects with which we can begin a logic, or "primitive ideas." These ideas, said Peano,[54] "must be acquired by experience or by induction; it is impossible to explain them by deduction." The primitive ideas of a science constitute the smallest dictionary which must be common to two men who speak different languages, in order that they may be able to understand one another on the subjects of this science.[55]

This determination of a primitive idea by a group of primitive propositions or postulates concerning it, was admitted under the name of "definitions by postulates" by Burali-Forti[56] as one of the three legitimate forms of definition in mathematics. To "define" an object *x* means: "to give one or many logical relations containing *x*, and such that, an element *y* being given, it is possible to affirm or deny the relation $x{=}y$."[57] In other words, *x* is defined

[52] Book IX, prop. 20.

[53] Thus, if by means of *a, b, c,* we can define *d,* and by means of *a, b, d,* we can define *c,* we can take for primitive ideas either *a, b, c,* or *a, b, d.* See also Russell, *op. cit.*, p. 111.

[54] *Op. cit.*, p. 50.

[55] Russell called the primitive ideas "indefinables" and enumerated them, and "logical constants" was the name he gave to all notions definable in terms of them (*op. cit.*, pp. 3, 4, 7-8, 11; Couturat, *op. cit.*, pp. 37-39).

[56] *Sur les différentes méthodes logiques pour la définitions du nombre réel,* (*Bibl. du Congrès Internat. de Phil.*, "Logique et histoire des sciences," III, pp. 294-307, especially pp. 294-296).

[57] This means (Burali-Forti, *loc. cit.*, pp. 292-293), that *every* property of *x* is also one of *y*. Certain relations which are reflexive, symmetrical, and transitive (like "is superposable on") have been denoted by =, but this was only with reference to all those properties *relative to our discourse*. Cf. also a preceding note on mathematical equality.

when one can ***deduce*** all the properties of x from the logical relations in question. The two other kinds of definition are (1) the "nominal definition" of x in the form $x=a$, which has already been described, and (2) the "definition by abstraction" of an operation ϕ by saying to what class a it is applicable and, x being an element of a, by establishing which are the y's of a such that $\phi y=\phi x$. This has also been described above.

* * *

Russell[58] urged against the validity of the above process of Peano's of using abstraction as a substitute for definition,[59] the fatal formal defect of not showing that only one object satisfies the definition. Thus, in the definition by abstraction of "powers" or "cardinal numbers," we consider two classes u and v which can be put in a one-one relation[60] with one another, or are equivalent. As equivalence is a reflexive, symmetrical, and transitive relation, Peano and common sense conclude that u and v have a common property, and *vice versa*; this common property we can then define as their cardinal number, so that the equality of the cardinal numbers of u and v consists in the equivalence of u and v. Instead of obtaining *one* common property[61] of similar classes, which is *the* cardinal number of the classes in question, we obtain a *class* of such properties, with no means of deciding how many terms this class contains.[62] In order to make this point clear, let us examine what is

[58] *Op. cit.*, pp. 114-115.

[59] This process of analyzing any reflexive, symmetrical, and transitive relation between the classes u and v into sameness of relation to an entity denoted by ϕu or ϕv to be obtained by abstraction, was called "definition by abstraction" by Burali-Forti in his *Logica Matematica*, published at Milan in 1894.

[60] A relation is one-one when, if x and x' have the relation in question to y, then x and x' are identical; while if x has the relation in question to y and y', then y and y' are identical. A one-one relation whose domains are u and v was denoted by Peano by $fv\phi$ placed partly between and partly after u and v (*Formulario*, 1905, p. 75).

The term "one-one" does not imply that the (as yet undefined) notion of "the number 1" is used in this definition, and such is not the case (cf. Russell, *op. cit.*, pp. 113, 305, and Couturat, *op. cit.*, pp. 31-32, 47-48).

[61] Cf. Cantor's definition by abstraction (1883) in *Zur Lehre vom Transfiniten*, pp. 23-24.

[62] Couturat (*op. cit.*, p. 48) pointed out that this class may, seemingly, be null; "a definition by abstraction shows neither the existence nor the uniqueness of the object defined."

meant in the present instance, by a common property. What is meant is, that any class has to a certain entity, its number, a relation which it has to nothing else, but which all equivalent classes (and no other entities) have to the said number. That is, there is many-one relation wihch every class has to its number and to nothing else. Thus, so far as the definition by abstraction can show, any set of entities to each of which some class has a certain many-one relation, and to one and only one of which any given class has this relation, and which are such that all classes equivalent to a given class have this relation to one and the same entity of the set, appear as the set of numbers, and any entity of this set is *the* number of some class. If then, there are many such sets of entities—and it is easy to prove that there are an infinite number of them—every class will have many numbers, and the definition wholly fails to define *the* number of a class. This argument is perfectly general, and shows that definition by abstraction is never a logically valid process.

The legitimacy of this process of Peano's requires[63] an axiom, namely that, if there is any instance of the relation in question—a transitive, symmetrical and (within its field) reflexive one between u and v—there is such a new entity as ϕu or ϕv such that our relation is analyzed into sameness of relation to the new term ϕu or ϕv. As the entity to be defined should be visible, at least to the mind's eye,[64] this axiom becomes, in the logic of relations, a proposition proved by Russell in his calculus of relations, and called by him "the principle of abstraction."[65] This principle is: "Any symmetrical and transitive relation R, of which there is at least one instance, can be expressed as the relative product of a many-one relation S and its converse, so that S subsists between each of the individuals x, y and a third term z in such a way[66] that xRy is equiv-

[63] Russell, *op. cit.*, p. 220

[64] *Ibid.*, p. 249.

[65] Russell, *op. cit.*, pp. 166-167, 116, 305; Couturat, *op. cit.*, pp. 33, 42-43, 48-50; and Russell's paper: "Sur la logique des relations," *Rev. de Math.*, VII, No. 2, § 1, Prop. 6, 2.

[66] An axiom virtually identical with this principle, but not stated with the necessary precision, or not demonstrated, is, according to Russell (*op. cit.*, p.

alent to the two propositions: xSz and ySz. It is this z which is Peano's ϕx or ϕy, and is the common property of x and y;[67] and all mathematical purposes of the supposed common property are completely served when it is replaced by this z. Russell actually constructed such a z by pointing out that the requirements were satisfied by the class of terms having the given relation to a given term.

Thus, if we apply the principle of abstraction to equivalent classes, we arrive[68] at a definition of the cardinal number of u as the class of the classes similar to u.[69]

* * *

The "definition by postulates" also is not a definition.[70] An aggregate of postulates only determines the meaning of the undefined symbols to a certain extent, for the same system of postulates can be verified by many interpretations given to the undefined symbols: a system of postulates is analogous to a system of equations between many unknowns; if our postulates really determine our undefined notions uniquely, a "resolution" with respect to these unknowns results in nominal or explicit definitions. When the system of postulates contains only one primitive idea, it is easy to extract the explicit definition of the latter, for we need only say that it is "such that" it verifies the system of postulates. But it

219n), to be found in a paper by De Morgan, *Camb. Phil. Trans*,. vol. X, p. 345.

[67] The principle, then asserts "that there *are* such entities, if only we know where to look for them" (Russell, *op. cit.*, p. 249).

[68] Russell, *op. cit.*, pp. 115, 304-307.

[69] It then becomes a strictly demonstrable proposition that any class u has a cardinal number. For u itself is a member of the class called the cardinal number of u, since u is similar to itself (equivalence is a reflexive relation) and hence the cardinal is not a null class (Russell, *op. cit.*, p. 305; first given in *Rev. de Math.*, VII, p. 121).
Cf. Frege, *Die Grundlagen der Arithmetik....*, Breslau, 1884, pp. 73-99.
Analogously, a nominal definition of ordinal types as a class of like relations was given by Russell (*op. cit.*, pp. 241, 313; cf. Couturat, *op. cit.*, pp. 76-77).

[70] Couturat, *op. cit.*, pp. 40-42, 57-58; Frege, "Ueber die Grundlagen der Geometrie," *Jahresber. d. deutsch. Math.-Ver.*, Bd. XII, 1903, pp. 319-324; 368-375). Cf. also Russell's criticism of Peano's way of defining finite integers, together with his proofs of Peano's primitive propositions in arithmetic, in *op. cit.*, pp. 124-128.

remains to prove the existence and uniqueness of this notion, as for every other explicit definition.

Every definition is, then, nominal;[71] the "definition by abstraction" is only necessitated by an incomplete logic which does not include a calculus of relations, while the introduction of primitive ideas other than those of logic into arithmetic can, as Russell[72] has shown, be avoided.

* * *

Hilbert later applied the same axiomatic method to the principles of arithmetic, and exposed his results at some length in a lecture *Ueber die Grundlagen der Logik und der Arithmetik* (*Verh. des. 3. internat. Math.-Kongresses in Heidelberg im August, 1904*, pp. 174-185, Leipsic, 1905; translated in *The Monist*, July, 1905, Vol. XV, pp. 338-352). At the beginning of this, he announced complacently that "to-day in researches on the foundations of geometry we are essentially agreed as to the procedures to be adopted." If the procedure is the procedure of Hilbert, in which the essential factor of existence of the object supposed to be defined by the axioms is disregarded, and consequently in which one cannot be sure that one is arguing about anything at all, this is most certainly not the case; in America, for instance, there is the important work on geometry of O. Veblen, who gives chains of axioms for various geometries, but proves the existence-theorems.

Hilbert's view is that there is an essential difference between an examination of the foundations of geometry and one of the foundations of arithmetic, because, in the former case, the mutual compatibility of the axioms can be proved by *arithmetical* constructions, while in the latter case, this is naturally impossible, and "in the founding of arithmetic, the appeal to another basal science seems unallowable." But (cf. on this point M. Pieri, *Sur la compatibilité des axiomes de l'arithmétique, Revue de métaphys. et de morale*, March, 1906, XIV, pp. 196-207) the basal science for arithmetic can be, as Russell's whole work has shown, logic—including the

[71] Russell, *op. cit.*, p. 112; Couturat, *op. cit.*, p. 43.

[72] *Op. cit.*, pp. 8-9, 497-498.

logic of relations, and logic alone is sufficient for the definition of *all* the conceptions of pure mathematics.

What is Hilbert's difficulty in the founding of arithmetic on logic appears from his criticism of Frege—a criticism which applies also to the earlier work of Russell. Hilbert quite correctly observes that "inasmuch as he (Frege), true to his plan, takes....as axiom, that a concept (an aggregate) is defined and immediately available, provided only it be determined for every object, whether it falls under the concept or not, and also in doing this subjects the concept "every" to no restriction [cf. also Cantor, *Math. Ann.*, Bd. XX, 1882, pp. 114-115], he exposes himself to just those paradoxes of the theory of aggregates, which lie, for instance, in the concept of the aggregate of all aggregates [cf. below], and which, it seems to me, show that the conceptions and means of investigation of logic, taken in the usual sense, are not adequate to the rigorous requirements set up by the theory of aggregates." Hilbert's aim, from the very outset, was to avoid such contradictions.

However, though Hilbert develops the conception of the various finite and transfinite numbers in order, and, at each stage, restricts the word *all* to apply only to those entities already introduced, and, by this method, which he has not been the only one to adopt, never gets to Burali-Forti's contradiction; yet he does not seem to me to avoid Russell's contradiction, since "non-existent" means, with him, non-entity, and consequently his "class" of the existent is "the class" of all things.

We will not, in this short account, attempt a detailed criticism of Hilbert's lecture; and will merely remark that the creation by the mind of various "thought-things" governed by certain axioms is, even if such creation is possible, at least unnecessary, for another way, which Frege and Russell had previously followed, is preferable if for no other reason than that Occam's principle is observed (cf. below). As our present object is solely the discrediting of "definitions by postulates," we may merely refer, for other criticisms, to Couturat, *Rev. de métaphys. et de morale,* March, 1906, XIV, pp. 234-235; and Pieri, *ibid.*, p. 200.

When we define a class[73] we must, in order to be able to reason on this class and investigate its properties, prove that there is at least one member of this class; in other words, that the class is not null, or "exists,"[74] so that the conditions which define it are not logically incompatible. Every definition must, then, be accompanied by an *existence-theorem* (or postulate),[75] and, if we have to speak of "the" member of a class, we must prove that, if two individuals are members of the class in question, they are identical.

V.

We must now consider the notion of equality (=) in logic and in mathematics. In mathematics the process is frequently adopted of defining equality for, say, whole numbers,[76] and then *redefining* equality for other classes of numbers, such as ratios and real numbers. If, as is usually the case, the same sign (=) is used for these different conceptions of equality, there may be confusion; but, altogether apart from this question, which merely concerns the symbols used, there is a real question of principle involved, which makes this redefinition of equality objectionable: the new meanings of equality imply, in fact, a lack of thoroughness in the analysis of these meanings, which always involve the *identity* (the original meaning of equality in logic) of the "equal" objects *in some respect.*

The meaning of "equality" in logic is identity; when we say there $a=b$ we mean that a and b are different names for the same thing;[77] or, in formal language, *every* property of the thing denoted

[73] Couturat (*op. cit.*, p. 39) stated that the term defined is *always* a class; Russell (*op. cit.*, pp. 63, 497) did not go as far as this, and it may be remarked that some of the different kinds of "number" defined in analysis are relations.

[74] Russell, *op. cit.*, pp. 21, 32; Couturat, *op. cit.*, pp. 25-26.

[75] Cf. Russell, *op. cit.*, pp. ix, 322; Couturat, *op. cit.*, pp. 39-40. Russell (*op. cit.*, p. 497) sketched the chain of proofs that the numbers and other classes defined in mathematics exist.

[76] Thus we may define the members of two classes u and v to be "equal," when there can be set up a one-one correspondence between the members of u and those of v.

The sign = is to be distinguished from "= D*f*."

[77] See Dedekind, *Was sind and was sollen die Zahlen?* 1887 and 1893, pp. 1-2 (translation in Dedekind's *Essays on Number*, Chicago, pp. 44-45); Schröder, *op. cit.*, Bd. I, Leipsic, 1890, pp. 184-186; Peano, *Formulaire de mathématiques*, t. II, § 1, prop. 80; Burali-Forti, *Bibl. du Congrès Internat. de Phil.*, t. III, p. 292; Frege, *Grundgesetze der Arithmetik*, Bd. I, Jena, 1893, p. ix, and II, 1902, p. 71.

by a is also one of the things denoted by b.[78] Now the notion of equality, so often used in mathematics, in which not *every* property of a is one of b, but the relation connecting a and b is, like equality, reflexive, symmetrical, and transitive, is always one of the equality as defined above of certain *functions* of a and b. Thus, some geometricians have extended the meaning of equality and have called a (plane and rectilinear) figure a "equal" to a figure b when a is superposable (after dissection, if necessary) on b; this relation of superposability is reflexive, symmetrical, and transitive, and this relation which we may write aSb, can be put into the form $\phi(a)=\phi(b)$ by letting "$\phi(\ )$" stand for "the area of ( )." Similarly, the relation of parallelism (analogous, in many ways, to equality) between two straight lines a and b transforms into an *identity* between certain functions of = the directions of a and b.[79]

It is better to avoid introducing new conceptions unless they are really necessary, and new conceptions of equality are not necessary, and have the disadvantage, further, of rendering confusion possible. The decisive factor is, here as in the question as to whether numbers and other mathematical conceptions are to be defined logically or to be regarded as entities created by our minds, that *entia non sunt multiplicanda praeter necessitatem,* and hence that the problem of first principles is a minimal problem.

VI.

Most mathematicians would say that "existence" is absence of contradiction; whereas we have defined logical (or, what is the same thing, mathematical) existence as an attribute applying to a *class* a which is not null.[80] The proof that a class "exists" or is not

[78] Supposing that properties (propositional functions) determine *classes* in the manner already explained, this may be also put in the form: "$a=b$" means that *every* class which contains the object a also contains b. On Frege's theory that equality is not an identity of names, but expresses an identity of what he calls "denotation" (*Bedeutung*) together with a diversity of "signification" (*Sinn*), see his essay "Ueber Sinn und Bedeutung" in *Zeitsch. für Phil.*, C, 1892. We shall return to this point.

[79] Cf. Frege, *Grundlagen*, pp. 76-77; Couturat, *op. cit.*, p. 49, note; Burali-Forti, "Sur l'égalité, et sur l'introduction des éléments dérivés dans la science," *Enseignement math.*, I, 1899, pp. 246-261, and the above-mentioned *Congrès* paper (pp. 289-307).

[80] Cf. Russell, *Mind*, N. S., XIV, 1905, p. 398.

null, is always brought about by the actual construction or indication of an individual belonging to the class, and to inquire if an individual "exists" has no meaning.[81] This, now, is the point: mathematicians require a proof of the "existence" of an individual, logicians reply that "existence is a property of classes alone.[82] And the logicians' reply is obviously not satisfactory:[83] it leads one to suspect that there may be individuals, which may be used to prove the existence of classes to which they belong, and which are self-contradictory.

Let us examine a case in which mathematicians have proved what they would call the non-existence of an individual, namely, the self-contradictory nature of a complex number with more than two independent unities which satisfies all the formal laws of ordinary algebra. But we may also express the result of this as: the class of such complex numbers is null, or non-existent, and such a number is not an entity at all. Mathematicians, in fact, have used the word "exists" in two senses: (1) A class exists when we can find a member of it;[84] and (2) an individual does not exist when it is self-contradictory. Logicians use "exists" in the first sense only, for the second sense is merely: a class of such individuals does not "exist," in the first sense. The question is merely a verbal one, the mathematician's usage is confusing, the logician's is not.[85]

What is of great importance in this connection is that, as we shall see in the second part, while we may define the null-class as $x \ni \phi x$, where ϕx is false for every entity x (such as $\phi x =$ "x is not identical with x"), we may also have a non-entity, which may be

[81] Of course, a class (even a null, or non-existent *class*) can be considered as an individual with respect to a class of classes.

[82] Cf. Couturat, *Rev. de métaphys. et de morale,* March, 1906, pp. 232-234; Poincaré, *ibid.,* September, 1906.

[83] Couturat's comments are not always accurately expressed: "On ne démontre pas l'existence d'un individu comme tel. Les individus, par cela même qu'ils sont des individus, sont toujours considerés comme existants; ou plutôt la question ne se pose pas pour eux." To talk of an *existent* individual (even though the epithet obviously was not meant to be taken literally) only increases confusion; the fact is that an entity, if it *is* an entity, is a self-contradictory entity (an entity which is a non-entity).

[84] Cf., for example, Dedekind, *Essays on Number,* 45, 49, 58.

[85] That "exists" in mathematics often means "has being" or "is an entity" is one of the discoveries whose genesis will be described in a later issue.

proved to be a member of the null-class, determined by $x \ni \psi x$, where ψx is *not* false for every entity x. Thus $\psi x =$ "x is not a member of x" is true for some (if not all) x's, but $x \ni \psi x$ is not an entity. And further, this $x \ni \psi x$ appears to be an existent class; a strange dilemma for those who rely on intuition.

As early as 1884, Frege,[86] when criticising Hankel's formalist theory of the "numbers" of analysis, gave, in an important passage, the modern logical view of existence, including the remark, that a contradictory concept is permissible—but has no extension,[87]—and that the process[88] of introducing new "signs" as numbers, conformably to the "principle of permanence," is an error.

The chain of the existence-theorems for cardinal and ordinal arithmetic is, now, as follows:[89]

It may be shown, to begin with, that no definite class embraces all terms: this results from the fact that, since 0 is a cardinal number, the number of numbers up to and including a finite number n is $n+1$. Further, if n be a finite number, $n+1$ is a new finite number different from all its predecessors. Hence finite cardinals form a "progression,"[90] and therefore the ordinal number ω and the cardinal number $\aleph_0$ exist (in the mathematical sense). Hence, by mere rearrangements of the series of finite cardinal numbers, we obtain all ordinal numbers of Cantor's second class. We may now define ω_1 as the class of serial relations such that, if u be a class contained in the field of one of them, to say that u has successors implies and is implied by saying that u has a finite number of, or $\aleph_0$, terms; and it is easy to show that the series of ordinal numbers of the first and second classes in order of magnitude is of this type. Hence the

[86] *Grundlagen*, pp. 105-106, 107-108.

[87] This may be illustrated by Euclid's "δε" (see above).

[88] This process is used in the formal theory, but there is no doubt that Cantor did not, in spite of a statement of Pringsheim's *Encykl. der math. Wiss.* (Bd. I, p. 69) consider (at least in his later works) his transfinite numbers as a generalization of the *a priori* given concept of finite number. Also Schönflies's use of the "principle of permanence" to obtain the concepts of infinite numbers and types (*Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipsic, 1900, pp. 3-4, 27) must be regarded as a mistake.

[89] *Op. cit.*, pp. 322-323; cf. pp. ix, 111-116, 277-281, 313, 321-322, 497-498; and *Hibbert Journal*, July, 1904, pp. 810-811.

[90] *Op. cit.*, p. 239.

existence of ω_1 is proved; and $\aleph_1$ is defined to be the cardinal number of terms in a series whose generating relation is of the type ω_1. Hence, we can advance to ω_2 and $\aleph_2$, to ω_ω, and $\aleph_\omega$, and so on. This process gives us a one-one correlation of ordinals with (some) cardinals: it is evident that, by extending the process, we can make each cardinal which can belong to a well-ordered series correspond to one and only one ordinal. Cantor assumed that every class is the field of some well-ordered series, and hence deduced that all cardinals are Alephs.[91] This assumption seemed to Russell unwarranted.

VII.

Another, and rather different, example of the use of this "principle of abstraction" was given by Russell[92] in his definition of a real number. A real number was, as we have seen above, defined by Peano by abstraction: but Russell gave a nominal definition of a real number as a class, which can be proved to be an" existent" class,[93] and which has all the mathematical properties commonly assigned to a real number.

Any class of rational numbers[94] which is not null, which does not comprise all rational numbers,[95] and which comprises all those less than any one of its elements, is called a *segment* of rationals. To each rational r belongs one segment (of rationals less than it),

[91] See below.

[92] *Op. cit.*, pp. 270-286; Couturat, *op. cit.*, pp. 85-89.

[93] It is only by defining a number nominally, and as a class that its "existence" can be proved.

[94] The rational numbers here used are *signless* ratios or *relations* of finite cardinal numbers (see Russell, *op. cit.*, pp. 149-150; Couturat, *op. cit.*, pp. 79-81), or again they may be defined as Frege (*Grundlagen*, pp. 114-115) seemed to have urged, as classes (which can be proved to "exist") of couples. In either case, these rationals must be carefully distinguished from the rationals *with sign* (positive and negative), in the same way that a cardinal number n (a class) is not to be confused with the "positive integer" $+n$ (a relation, see Russell, *op. cit.*, p. 229; Couturat, *op. cit.*, pp. 80, 89). Cf. the distinction carried out between integers, integers with sign, rationals, rationals with sign, and so on, by Peano, with only minor mistakes in his *Formulario de mathematica*, V, 1905, pp. 83, 95-100, etc.

[95] By dropping the first condition alone, we may introduce zero, and, by dropping the second condition alone, we may introduce infinity, as limiting cases of segments (Russell, *op. cit.*, pp. 273-274; Couturat, *op. cit.*, p. 89; cf. Jourdain, *Journ. für Math.*, Bd. CXXVIII, 1905, p. 186).

but not inversely;[96] and, indeed, the class of segments is not capable of a one-one correlation with the class of rationals.

If, now, we confine our attention to those segments which have no rational maximum, or, in other words, the segments (u) such that every term of u is less than some other term of u, and consider those other classes v of rationals such that, if x is any member of u, there is a member of v greater than it, and, if y is any member of v, there is a member of u greater than it. This relation of v to u may be expressed:[97] "v is coherent to u"; and Cantor[98] considered this relation of being coherent (*zusammengehörig*) when, as is sufficient when u is any segment of rationals (of type η), v is a "fundamental" series (of type ω)

$$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots;$$

while the series cannot be of *finite* type if u has no maximum. Another class w (arranged in type $\omega . 2$, for example, or again in type ω), may also be coherent to u, and the relation of being coherent may be proved to be symmetrical and transitive. From this we infer that both v and w have to some third term (the "common property") a relation which neither has to any other term; and this third term may be taken to be the segment u which both define, and thus u is said to be the real number which all classes coherent to u define.[99]

Now there is a difference between the use of the "principle of abstraction" here and its use in defining the cardinal number of a class. Here a class of rationals has the relation of "being coherent to" its real number, there a class had the relation ϵ to its cardinal number. And we may frame a definition of a real number like, in this respect, that of a cardinal number. If a is a class (finite or infinite) of rationals, we may define the real number be-

[96] This fact may be described by saying that there are irrational segments.

[97] Russell, *op. cit.*, p. 274.

[98] *Math. Ann.*, Bd. XLVI, 1895, p. 508.

[99] Peano (*Formulario*, 1905, p. 10) defined a real number $l'u$, the "upper limit of u," where u is a class of rationals, by the abstraction:

$$l'u = l'v \,.=.\, \eta u = \eta v \qquad Df.,$$

where ηu and ηv are segments. Russell's definition is:

$$l'u = \eta u,$$

which obviously satisfies the above equation, and does not require any new meaning of $=$ besides logical identity (cf. above § 5).

longing to u ($l'u$) as "the class of all those classes which are coherent to u."

We shall find an analogue to these two definitions of real number in the definitions of cardinal and ordinal number which I propose (in what I call my "second theory"), as seeming, for a certain reason, preferable to Russell's though the new definitions are not essential to the theory.[100] With Russell's the relation of a class v to its cardinal number is ϵ, with mine this relation is "is similar to"; and both definitions satisfy the necessary requirements of being nominal and not requiring the re-definition of $=$, which has thus the same meaning (of identity) throughout all logic and mathematics.

[TO BE CONTINUED.]

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[100] This "second theory," in which cardinal numbers are defined, by an extended induction, to be classes of the preceding cardinal numbers, seems necessary if we are to avoid—what we must in what will be referred to as the "limitation-of-size theory"—defining a number as a class equivalent to the class of all classes. However, it must be acknowledged that Russell, by his "no-classes theory," has made such an attempt to improve the older theory superfluous.