Philosophical Magazine Series 5
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm16

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Version of record first published: 08 May 2009.

To cite this article: Lord Rayleigh Sec. R.S. (1892): LVI. On the influence of obstacles arranged in rectangular order upon the properties of a medium , Philosophical Magazine Series 5, 34:211, 481-502

To link to this article: http://dx.doi.org/10.1080/14786449208620364

limited to two particles only. Given the proper temperature with corresponding conditions of mass, shape, and distribution of charge on the particles, and, as it seems to me, almost any amount of molecular complexity is possible. That I have not taken this possibility into account does not, however, vitiate the results here brought forward, as they do not pretend to greater accuracy than that of their order of magnitude.

It is the cumulative value of these results which will, I hope, be regarded as sufficient reason for the publication of what is at best an incomplete piece of theory.

Univ. Coll. Bristol.

LVI. *On the Influence of Obstacles arranged in Rectangular Order upon the Properties of a Medium.* *By* Lord RAYLEIGH, *Sec. R.S.**

THE remarkable formula, arrived at almost simultaneously by L. Lorenz † and H. A. Lorentz ‡, and expressing the relation between refractive index and density, is well known; but the demonstrations are rather difficult to follow, and the limits of application are far from obvious. Indeed, in some discussions the necessity for any limitation at all is ignored. I have thought that it might be worth while to consider the problem in the more definite form which it assumes when the obstacles are supposed to be arranged in rectangular or square order, and to show how the approximation may be pursued when the dimensions of the obstacles are no longer very small in comparison with the distances between them.

Taking, first, the case of two dimensions, let us investigate the conductivity for heat, or electricity, of an otherwise uniform medium interrupted by cylindrical obstacles which are arranged in rectangular order. The sides of the rectangle will be denoted by α, β, and the radius of the cylinders by a. The simplest cases would be obtained by supposing the material composing the cylinders to be either non-conducting or perfectly conducting; but it will be sufficient to suppose that it has a definite conductivity different from that of the remainder of the medium.

By the principle of superposition the conductivity of the interrupted medium for a current in any direction can be deduced from its conductivities in the three principal directions.

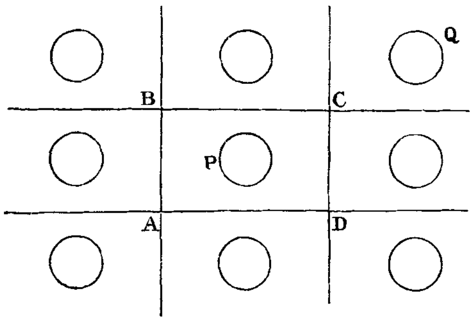
* Communicated by the Author.

† Wied. *Ann.* xi. p. 70 (1880).

‡ Wied. *Ann.* ix. p. 641 (1880).

Since conduction parallel to the axes of the cylinders presents nothing special for our consideration, we may limit our attention to conduction parallel to one of the sides (α) of the rectangular structure. In this case lines parallel to α,

Fig. 1.



symmetrically situated between the cylinders, such as AD, BC, are lines of flow, and the perpendicular lines AB, CD are equipotential.

If we take the centre of one of the cylinders P as origin of polar coordinates, the potential external to the cylinder may be expanded in the series

$$V = A_0 + (A_1 r + B_1 r^{-1}) \cos\theta + (A_3 r^3 + B_3 r^{-3}) \cos 3\theta + \ldots, \quad (1)$$

and at points within the cylinder in the series

$$V' = C_0 + C_1 r \cos\theta + C_3 r^3 \cos 3\theta + \ldots, \quad . \quad . \quad (2)$$

θ being measured from the direction of α. The sines of θ and its multiples are excluded by the symmetry with respect to $\theta = 0$, and the cosines of the even multiples by the symmetry with respect to $\theta = \frac{1}{2}\pi$. At the bounding surface, where $r = a$, we have the conditions

$$V = V', \quad \nu dV'/dr = dV/dr,$$

ν denoting the conductivity of the material composing the cylinders in terms of that of the remainder reckoned as unity. The application of these conditions to the term in $\cos n\theta$ gives

$$B_n = \frac{1-\nu}{1+\nu} a^{2n} A_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the case where the cylinders are perfectly conducting, $\nu = \infty$. If they are non-conducting, $\nu = 0$.

The values of the coefficients A_1,B_1, $A_3,B_3\ldots$ are necessarily the same for all the cylinders, and each may be regarded as a similar multiple source of potential. The first term A_0, however, varies from cylinder to cylinder, as we pass up or down the stream.

Let us now apply Green's theorem,

$$\int\left(\mathrm{U}\frac{d\mathrm{V}}{dn}-\mathrm{V}\frac{d\mathrm{U}}{dn}\right)ds=0 \quad . \quad . \quad . \quad . \quad (4)$$

to the contour of the region between the rectangle ABCD and the cylinder P. Within this region V satisfies Laplace's equation, as also will U, if we assume

$$\mathrm{U}=x=r\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Over the sides BC, AD, $d\mathrm{U}/dn$, $d\mathrm{V}/dn$ both vanish. On CD, $\int d\mathrm{V}/dn\, ds$ represents the total current across the rectangle, which we may denote by C. The value of this part of the integral over CD, AB is thus αC. The value of the remainder of the integral over the same lines is $-\mathrm{V}_1\beta$, where V_1 is the fall in potential corresponding to one rectangle, as between CD and AB.

On the circular part of the contour,

$$\mathrm{U}=a\cos\theta,\quad d\mathrm{U}/dn=-d\mathrm{U}/dr=-\cos\theta\,;$$

and thus the only terms in (1) which will contribute to the result are those in $\cos\theta$. Thus we may write

$$\mathrm{V}=(\mathrm{A}_1a+\mathrm{B}_1a^{-1})\cos\theta,$$
$$d\mathrm{V}/dn=-(\mathrm{A}_1-\mathrm{B}_1a^{-2})\cos\theta\,;$$

so that this part of the integral is $2\pi\mathrm{B}_1$. The final result from the application of (4) is thus

$$\alpha\mathrm{C}-\beta\mathrm{V}_1+2\pi\mathrm{B}_1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $\mathrm{B}_1=0$, we fall back upon the uninterrupted medium of which the conductivity is unity. For the case of the actual medium we require a further relation between B_1 and V_1.

The potential V at any point may be regarded as due to external sources at infinity (by which the flow is caused) and to multiple sources situated on the axes of the cylinders. The first part may be denoted by Hx. In considering the second it will conduce to clearness if we imagine the (infinite) region occupied by the cylinders to have a rectangular boundary parallel to α and β. Even then the manner in which the infinite system of sources is to be taken into account will depend upon the shape of the rectangle. The simplest case, which suffices for our purpose, is when we suppose the rectangular boundary to be extended infinitely more parallel to α than parallel to β. It is then evident that the periodic difference V_1 may be reckoned as due entirely to Hx, and equated to $H\alpha$. For the difference due to the sources upon the axes will be equivalent to the addition of one extra column at $+\infty$, and the removal of one at $-\infty$, and in the case supposed such a transference is immaterial*. Thus

$$V_1 = H\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

simply, and it remains to connect H with B_1.

This we may do by equating two forms of the expression for the potential at a point x, y near P. The part of the potential due to Hx and to the multiple sources Q (P not included) is

$$A_0 + A_1 r \cos\theta + A_3 r^3 \cos 3\theta + \ldots ;$$

or, if we subtract Hx, we may say that the potential at x, y due to the multiple sources at Q is the real part of

$$A_0 + (A_1 - H)(x+iy) + A_3(x+iy)^3 + A_5(x+iy)^5 + \ldots \quad . \quad (8)$$

But if x', y' are the coordinates of the same point when referred to the centre of one of the Q's, the same potential may be expressed by

$$\Sigma\{B_1(x'+iy')^{-1} + B_3(x'+iy')^{-3} + \ldots\}, \quad . \quad . \quad (9)$$

the summation being extended over all the Q's. If ξ, η be the coordinates of a Q referred to P,

$$x' = x - \xi, \quad y' = y - \eta ;$$

so that

$$B_n(x'+iy')^{-n} = B_n(x+iy-\xi-i\eta)^{-n}.$$

Since (8) is the expansion of (9) in rising powers of $(x+iy)$, we obtain, equating term to term,

$$\left.\begin{aligned} H - A_1 &= B_1\Sigma_2 + 3B_3\Sigma_4 + 5B_5\Sigma_6 + \ldots \\ -1.2.3\,A_3 &= 1.2.3\,B_1\Sigma_4 + 3.4.5\,B_3\Sigma_6 + \ldots \\ -1.2.3.4.5\,A_5 &= 1.2.3.4.5\,B_1\Sigma_6 + 3.4.5.6.7\,B_5\Sigma_8 + \ldots \end{aligned}\right\}$$

and so on, where

$$\Sigma_{2n} = \Sigma(\xi+i\eta)^{-2n}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the summation extending over all the Q's.

* It would be otherwise if the infinite rectangle were supposed to be of another shape, *e. g.* to be square.

By (3) each B can be expressed in terms of the corresponding A. For brevity, we will write

$$A_n = \nu' a^{-2n} B_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$\nu' = (1+\nu)/(1-\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We are now prepared to find the approximate value of the conductivity. From (6) the conductivity of the rectangle is

$$\frac{C}{V_1} = \frac{\beta}{\alpha}\left\{1 - \frac{2\pi B_1}{\beta V_1}\right\} = \frac{\beta}{\alpha}\left\{1 - \frac{2\pi B_1}{\alpha\beta H}\right\};$$

so that the specific conductivity of the actual medium for currents parallel to α is

$$1 - \frac{2\pi B_1}{\alpha\beta H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the ratio of H to B_1 is given approximately by (10) and (12).

In the first approximation we neglect $\Sigma_4, \Sigma_6 \ldots$, so that $A_3, A_5 \ldots B_3, B_5 \ldots$ vanish. In this case

$$H = A_1 + B_1\Sigma_2 = B_1(\nu' a^{-2} + \Sigma_2), \quad . \quad . \quad . \quad (15)$$

and the conductivity is

$$1 - \frac{2\pi a^2}{\alpha\beta(\nu' + a^2\Sigma_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The second approximation gives

$$\frac{Ha^2}{B_1} = \nu' + a^2\Sigma_2 - \frac{3}{\nu'} a^8 \Sigma_4^2, \quad . \quad . \quad . \quad . \quad (17)$$

and the series may be continued as far as desired.

The problem is thus reduced to the evaluation of the quantities $\Sigma_2, \Sigma_4, \ldots$ We will consider first the important particular case which arises when the cylinders are in *square* order, that is when $\beta = \alpha$. ξ and η in (11) are then both multiples of α, and we may write

$$\Sigma_n = \alpha^{-n} S_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where

$$S_n = \Sigma (m' + im)^{-n}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

the summation being extended to all integral values of m, m', positive or negative, except the pair $m=0$, $m'=0$. The quantities S are thus purely numerical, and real.

The next thing to be remarked is that, since m, m' are as much positive as negative, S_n vanishes for every odd value of n. This holds even when α and β are unequal.

Again,

$$S_{2n} = \Sigma(m'+im)^{-2n} = i^{-2n}\,\Sigma(-im'+m)^{-2n}$$
$$= (-1)^n\Sigma(-im'+m)^{-2n} = (-1)^n\, S_{2n}.$$

Whenever n is odd, $S_{2n} = -S_{2n}$, or S_{2n} vanishes. Thus for square order,

$$S_6 = S_{10} = S_{14} = \ldots = 0. \quad . \quad . \quad . \quad . \quad (20)$$

This argument does not, without reservation, apply to S_2. In that case the sum is not convergent; and the symmetry between m and m', essential to the proof of evanescence, only holds under the restriction that the infinite region over which the summation takes place is symmetrical with respect to the two directions α and β—is, for example, square or circular. On the contrary, we have supposed, and must of course continue to suppose, that the region in question is infinitely elongated in the direction of α.

The question of convergency may be tested by replacing the parts of the sum relating to a great distance by the corresponding integral. This is

$$\iint \frac{dx\,dy}{(x+iy)^{2n}} = \iint \frac{\cos 2n\theta\; r\,dr\,d\theta}{r^{2n}};$$

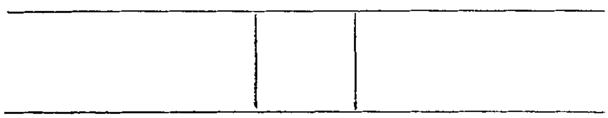
and herein

$$\int r^{-2n+1}\,dr = r^{-2n+2}/(-2n+2)\;;$$

so that if $n>1$ there is convergency, but if $n=1$ the integral contains an infinite logarithm.

We have now to investigate the value of S_2 appropriate to our purpose; that is, when the summation extends over the region bounded by $x=\pm u$, $y=\pm v$, where u and v are both infinite, but so that $v/u=0$. If we suppose that the region of summation is that bounded by $x=\pm v$, $y=\pm v$, the sum

Fig. 2.



vanishes by symmetry. We may therefore regard the summation as extending over the region bounded externally by $x=\pm\infty$, $y=\pm v$, and internally by $x=\pm v$ (fig. 2). When v is very great, the sum may be replaced by the corresponding integral. Hence

$$S_2 = 2 \iint \frac{dx\,dy}{(x+iy)^2}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the limits for y being $\pm v$, and those for x being v and ∞. Ultimately v is to be made infinite.

We have

$$\int_{-v}^{+v} \frac{dy}{(x+iy)^2} = \frac{i}{x+iv} - \frac{i}{x-iv} = \frac{2v}{x^2+v^2};$$

and

$$\int_{v}^{\infty} \frac{2v\,dx}{x^2+v^2} = 2\tan^{-1}\infty - 2\tan^{-1}1 = \tfrac{1}{2}\pi.$$

Accordingly

$$S_2 = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the case of square order, equations (10) (12) give

$$\frac{Ha^2}{B_1} = \nu' + a^2\Sigma_2 - \frac{3}{\nu'}a^8\Sigma_4{}^2 - \frac{7}{\nu'}a^{16}\Sigma_8{}^2 - \ldots.$$

$$= \nu' + \frac{\pi a^2}{\alpha^2} - \frac{3}{\nu'}\frac{a^8}{\alpha^8}S_4{}^2 - \frac{7}{\nu'}\frac{a^{16}}{\alpha^{16}}S_8{}^2 - \ldots; \quad . \quad (23)$$

and by (14)

$$\text{Conductivity} = 1 - \frac{2\pi a^2}{\alpha^2} \cdot \frac{B_1}{Ha^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If p denote the proportional space occupied by the cylinders,

$$p = \pi a^2/\alpha^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and

$$\text{Conductivity} = 1 - \frac{2p}{\nu' + p - \frac{3p^4}{\nu'\pi^4}S_4{}^2 - \frac{7p^8}{\nu'\pi^8}S_8{}^2}. \quad . \quad . \quad . \quad (26)$$

Of the double summation indicated in (19) one part can be effected without difficulty. Consider the roots of

$$\sin(\xi - im\pi) = 0.$$

They are all included in the form

$$\xi = m'\pi + im\pi,$$

where m' is any integer, positive, negative, or zero. Hence we see that $\sin(\xi - im\pi)$ may be written in the form

$$A\left(1 - \frac{\xi}{im\pi}\right)\left(1 - \frac{\xi}{im\pi+\pi}\right)\left(1 - \frac{\xi}{im\pi-\pi}\right)\left(1 - \frac{\xi}{im\pi+2\pi}\right)\ldots,$$

in which

$$A = -\sin im\pi.$$

Thus

$$\log\{\cos\xi - \cot im\pi \sin\xi\} = \log\left(1-\frac{\xi}{im\pi}\right)$$
$$+\log\left(1-\frac{\xi}{im\pi+\pi}\right)+\ldots.$$

If we change the sign of m, and add the two equations, we get

$$\log\left\{1-\frac{\sin^2\xi}{\sin^2 im\pi}\right\}=\log\left\{1-\frac{\xi^2}{(im\pi)^2}\right\}$$
$$+\log\left\{1-\frac{\xi^2}{(im\pi+\pi)^2}\right\}+\log\left\{1-\frac{\xi^2}{(im\pi-\pi)^2}\right\}+\ldots;$$

whence, on expansion of the logarithms,

$$\frac{\sin^2\xi}{\sin^2 im\pi}+\frac{\sin^4\xi}{2\sin^4 im\pi}+\frac{\sin^6\xi}{3\sin^6 im\pi}+\ldots.$$
$$=\xi^2\left\{\frac{1}{(im\pi)^2}+\frac{1}{(im\pi+\pi)^2}+\frac{1}{(im\pi-\pi)^2}+\ldots\right\}$$
$$+\tfrac{1}{2}\xi^4\left\{\frac{1}{(im\pi)^4}+\frac{1}{(im\pi+\pi)^4}+\frac{1}{(im\pi-\pi)^4}+\ldots\right\}$$
$$+\tfrac{1}{3}\xi^6\left\{\frac{1}{(im\pi)^6}+\frac{1}{(im\pi+\pi)^6}+\ldots\right\}+\ldots.$$

By expanding the sines on the left and equating the corresponding powers of ξ, we find

$$\frac{1}{(im)^2}+\frac{1}{(im+1)^2}+\frac{1}{(im-1)^2}+\frac{1}{(im+2)^2}+\ldots=\frac{\pi^2}{\sin^2 im\pi}, \quad (27)$$

$$\frac{1}{(im)^4}+\frac{1}{(im+1)^4}+\ldots=-\frac{2\pi^4}{3\sin^2 im\pi}+\frac{\pi^4}{\sin^4 im\pi}, \quad . \quad (28)$$

$$\frac{1}{(im)^6}+\frac{1}{(im+1)^6}+\ldots=\frac{2\pi^6}{15\sin^2 im\pi}-\frac{\pi^6}{\sin^4 im\pi}+\frac{\pi^6}{\sin^6 im\pi}. \quad (29)$$

In the summation with respect to m required in (19) we are to take all positive and negative integral values. But in the case of $m=0$ we are to leave out the first term, corresponding to $m'=0$. When $m=0$,

$$\frac{\pi^2}{\sin^2 im\pi}-\frac{1}{(im)^2}=\frac{\pi^2}{3},$$

which, as is well known, is the value of

$$\frac{1}{1^2}+\frac{1}{(-1)^2}+\frac{1}{2^2}+\frac{1}{(-2)^2}+\ldots.$$

Hence

$$S_2=2\pi^2\sum_{m=1}^{m=\infty}\sin^{-2}im\pi+\tfrac{1}{3}\pi^2\,; \quad . \quad . \quad . \quad . \quad (30)$$

and in like manner

$$S_4=\frac{\pi^4}{45}+2\pi^4\sum_{m=1}^{m=\infty}\{-\tfrac{2}{3}\sin^{-2}im\pi+\sin^{-4}im\pi\}, \quad . \quad . \quad (31)$$

$$S_6=\frac{2\pi^6}{27\,.\,35}+2\pi^6\sum_{m=1}^{m=\infty}\{\tfrac{2}{15}\sin^{-2}im\pi$$
$$-\sin^{-4}im\pi+\sin^{-6}im\pi\}. \quad . \quad . \quad (32)$$

We have seen already that $S_6=0$, and that $S_2=\pi$. The comparison of the latter with (30) gives

$$\sum_{m=1}^{m=\infty}\sin^{-2}im\pi=\frac{1}{2\pi}-\frac{1}{6}. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

We will now apply (31) to the numerical calculation of S_4. We find :

m.	$-\sin^{-2}im\pi$.	$\sin^{-4}im\pi$.
1	·00749767	·0000562150
2	· 1395	2
3	· 3	
Sum	·00751165	·0000562152

so that

$$S_4=\pi^4\times{\cdot}03235020. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

In the same way we may verify (33), and that (32) $=0$.

If we introduce this value into (26), taking for example the case where the cylinders are non-conductive ($\nu'=1$), we get

$$1-\frac{2p}{1+p-{\cdot}3058p^4}. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

From the above example it appears that in the summation with respect to m there is a high degree of convergency.

The reason for this will appear more clearly if we consider the nature of the first summation (with respect to m'). In (19) we have to deal with the sum of $(x+iy)^{-n}$, where y is for the moment regarded as constant, while x receives the values $x=m'$. If, instead of being concentrated at equidistant points, the values of x were uniformly distributed, the sum would become

$$\int_{-\infty}^{+\infty} \frac{dx}{(x+iy)^n}.$$

Now, n being greater than 1, the value of this integral is zero. We see, then, that the finite value of the sum depends entirely upon the discontinuity of its formation, and thus a high degree of convergency when y increases may be expected.

The same mode of calculation may be applied without difficulty to any particular case of a rectangular arrangement. For example, in (11)

$$\Sigma_2 = \Sigma(m'\alpha + im\beta)^{-2} = \alpha^{-2}\Sigma(m' + im\beta/\alpha)^{-2}.$$

If m be given, the summation with respect to m' leads, as before, to

$$\Sigma(m' + im\beta/\alpha)^{-2} = \frac{\pi^2}{\sin^2(im\pi\beta/\alpha)};$$

and thus

$$\alpha^2\Sigma_2 = 2\pi^2 \sum_{m=1}^{m=\infty} \sin^{-2}(im\pi\beta/\alpha) + \tfrac{1}{3}\pi^2. \quad . \quad . \quad (36)$$

The numerical calculation would now proceed as before, and the final approximate result for the conductivity is given by (16). Since (36) is not symmetrical with respect to α and β, the conductivity of the medium is different in the two principal directions.

When $\beta=\alpha$, we know that $\alpha^{-2}\Sigma_2=\pi$. And since this does not differ much from $\frac{1}{3}\pi^2$, it follows that the series on the right of (36) contributes but little to the total. The same will be true, even though β be not equal to α, provided the ratio of the two quantities be moderate. We may then identify $\alpha^{-2}\Sigma_2$ with π, or with $\frac{1}{3}\pi^2$, if we are content with a very rough approximation.

The question of the values of the sums denoted by Σ_{2n} is intimately connected with the theory of the θ-functions *, inasmuch as the roots of $H(u)$, or $\theta_1(\pi u/2K)$, are of the form

$$2mK + 2m'iK'.$$

* Cayley's 'Elliptic Functions,' p. 300. The notation is that of Jacobi.

The analytical question is accordingly that of the expansion of log $\theta_1(x)$ in ascending powers of x. Now, Jacobi * has himself investigated the expansion in powers of x of

$$\theta_1(x)=2\{q^{1/4}\sin x-q^{9/4}\sin 3x+q^{25/4}\sin 5x-\ldots\}, \quad (37)$$

where

$$q=e^{-\pi K'/K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

So far as the cube of x the result is

$$\frac{\theta_1(x)}{D.x}=1-\frac{2x^2}{3\pi^2}\left\{3KE-(2-k^2)K^2\right\}+\ldots, \quad (39)$$

D being a constant which it is not necessary further to specify. K and E are the elliptic functions of k usually so denoted. By what has been stated above the roots of $\theta_1(x)$ are of the form

$$\pi\,(m+m'iK'/K)\,; \quad . \quad . \quad . \quad . \quad . \quad (40)$$

so that

$$\Sigma\{m+im'K'/K\}^{-2}=\tfrac{4}{3}\{3KE-(2-k^2)K^2\}, \quad . \quad (41)$$

the summation on the left being extended to all integral values of m and m', except $m=0$, $m'=0$.

This is the general solution for Σ_2. If $K'=K$, $k^2=\frac{1}{2}$, and

$$\Sigma\{m+im'\}^{-2}=2\{2KE-K^2\}=\pi,$$

since in general †,

$$EK'+E'K-KK'=\tfrac{1}{2}\pi.$$

In proceeding further it is convenient to use the form in which an exponential factor is removed from the series. This is

$$\theta_1(x)=A^{\frac{1}{2}}e^{-\frac{1}{2}ABx^2}\left\{s_0Ax-s_1\frac{A^3x^3}{3\,!}+s_2\frac{A^5x^5}{5\,!}-s_3\frac{A^7x^7}{7\,!}+\ldots\right\}, \quad (42)$$

in which

$$A=\frac{2K}{\pi},\quad B=\frac{2E}{\pi}-k'^2\frac{2K}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$s_0=\beta,\quad s_1=\alpha\beta,\quad s_2=\beta(\alpha^2-2\beta^4),\quad s_3=\alpha\beta(\alpha^2-6\beta^4),$$

the law of formation of s being

$$s_{m+1}=2m(2m+1)\beta^4 s_{m-1}+\alpha\beta\, ds_m/d\beta-8\beta^4 ds_m/d\alpha, \quad (44)$$

* 'Crelle,' Bd. 54, p. 82.

† Cayley's 'Elliptic Functions,' p. 49.

while

$$\alpha=k'^2-k^2, \quad \beta=\sqrt{(kk')}. \quad . \quad . \quad . \quad (45)$$

I have thought it worth while to quote these expressions, as they do not seem to be easily accessible; but I propose to apply them only to the case of square order, $K'=K$, $k'^2=k^2=\frac{1}{2}$. Thus

$$AB=1/\pi, \quad \alpha=0, \quad \beta=1/\sqrt{2}\,; \quad . \quad . \quad . \quad (46)$$

$$s_0=\beta, \quad s_1=0, \quad s_2=-2\beta^5, \quad s_3=0, \quad s_4=-36\beta^9,$$

and

$$\theta_1(x)=\frac{A^{\frac{3}{2}}x}{\sqrt{2}}e^{-x^2/2\pi}\left\{1-\frac{A^4x^4}{2.5!}-\frac{A^8x^8}{4.8!}+\dots\right\}. . \quad (47)$$

Hence

$$\log\frac{\theta_1(x)}{D.x}=-\frac{x^2}{2\pi}-\frac{A^4x^4}{2.5!}-\frac{A^8x^8}{16.35.5!}-\dots \quad (48)$$

If $\pm\lambda_1$, $\pm\lambda_2$, ... are the roots of $\theta_1(x)/x=0$, we have

$$\Sigma\lambda^{-2}=\frac{1}{2\pi}, \quad \Sigma\lambda^{-4}=\frac{A^4}{5!}, \quad \Sigma\lambda^{-6}=0, \quad \Sigma\lambda^{-8}=\frac{A^8}{7.5.4.5!}.$$

Now by (40) the roots in question are $\pi(m+im')$, and thus

$$S_2=\pi, \quad S_4=\frac{\pi^4}{60}A^4, \quad S_8=\frac{\pi^8A^8}{70.5!}, \quad . \quad . \quad (49)$$

in which

$$A=\frac{2}{\pi}K=1+\frac{1^2}{2^2}\cdot\frac{1}{2}+\frac{1^2.3^2}{2^2.4^2}\cdot\frac{1}{4}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cdot\frac{1}{8}+\dots$$

$$=1\cdot18034.$$

Leaving the two-dimensional problem, I will now pass on to the case of a medium interrupted by *spherical* obstacles arranged in rectangular order. As before, we may suppose that the side of the rectangle in the direction of flow is α, the two others being β and γ. The radius of the sphere is a.

The course of the investigation runs so nearly parallel to that already given, that it will suffice to indicate some of the steps with brevity. In place of (1) and (2) we have the expansions

$$V=A_0+(A_1r+B_1r^{-2})Y_1+\dots$$
$$+(A_nr^n+B_nr^{-n-1})Y_n+\dots, \quad . \quad . \quad (50)$$

$$V'=C_0+C_1Y_1r+\dots+C_nY_nr^n+\dots, \quad . \quad (51)$$

Y_n denoting the spherical surface harmonic of order n. And from the surface conditions

$$V=V', \quad \nu dV'/dn = dV/dn,$$

we find

$$B_n = \frac{1-\nu}{1+\nu+1/n} a^{2n+1} A_n. \quad . \quad . \quad . \quad (52)$$

We must now consider the limitations to be imposed upon Y_n. In general,

$$Y_n = \sum_{s=0}^{s=n} \Theta_n^s (H_s \cos s\phi + K_s \sin s\phi), \quad . \quad . \quad (53)$$

where

$$\Theta_n^s = \sin^s\theta \left(\cos^{n-s}\theta - \frac{(n-s)(n-s-1)}{2(2n-1)} \cos^{n-s-2}\theta + \ldots \right), \quad (54)$$

θ being supposed to be measured from the axis of x (parallel to α), and ϕ from the plane of xz. In the present application symmetry requires that s should be even, and that Y_n (except when $n=0$) should be reversed when $(\pi-\theta)$ is written for θ. Hence even values of n are to be excluded altogether. Further, no sines of $s\phi$ are admissible. Thus we may take

$$Y_1 = \cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$Y_3 = \cos^3\theta - \tfrac{3}{5}\cos\theta + H_2 \sin^2\theta \cos\theta \cos 2\phi, \quad (56)$$

$$\begin{aligned} Y_5 = \cos^5\theta - \tfrac{10}{9}\cos^3\theta + \tfrac{5}{21}\cos\theta \\ + L_2 \sin^2\theta (\cos^3\theta - \tfrac{1}{3}\cos\theta)\cos 2\phi \\ + L_4 \sin^4\theta \cos\theta \cos 4\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57) \end{aligned}$$

In the case where $\beta=\gamma$ symmetry further requires that

$$H_2=0, \quad L_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In applying Green's theorem (4) the only difference is that we must now understand by s the area of the *surface* bounding the region of integration. If C denote the total current flowing across the faces $\beta\gamma$, V_1 the periodic difference of potential, the analogue of (6) is

$$\alpha C - \beta\gamma V_1 + 4\pi B_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

We suppose, as before, that the system of obstacles, extended without limit in every direction, is yet infinitely more extended in the direction of α than in the directions of β and γ.

Then, if Hx be the potential due to the sources at infinity other than the spheres, $V_1 = H\alpha$, and

$$\frac{C}{V_1} = \frac{\beta\gamma}{\alpha}\left\{1 - \frac{4\pi B_1}{\alpha\beta\gamma H}\right\};$$

so that the specific conductivity of the compound medium parallel to α is

$$1 - \frac{4\pi B_1}{\alpha\beta\gamma H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

We will now show how the ratio B_1/H is to be calculated approximately, limiting ourselves, however, for the sake of simplicity to the case of cubic order, where $\alpha = \beta = \gamma$. The potential round P, viz.

$$A_0 + A_1 r . Y_1 + A_3 r^3 Y_3 + \ldots,$$

may be regarded as due to Hx and to the other spheres Q acting as sources of potential. Thus, if we revert to rectangular coordinates and denote the coordinates of a point relatively to P by x, y, z, and relatively to one of the Q's by x', y', z', we have

$$A_0 + (A_1 - H)x + A_3(x^3 - \tfrac{3}{5}xr^2) + \ldots$$
$$= B_1 \Sigma \frac{x'}{r'^3} + B_3 \Sigma \frac{x'^3 - \frac{3}{5}x'r'^2}{r'^7} + \ldots, \quad . \quad (61)$$

in which

$$x' = x - \xi, \quad y' = y - \eta, \quad z' = z - \zeta,$$

if ξ, η, ζ be the coordinates of Q referred to P. The left side of (61) is thus the expansion of the right in ascending powers of x, y, z. Accordingly, $A_1 - H$ is found by taking d/dx of the right-hand member and then making x, y, z vanish. In like manner $6A_3$ will be found from the third differential coefficient. Now, at the origin,

$$\frac{d}{dx}\frac{x'}{r'^3} = -\frac{d}{d\xi}\frac{x'}{r'^3} = -\frac{d}{d\xi}\frac{-\xi}{\rho^3} = \frac{\rho^2 - 3\xi^2}{\rho^5},$$

in which

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2.$$

It will be observed that we start with a harmonic of order 1 and that the differentiation raises the order to 2. The law that each differentiation raises the order by unity is general; and, so far as we shall proceed, the harmonics are all zonal, and may be expressed in the usual way as functions $P_n(\mu)$ of μ, where $\mu=\xi/\rho$. Thus

$$\frac{d}{dx}\frac{x'}{r'^3}=-2\rho^{-3}\,P_2(\mu).$$

In like manner,

$$\frac{d}{dx}\frac{x'^3-\frac{3}{5}x'r'^2}{r'^7}=\frac{d}{d\xi}\frac{\xi^3-\frac{3}{5}\xi\rho^2}{\rho^7}=-\frac{8}{5}\rho^{-5}\,P_4(\mu),$$

and

$$\frac{d^3}{dx^3}\frac{x'}{r'^3}=\frac{d^3}{d\xi^3}\frac{\xi}{\rho^3}=-24\,\rho^{-5}\,P_4(\mu).$$

The comparison of terms in (61) thus gives

$$\left.\begin{aligned}A_1-H&=-2B_1\,\Sigma\rho^{-3}\,P_2-\tfrac{8}{5}\,B_3\,\Sigma\rho^{-5}\,P_4+\dots\\ A_3&=-4B_1\,\Sigma\rho^{-5}\,P_4+\dots.\\ \dots&=\dots..\end{aligned}\right\}\quad . \quad . \quad (62)$$

In each of the quantities, such as $\Sigma\rho^{-3}P_2$, the summation is to be extended to all the points whose coordinates are of the form

$$l\alpha,\ m\alpha,\ n\alpha,$$

where l, m, n are any set of integers, positive or negative, except 0, 0, 0. If we take $\alpha=1$, and denote the corresponding sums by S_2, S_4, ... , these quantities will be purely numerical, and

$$\Sigma\rho^{-n-1}P_n=\alpha^{-n-1}S_n. \quad . \quad . \quad . \quad . \quad . \quad (63)$$

From (52) (62) we now obtain

$$\frac{Ha^3}{B_1}=\frac{2+\nu}{1-\nu}+2S_2\frac{a^3}{\alpha^3}-\frac{32}{5}\frac{1-\nu}{\frac{4}{3}+\nu}S_4^{\,2}\frac{a^{10}}{\alpha^{10}}+\dots, \quad (64)$$

which with (60) gives the desired result for the conductivity of the medium.

We now proceed to the calculation of S_2. We have

$$S_2=\Sigma\frac{3\mu^2-1}{2\rho^3}=\Sigma\frac{2\xi^2-\eta^2-\zeta^2}{2\rho^5}=-\tfrac{1}{2}\Sigma\frac{d}{d\xi}\left(\frac{\xi}{\rho^3}\right).$$

By the symmetry of a cubical arrangement, it follows that

$$\Sigma\,(\xi^2/\rho^5)=\Sigma\,(\eta^2/\rho^5)=\Sigma\,(\zeta^2/\rho^5)\,;$$

so that if S were calculated for a space bounded by a cube, it would necessarily vanish. But for our purpose S_2 is to be calculated over the space bounded by $\xi=\pm\infty$, $\eta=\pm v$,

$\zeta = \pm v$, where v is finally to be made infinite; and, as we have just seen, we may exclude the space bounded by

$$\xi = \pm v, \quad \eta = \pm v, \quad \zeta = \pm v;$$

so that $\frac{1}{2}S_2$ will be obtained from the space bounded by

$$\xi = v, \quad \xi = \infty, \quad \eta = \pm v, \quad \zeta = \pm v.$$

Now when ρ is sufficiently great, the summation may be replaced by an integration; thus

$$S_2 = -\iiint \frac{d}{d\xi}\left(\frac{\xi}{\rho^3}\right) d\xi \, d\eta \, d\zeta.$$

In this,

$$\int_v^\infty \frac{d}{d\xi}\frac{\xi}{\rho^3}\, d\xi = -\frac{v}{(v^2+\eta^2+\zeta^2)^{\frac{3}{2}}},$$

$$\int_{-v}^{+v} \frac{v\, d\eta}{(v^2+\eta^2+\zeta^2)^{\frac{3}{2}}} = \frac{2v^2}{(v^2+\zeta^2)(2v^2+\zeta^2)^{\frac{1}{2}}},$$

and finally

$$\int_{-v}^{+v} \frac{v^2 d\zeta}{(v^2+\zeta^2)(2v^2+\zeta^2)^{\frac{1}{2}}} = \int_0^1 \frac{2d\theta}{\sqrt{(2+\tan^2\theta)}}$$

$$= \int_0^{1/\sqrt{2}} \frac{2ds}{\sqrt{(2-s^2)}} = \frac{\pi}{3}.$$

Thus

$$S_2 = \frac{2\pi}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

If we neglect a^{10}/α^{10}, and write p for the ratio of volumes, viz.

$$p = \frac{4\pi a^3}{3\alpha^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

we have by (60) for the conductivity

$$\frac{(2+\nu)/(1-\nu)-2p}{(2+\nu)/(1-\nu)+p}, \quad . \quad . \quad . \quad . \quad . \quad (67)^*$$

or in the particular case of non-conducting obstacles ($\nu = 0$)

$$\frac{1-p}{1+\frac{1}{2}p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

In order to carry on the approximation we must calculate S_4 &c. Not seeing any general analytical method, such as was available in the former problem, I have calculated an

* Compare Maxwell's 'Electricity,' § 314.

approximate value of S_4 by direct summation from the formula

$$S_4 = \Sigma \frac{35\xi^4 - 30\xi^2\rho^2 + 3\rho^4}{8\rho^9}.$$

We may limit ourselves to the consideration of positive and zero values of ξ, η, ζ. Every term for which ξ, η, ζ are finite is repeated in each octant, that is 8 times. If one of the three coordinates vanish, the repetition is fourfold, and if two vanish, twofold.

The following table contains the result for all points which lie within $\rho^2 = 18$. This repetition in the case, for example, of $\rho^2 = 9$ represents two kinds of composition. In the first

$$\rho^2 = 2^2 + 2^2 + 1^2 = 9,$$

and in the second

$$\rho^2 = 3^2 + 0^2 + 0^2 = 9.$$

The success of the approximation is favoured by the fact that P vanishes when integrated over the complete sphere, so that the sum required is only a kind of residue depending upon the discontinuity of the summation.

The result is

$$S_4 = 3{\cdot}11. \quad \ldots \ldots \quad (69)$$

	ρ^2			ρ^2	
0, 0, 1	1	+ 3·5000	0, 0, 3	9	+ ·0144
0, 1, 1	2	− ·3094	0, 1, 3	10	+ ·0243
1, 1, 1	3	− ·1996	1, 1, 3	11	+ ·0075
0, 0, 2	4	+ ·1094	2, 2, 2	12	− ·0062
0, 1, 2	5	+ ·0501	0, 2, 3	13	− ·0015
1, 1, 2	6	− ·0397	1, 2, 3	14	− ·0095
0, 2, 2	8	− ·0097	0, 0, 4	16	+ ·0034
1, 2, 2	9	− ·0277	2, 2, 3	17	− ·0061
			0, 1, 4	17	+ ·0085

The results of our investigation have been expressed for the sake of simplicity in electrical language as the conductivity of a compound medium, but they may now be applied to certain problems of vibration. The simplest of these is the problem of wave-motion in a gaseous medium obstructed by rigid and fixed cylinders or spheres. It is assumed that the wave-length is very great in comparison with the period (α, β, γ) of the structure. Under these circumstances the flow of gas round the obstacles follows the same law as that of electricity, and the kinetic energy of the motion is at once given by the expressions already obtained. In fact the kinetic energy corresponding to a given total flow is increased by the obstacles in the same proportion as the electrical resistances of the original problem, so that the influence of the obstacles is taken into account if we suppose that the density of the gas is increased in the above ratio of resistances. In the case of cylinders in square order (35), the ratio is approximately $(1+p)/(1-p)$, and in the case of spheres in cubic order by (68) it is approximately $(1+\frac{1}{2}p)/(1-p)$.

But this is not the only effect of the obstacles which we must take into account in considering the velocity of propagation. The potential energy also undergoes a change. The space available for compression or rarefaction is now $(1-p)$ only instead of 1; and in this proportion is increased the potential energy corresponding to a given accumulation of gas*. For cylindrical obstruction the square of the velocity of propagation is thus altered in the ratio

$$\frac{1-p}{1+p} \div (1-p) = \frac{1}{1+p};$$

so that if μ denote the refractive index, referred to that of the unobstructed medium as unity, we find

$$\mu^2 = 1+p,$$

or

$$(\mu^2-1)/p = \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

which shows that a medium thus constituted would follow Newton's law as to the relation between refraction and density of obstructing matter. The same law (70) obtains also in the case of spherical obstacles; but reckoned absolutely the effect of spheres is only that of cylinders of halved density. It must be remembered, however, that while the velocity in the last case is the same in all directions, in the case of cylinders it is otherwise. For waves propagated parallel to the cylinders the velocity is uninfluenced by their presence. The medium containing the cylinders has therefore some of the properties which we are accustomed to associate with double refraction, although here the refraction is necessarily single. To this point we shall presently return, but in the meantime it may be well to apply the formulæ to the more general case where the obstacles have the properties of fluid, with finite density and compressibility.

* 'Theory of Sound,' § 303.

To deduce the formula for the kinetic energy we have only to bear in mind that density corresponds to electrical *resistance*. Hence, by (26), if σ denote the density of the cylindrical obstacle, that of the remainder of the medium being unity, the kinetic energy is altered by the obstacles in the approximate ratio

$$\frac{(\sigma+1)/(\sigma-1)+p}{(\sigma+1)/(\sigma-1)-p}. \quad . \quad . \quad . \quad . \quad . \quad (71)$$

The effect of this is the same as if the density of the whole medium were increased in the like ratio.

The change in the potential energy depends upon the "compressibility" of the obstacles. If the material composing them resists compression m times as much as the remainder of the medium, the volume p counts only as p/m, and the whole space available may be reckoned as $1-p+p/m$ instead of 1. In this proportion is the potential energy of a given accumulation reduced. Accordingly, if μ be the refractive index as altered by the obstacles,

$$\mu^2 = (71) \times (1-p+p/m). \quad . \quad . \quad . \quad (72)$$

The compressibilities of all actual gases are nearly the same, so that if we suppose ourselves to be thus limited, we may set $m=1$, and

$$\mu^2 = \frac{(\sigma+1)/(\sigma-1)+p}{(\sigma+1)/(\sigma-1)-p}; \quad . \quad . \quad . \quad . \quad (73)$$

or, as it may also be written,

$$\frac{\mu^2-1}{\mu^2+1}\,\frac{1}{p} = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

In the case of spherical obstacles of density σ we obtain in like manner ($m=1$),

$$\mu^2 = \frac{(2\sigma+1)/(\sigma-1)+p}{(2\sigma+1)/(\sigma-1)-2p}, \quad . \quad . \quad . \quad (75)$$

or

$$\frac{\mu^2-1}{\mu^2+\frac{1}{2}}\,\frac{1}{p} = \text{constant}. \quad . \quad . \quad . \quad . \quad (76)$$

In the general case, where m is arbitrary, the equation expressing p in terms of μ^2 is a quadratic, and there are no simple formulæ analogous to (74) and (76).

It must not be forgotten that the application of these formulæ is limited to moderately small values of p. If it be desired to push the application as far as possible, we must employ closer approximations to (26) &c. It may be remarked that however far we may go in this direction, the final formula will always give μ^2 explicitly as a function of p. For example, in the case of rigid cylindrical obstacles, we have from (35)

$$\mu^2 = (1-p)\frac{1+p-\cdot 3058p^4+\ldots}{1-p-\cdot 3058p^4+\ldots}. \quad . \quad . \quad (77)$$

It will be evident that results such as these afford no foundation for a theory by which the refractive properties of a mixture are to be deduced by addition from the corresponding properties of the components. Such theories require formulæ in which p occurs in the first power only, as in (76).

If the obstacles are themselves elongated, or, even though their form be spherical, if they are disposed in a rectangular order which is not cubic, the velocity of wave-propagation becomes a function of the direction of the wave-normal. As in Optics, we may regard the character of the refraction as determined by the form of the *wave-surface*.

The æolotropy of the structure will not introduce any corresponding property into the potential energy, which depends only upon the volumes and compressibilities concerned. The present question, therefore, reduces itself to the consideration of the kinetic energy as influenced by the direction of wave-propagation. And this, as we have seen, is a matter of the electrical resistance of certain compound conductors, on the supposition, which we continue to make, that the wave-length is very large in comparison with the periods of the structure. The theory of electrical conduction in general has been treated by Maxwell ('Electricity,' § 297). A parallel treatment of the present question shows that in all cases it is possible to assign a system of principal axes, having the property that if the wave-normal coincide with any one of them the direction of flow will also lie in the same direction, whereas in general there would be a divergence. To each principal axis corresponds an efficient "density," and the equations of motion, applicable to the medium in the gross, take the form

$$\sigma_x\frac{d^2\xi}{dt^2}=m_1\frac{d\delta}{dx}, \quad \sigma_y\frac{d^2\eta}{dt^2}=m_1\frac{d\delta}{dy}, \quad \sigma_z\frac{d^2\zeta}{dt^2}=m_1\frac{d\delta}{dz},$$

where ξ, η, ζ are the displacements parallel to the axes, m_1 is the compressibility, and

$$\delta=\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}.$$

If λ, μ, ν are the direction-cosines of the displacement, l, m, n of the wave-normal, we may take

$$\xi = \lambda\theta, \quad \eta = \mu\theta, \quad \zeta = \nu\theta,$$

where

$$\theta = e^{i(lx+my+nz-\mathrm{V}t)}.$$

Thus

$$d\delta/dx = -l\theta(l\lambda+m\mu+n\nu), \text{ \&c.},$$

and the equations become

$$\sigma_x\lambda\mathrm{V}^2 = m_1 l(l\lambda+m\mu+n\nu),$$
$$\sigma_y\mu\mathrm{V}^2 = m_1 m(l\lambda+m\mu+n\nu),$$
$$\sigma_z\nu\mathrm{V}^2 = m_1 n(l\lambda+m\mu+n\nu),$$

from which, on elimination of $\lambda : \mu : \nu$, we get

$$\mathrm{V}^2 = m_1\left(\frac{l^2}{\sigma_x}+\frac{m^2}{\sigma_y}+\frac{n^2}{\sigma_z}\right) = a^2l^2+b^2m^2+c^2n^2, \quad . \quad (78)$$

if a, b, c denote the velocities in the principal directions x, y, z.

The wave-surface after unit time is accordingly the ellipsoid whose axes are a, b, c.

As an example, if the medium, otherwise uniform, be obstructed by rigid cylinders occupying a moderate fraction (p) of the whole space, the velocity in the direction z, parallel to the cylinders, is unaltered; so that

$$c^2 = 1, \quad a^2 = b^2 = 1/(1+p).$$

In the application of our results to the electric theory of light we contemplate a medium interrupted by spherical, or cylindrical, obstacles, whose inductive capacity is different from that of the undisturbed medium. On the other hand, the magnetic constant is supposed to retain its value unbroken. This being so, the kinetic energy of the electric currents for the same total flux is the same as if there were no obstacles, at least if we regard the wave-length as infinitely great*. And the potential energy of electric displacement is subject to the same mathematical laws as the resistance of our compound electrical conductor, specific inductive capacity in the one question corresponding to electrical conductivity in the other.

Accordingly, if ν denote the inductive capacity of the material composing the spherical obstacles, that of the undisturbed medium being unity, then the approximate value of μ^2 is

* See Prof. Willard Gibbs's "Comparison of the Elastic and Electric Theories of Light." Am. Journ. Sci. xxxv. (1888).

given at once by (67). The equation may also be written in the form given by Lorentz,

$$\frac{\mu^2-1}{\mu^2+2}\frac{1}{p}=\frac{\nu-1}{\nu+2}=\text{constant}; \quad . \quad . \quad . \quad (79)$$

and, indeed, it appears to have been by the above argument that (79) was originally discovered.

The above formula applies in strictness only when the spheres are arranged in cubic order*, and, further, when p is moderate. The next approximation is

$$\mu^2 = 1+\frac{3p}{\frac{\nu+2}{\nu-1}-p-1{\cdot}65\frac{\nu-1}{\nu+4/3}p^{10/3}} \quad . \quad . \quad . \quad (80)$$

If the obstacles be cylindrical, and arranged in square order, the compound medium is doubly refracting, as in the usual electric theory of light, in which the medium is supposed to have an inductive capacity variable with the direction of displacement, independently of any discontinuity in its structure. The double refraction is of course of the uniaxal kind, and the wave-surface is the sphere and ellipsoid of Huygens.

For displacements parallel to the cylinders the resultant inductive capacity (analogous to conductivity in the conduction problem) is clearly $1-p+\nu p$; so that the value of μ^2 for the principal extraordinary index is

$$\mu^2 = 1+(\nu-1)p, \quad . \quad . \quad . \quad . \quad . \quad (81)$$

giving Newton's law for the relation between index and density.

For the ordinary index we have

$$\mu^2 = (26),$$

in which $\nu'=(1+\nu)/(1-\nu)$, while S_4, S_8 ... have the values given by (49). If we omit p^4 &c. we get

$$\mu^2=\frac{\nu'-p}{\nu'+p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

or

$$\frac{\mu^2-1}{\mu^2+1}\frac{1}{p}=-\frac{1}{\nu'}=\frac{\nu-1}{\nu+1}. \quad . \quad . \quad . \quad (83)$$

The general conclusion as regards the optical application is that, even if we may neglect dispersion, we must not expect such formulæ as (79) to be more than approximately correct in the case of dense fluid and solid bodies.

* An irregular isotropic arrangement would, doubtless, give the same result.